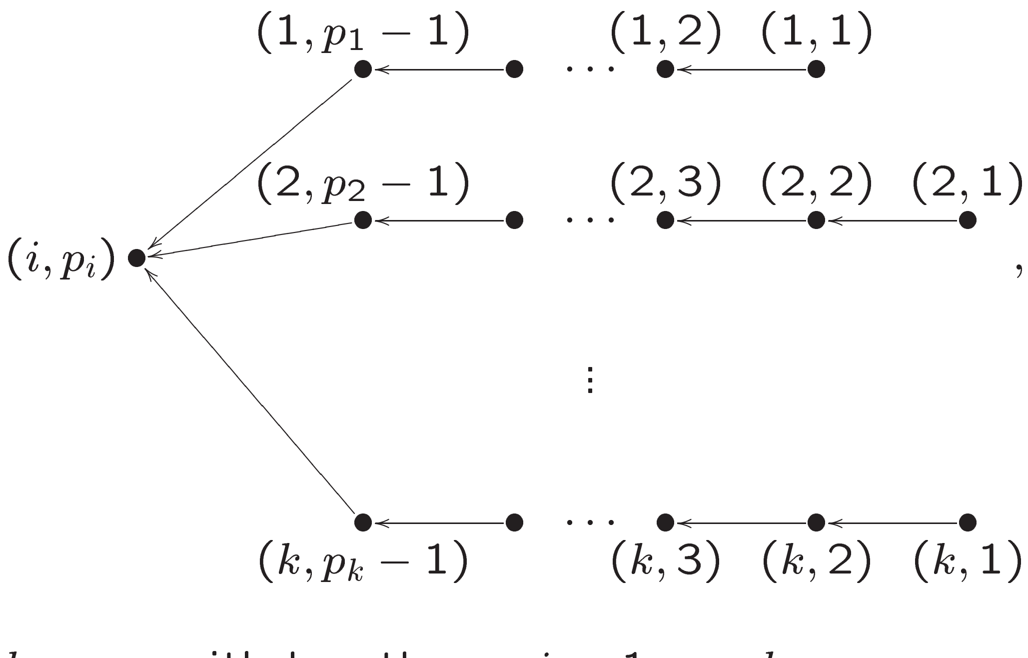
The s-tame dimension vectors for stars

Magyar, Weyman, Zelevinsky: Multiple Flag varieties of Finite Type (1999)

Classification of all dimension vectors for stars with only finitely many isomorphism classes of subspace representations

§1: Introduction

star: quiver of the following shape



, $i = 1, \dots, k$

k arms with lengths p_i, $i = 1, \dots, k$

subspace representation: all maps of the representation are injective

s-vectors: dimension vectors of subspace representations

An s-vector is **s-tame** $:\Leftrightarrow$ There exists an indecomposable one parameter family of subspace representations, but there is no decomposition of the s-vector into a sum of s-vectors with an indecomposable two parameter family of subspace representations for one of the summands.

$\{\text{s-vectors}\} \overset{1-1}{\longleftrightarrow} \{\text{tuples of compositions of a number}\}$

$$
\begin{array}{lllll}
 & d_{1,p_1-1} & \cdots & d_{12} & \leftarrow d_{11} \\
 & \swarrow & & & \\
d_{i,p_i} & \leftarrow d_{2,p_2-1} & \cdots & d_{22} & \leftarrow d_{21} \\
 & \nwarrow & \vdots & & \\
 & d_{k,p_k-1} & \cdots & d_{k2} & \leftarrow d_{k1}
\end{array}
$$

$$
\overset{\delta}{\mapsto} \quad ((d_{11}, d_{12}-d_{11}, \ldots, d_{1,p_1}-d_{1,p_1-1}),
$$
$$
\ldots, (d_{k1}, d_{k2}-d_{k1}, \ldots, d_{k,p_k}-d_{k,p_k-1}))
$$

tuples of compositions: have non negative entries (since the "dimension jumps" are non negative).

$n := d_{i,p_i}$

Tits form for dimension vectors of quivers

quiver $Q = (Q_0, Q_1, s, t)$;
Q_0: set of vertices, Q_1: set of arrows, $s(\alpha)$: starting point of $\alpha \in Q_1$, $t(\alpha)$: terminating point, $\mathbf{x} \in \mathbb{N}_0^{Q_0}$

$$q(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$$

Tits form for tuples of compositions of a number n

$\mathbf{a_i} = (a_{i,1}, \ldots, a_{i,p_i})$, $i = 1, \ldots, k$, compositions of n

$$\overline{q}(\mathbf{a_1}, \ldots, \mathbf{a_k}) = \frac{1}{2}\left(\sum_{i=1}^{k} \sum_{j=1}^{p_i} a_{ij}^2 + (2-k)n^2\right)$$

Properties of the Tits form

• independent of the ordering of the “dimension jumps” along the arms

• becomes minimal for fixed central dimension, if the “dimension jumps” are distributed as equally as possible in every arm

Order on tuples of compositions

For k and p_i, $i = 1, \ldots, k$, fixed

$(\mathbf{a_1}, \ldots, \mathbf{a_k}) \leq (\mathbf{b_1}, \ldots, \mathbf{b_k}) :\Leftrightarrow a_{i,j} \leq b_{i,j} \ \forall (i,j)$

$(\mathbf{a_1}, \ldots, \mathbf{a_k}) < (\mathbf{b_1}, \ldots, \mathbf{b_k}) :\Leftrightarrow (\mathbf{a_1}, \ldots, \mathbf{a_k}) \leq (\mathbf{b_1}, \ldots, \mathbf{b_k})$, and $\exists (i,j)$ with $a_{i,j} < b_{i,j}$

§2: Classification of s-tame vectors

From now on, assume the underlying quivers not to be of Dynkin or Euclidean type.

Theorem

A strict tuple $(\mathbf{a_1}, \dots, \mathbf{a_k})$ of compositions of a number is s-tame if and only if

- $\overline{q}(\mathbf{a_1}, \dots, \mathbf{a_k}) = 0$, and
- $\overline{q}(\mathbf{b_1}, \dots, \mathbf{b_k}) \geq 0$ for all $(\mathbf{b_1}, \dots, \mathbf{b_k}) \leq (\mathbf{a_1}, \dots, \mathbf{a_k})$

Properties of the s-tame tuples of compositions

The list consists of 49 tuples of compositions – when ordered increasingly along their arms –, and all of them have the following properties:

- $3 \leq k \leq 4$
- The central dimension is at most 14.
- The following arm lengths occur for the underlying quiver:
$(2,2,2,3)$, $(2,2,2,4)$ (for $k=4$); and $(3,3,4)$, $(3,3,5)$, $(3,3,6)$; $(2,4,5)$, $(2,4,6)$, $(2,4,7)$, $(2,4,8)$; $(2,3,7)$, $(2,3,8)$, $(2,3,9)$ (for $k=3$)

Finding s-tame vectors

First construct the list of all **s-hypercritical** tuples of compositions of a number, i.e. tuples of compositions with the properties

- $\overline{q}(\mathbf{a_1}, \ldots, \mathbf{a_k}) < 0$, and

- $\overline{q}(\mathbf{b_1}, \ldots, \mathbf{b_k}) \geq 0$ for all $(\mathbf{b_1}, \ldots, \mathbf{b_k}) < (\mathbf{a_1}, \ldots, \mathbf{a_k})$

Properties of the s-hypercritical tuples of compositions

The list consists of 19 tuples of compositions – when ordered increasingly along their arms –, and all of them have the following properties:

- $3 \leq k \leq 5$

- The central dimension is at most 12.

- The arm lengths for the underlying quiver are bounded by $(2, 2, 2, 2, 2)$ for $k = 5$, $(3, 3, 3, 4)$ for $k = 4$ and $(4, 5, 8)$ for $k = 3$.

Finding all s-tame tuples of compositions means now:

- Finding the smaller ones $(\mathbf{b_1}, \ldots, \mathbf{b_k})$ with $\overline{q}(\mathbf{b_1}, \ldots, \mathbf{b_k}) = 0$, and

- Finding the incomparable ones $(\mathbf{b_1}, \ldots, \mathbf{b_k})$ with $\overline{q}(\mathbf{b_1}, \ldots, \mathbf{b_k}) = 0$.

Example

One of the s-hypercritical tuples of compositions of 12 is the following:

$((6,6),(4,4,4),(1,1,2,2,2,2,2))$

This leads to the following s-tame tuples of compositions:

smaller ones:

- $((3,4),(2,2,3),(1,1,1,1,1,1,1))$
- $((4,4),(2,3,3),(1,1,1,1,1,1,2))$
- $((4,5),(3,3,3),(1,1,1,1,1,2,2))$
- $((5,5),(3,3,4),(1,1,1,1,2,2,2))$
- $((5,6),(3,4,4),(1,1,1,2,2,2,2))$

and incomparable ones:

- $((5,7),(4,4,4),(1,1,2,2,2,2,2))$
- $((6,6),(3,4,5),(1,1,2,2,2,2,2))$
- $((6,6),(4,4,4),(1,1,1,2,2,2,3))$

- $((6,7),(3,5,5),(1,2,2,2,2,2,2))$
- $((7,7),(3,5,6),(2,2,2,2,2,2,2))$

§3: Construction of indecomposable one parameter families of subspace representations for the s-tame vectors

• Take "orbits" under the Auslander-Reiten translate (indecomposable representations go to indecomposable representations, numbers of parameters are preserved)

• Restrict to smaller quivers (e.g. of Euclidean or Dynkin type), find an appropriate decomposition of the new tuple of compositions into tuples of compositions for indecomposable representations and find the "right embeddings"

Examples

The following tuples of compositions of 6 (resp. 9) are s-tame.

1. $((3,3),(1,1,2,2),(1,1,1,1,2))$

$$\rightsquigarrow \begin{array}{cccccccc} & & & 3 & & & & \\ 1 & 2 & 4 & 6 & 4 & 3 & 2 & 1 \end{array}$$

2. $((5,4),(1,2,3,3),(2,2,2,2,1))$

$$\rightsquigarrow \begin{array}{cccccccc} & & & 5 & & & & \\ 1 & 3 & 6 & 9 & 8 & 6 & 4 & 2 \end{array}$$

3. $((5,4),(1,3,2,3),(2,1,2,2,2))$

$$\rightsquigarrow \begin{array}{cccccccc} & & & 5 & & & & \\ 1 & 4 & 6 & 9 & 7 & 5 & 3 & 2 \end{array}$$

§4: s-tame $\neq$ tame

Take for example $((3,1),(2,2),(2,2),(1,1,2))$.

This is s-tame, but not tame. For example, there is a two parameter family of representations for this tuple of compositions of 4.

One can construct it by decomposing the corresponding dimension vector

$$\begin{matrix} & 3 & \\ 2 & 4 & 2 \\ & 2 & \\ & 1 & \end{matrix}$$

into

$$\begin{matrix} & 2 & \\ 2 & 4 & 2 \\ & 2 & \\ & 1 & \end{matrix} \quad \oplus \quad \begin{matrix} & 1 & \\ 0 & 0 & 0 \\ & 0 & \\ & 0 & \end{matrix} .$$

For the first dimension vector there is an indecomposable two parameter family of representations, but one can also see that the family of representations constructed in this way is *not* a family of *subspace* representations.