Numerical approximation of stochastic PDEs

Peter Kloeden

Johann Wolfgang Goethe Universität
Frankfurt am Main

joint work with Arnulf Jentzen

Supported by DFG project:

"Pathwise numerical analysis of stochastic evolution equations"

Stochastic PDEs

Consider a stochastic reaction-diffusion equation with a Dirichlet boundary condition on a bounded domain $\mathcal{D}$ in $\mathbb{R}^d$ with smooth boundary.

$$\boxed{dU_t = \{AU_t + f(U_t)\}\ dt + g(U_t)\, dW_t, \qquad (1)}$$

Assume that

- the eigenvalues λ_j and the corresponding eigenfunctions $\phi_j \in H_0^{1,2}(\mathcal{D})$ of the operator $-A$, i.e.,

$$-A\phi_j = \lambda_j \phi_j, \qquad j = 1, 2, \ldots,$$

form an orthonormal basis in $L_2(\mathcal{D})$ with $\lambda_j \to \infty$ as $j \to \infty$

- W_t is a cylindrical Q-Wiener process – more details later

An old result

W. Grecksch and P.E. Kloeden, Time-discretised Galerkin approximation of parabolic stochastic PDEs, *Bull. Austral. Math. Soc.*, **54** (1996), 79-85.

• Assume that W_t is a scalar Wiener process, i.e. one-dimensional

• Consider the Galerkin approximation of the SPDE (1), i.e. an N–dimensional Ito SDE obtained by projecting the SPDE onto the N–dimensional subspace $\mathcal{X}_N$ of $L_2(\mathcal{D})$ spanned by the $\{\phi_1, \cdots, \phi_N\}$

$$\boxed{dU_t^N = \left\{A_N U_t^N + f_N(U_t^N)\right\}\, dt + g_N(U_t^N)\, dW_t \qquad (2)}$$

where we write U^N synonomously for $(U^{N,1}, \cdots, U^{N,N})^\top \in \mathbb{R}^N$ or $\sum_{j=1}^N U^{N,j}\phi_j \in \mathcal{X}_N$ according to the context

Here $A_N = P_N A\big|_{\mathcal{X}_N}$, $f_N = P_N f\big|_{\mathcal{X}_N}$ and $g_N = P_N g\big|_{\mathcal{X}_N}$ where f and g are now interpreted as mappings of $L_2(\mathcal{D})$ or $H_0^{1,2}(\mathcal{D})$ into itself, where P_N is the projection of $L_2(\mathcal{D})$ or $H_0^{1,2}(\mathcal{D})$ onto $\mathcal{X}_N$

• Apply an order γ strong Taylor scheme with constant time–step h to the Ito-Galerkin SDE (2)

$$\boxed{Y_{k+1}^N = Y_k^N + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v\}} F_\alpha^N \left(Y_k^N \right) I_{\alpha,k,h}, \qquad (3)}$$

with coefficient functions F_α^N and multiple stochastic integrals $I_{\alpha,k,\Delta}$.

$$\text{e.g. Euler-Maruyama scheme } \gamma = \frac{1}{2}, \quad \text{Milstein scheme } \gamma = 1$$

Theorem 1 *The global space–time discretization error of the order γ strong Taylor scheme (3) with constant time–step h applied to the N–dimensional Ito–Galerkin SDE (2) of the SPDE(1) has the form*

$$\mathbb{E}|U_{k\Delta} - Y_k^N| \leq K \left(\lambda_{N+1}^{-\frac{1}{2}} + \lambda_N^{[\gamma+\frac{1}{2}]+1} h^\gamma \right), \qquad (4)$$

where $[x]$ is the integer part of the real number x and the constant K depends on $E\|U_0\|^2$, bounds on the f and g coefficients of the SPDE and the length of the time interval $[0,T]$ under consideration.

Similar results in

E. Hausenblas, Approximation for semilinear stochastic evolution equations, *Potential Analysis*, **18** (2003), 141-186.

Shortcomings

- W_t is only one-dimensional in Grecksch & Kloeden — more general in Hausenblas

- proofs of the convergence of Taylor schemes for SDE in the Kloeden & Platen and Milstein monographs assume that partial derivatives of the coefficient functions of the SDE are <u>uniformly bounded</u> on $\mathbb{R}^N$.

- to obtain an overall convergence rate we need to balance the two components of the error bound. This requires the time–step h to become very small as N increases due to product

$$\lambda_N^{[\gamma+\frac{1}{2}]+1} h^{\gamma}$$

since $\lambda_N \to \infty$ as $N \to \infty$.

Overall rate

Suppose that a numerical scheme with time–step $h = \frac{T}{M}$ requires N arithmetical operations, random number and function evaluations per time–step to calculate the next iterate $Y_k^{N,M}$, then the computational cost of the scheme is $K = N \cdot M$.

If the scheme has error bound

$$\sup_{k=0,\dots,M} \left(\mathbb{E} \left| U_{t_k} - Y_k^{N,M} \right|^2 \right)^{\frac{1}{2}} \leq C \left(\frac{1}{N^\alpha} + \frac{1}{M^\beta} \right) \tag{5}$$

for $\alpha, \beta > 0$, then the optimal overall rate is $\dfrac{\alpha\beta}{\alpha + \beta}$ with respect to the computational cost, i.e.

$$\sup_{k=0,\dots,M} \left(\mathbb{E} \left| U_{t_k} - Y_k^{N,M} \right|^2 \right)^{\frac{1}{2}} \leq C \cdot K^{-\dfrac{\alpha\beta}{\alpha+\beta}}.$$

For example, if $\alpha = \dfrac{1}{2}$ and $\beta = 1$, then obtain the overall rate is $\dfrac{1}{3}$

Other results in the literature

Much of the literature is concerned with consider a semilinear stochastic heat equation with additive space-time white noise on the one dimensional domain $[0,1]$ over the time interval $[0,T]$ with $T=1$, i.e.

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) + \dot{W}_t \qquad (6)}$$

with the Dirichlet boundary condition.

• Gyöngy & Nualart (1995) introduced an implicit numerical scheme for the SPDE (6) and showed that it converges strongly to the exact solution without giving a rate.

• Shardlow (1999) applied finite differences to the SPDE (6) to obtain a spatial discretization which he then discretized in time with a θ-method. This had an overall convergence rate $\frac{1}{6}$ with respect to the computational cost.

• Gyöngy (1998, 1999) also applied finite differences an SPDE driven by space-time white noise and then used several temporal implicit and explicit schemes, in particular, the linear-implicit Euler scheme. He showed that these schemes converge with order $\frac{1}{2}$ in the space and with order $\frac{1}{4}$ in time (assuming a smooth initial value). Hence, he obtained an overall convergence rate of $\frac{1}{6}$ with respect to the computational cost in space and time.

• Davie & Gaines (2000) showed that any numerical scheme applied to the SPDE (6) with $f=0$ which uses only values of the noise W_t cannot converge faster than the rate of $\frac{1}{6}$ with respect to the computational cost.

A.M. Davie and J.G. Gaines, Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations, *Mathematics of Computation*, **70** (2000), 121-134.

• Müller-Gronbach & Ritter (2007) also showed that this is a lower bound for the convergence rate. They even showed that one cannot improve this rate of convergence by choosing non-uniform time steps.

T. Müller-Gronbach and K. Ritter, Lower bounds and nonuniform time discretization for approximation of stochastic heat equations. *Found. Comput. Math.* **7** (2007), no. 2, 135–181.

• Hausenblas (2003) applied the linear-implicit and explicit Euler scheme and the Crank-Nicholson scheme to an SPDE (6) driven by an infinite dimensional noise. In the case of a smoother noise, i.e. trace-class noise, she obtained the order $\frac{1}{4}$ with respect to the computational cost. However, in the general case of space-time white noise the convergence rate is no better $\frac{1}{6}$.

• Lord & Rougemont (2004) also considered the SPDE (6) with a smoother noise. They discretized the Galerkin-SDE in time with numerical scheme that uses the factor $e^{A_N t}$ with $A_N = P_N A$ for $A = \Delta$.

$$\boxed{X_{k+1}^{N,M} = e^{A_N h}\left(X_k^{N,M} + h f_N(X_k^{N,M}) + W_{t_{k+1}}^N - W_{t_k}^N\right) \quad (7)}$$

They showed that this scheme is useful when the noise is very smooth in space, in particular with Gevrey regularity. However, in the general case of space-time white noise the scheme (7) converges also only with the rate of $\frac{1}{6}$ with respect to the computational cost as a consequence of the work of Davie & Gaines (2000).

A numerical scheme of higher order

Davie & Gaines (2000, page 129) remarked that it may be possible to improve the convergence rate by using suitable linear functionals of the noise. This suggestion was used in the following paper.

A. Jentzen and P.E. Kloeden, Overcoming the order barrier in the numerical approximation of SPDEs with additive space-time noise, *Proc. Roy. Soc. London* (to appear)

Let fix $T > 0$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $\mathcal{F}_t$, $t \in [0, T]$, and let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space with norm $|\cdot|$.

We consider a parabolic SPDE with additive noise,

$$\boxed{dU_t = [AU_t + f(U_t)] \; dt + dW_t, \qquad U_0 = u_0, \qquad (8)}$$

where A is an in general unbounded operator (for example $A = \Delta$), f is a nonlinear continuous function and W_t is a cylindrical Wiener process.

We interpret the SPDE (8) in such a space H in the mild sense, i.e. as satisfying the integral equation

$$\boxed{U_t = e^{At} u_0 + \int_0^t e^{A(t-s)} f(U_s) \, ds + \int_0^t e^{A(t-s)} \, dW_s. \qquad (9)}$$

and introduce a finite dimensional SDE in the space $H_N := P_N H$ (or, equivalently, in $\mathbb{R}^N$) by

$$\boxed{dU_t^N = \left(A_N U_t^N + f_N(U_t^N) \right) \, dt + dW_t^N, \qquad (10)}$$

which is the Galerkin projection of the SPDE (8) onto H_N, where $A_N = P_N A$ is the (matrix) operator $A_N : H_N \to H_H$.

Assumptions

(A1) Assumption: Linear operator A

There exist sequences of real eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and eigenfunctions $\{e_n\}_{n\geq 1}$ of A such that the linear operator $A : D(A) \subset H \to H$ is given by

$$Av = \sum_{n=1}^{\infty} -\lambda_n \langle e_n, v \rangle e_n$$

for all $v \in D(A)$ with $D(A) = \left\{ v \in H \middle| \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle e_n, v \rangle|^2 < \infty \right\}$.

(A2) Assumption: Cylindrical Brownian motion W_t

There exist a sequence $q_n \geq 0$, $n \geq 1$, of positive real numbers, a real number $\gamma > 0$ such that

$$\sum_{n=1}^{\infty} (\lambda_n)^{2\gamma - 1} q_n < \infty$$

and independent real valued $\mathcal{F}_t$-Brownian motions β_t^n, $t \geq 0$, for $n \geq 1$, i.e. each β_t^n is $\mathcal{F}_t$-adapted and the increments $\beta_{t+h}^n - \beta_t^n$, $h > 0$, are independent of $\mathcal{F}_t$. Then, the cylindrical Brownian motion W_t is given by

$$W_t = \sum_{n=1}^{\infty} \sqrt{q_n}\, e_n\, \beta_t^n. \tag{11}$$

(A3) Assumption: Nonlinearity f

The nonlinearity $f : H \to H$ is two times continuously Frechet differentiable and its derivatives satisfy

$$|f'(x) - f'(y)| \leq L\,|x-y|\,, \qquad \left|(-A)^{(-r)} f'(x)(-A)^r v\right| \leq L\,|v|$$

for all $x, y \in H$, $v \in D((-A)^r)$ and $r = 0, \frac{1}{2}, 1$, and

$$\left|A^{-1} f''(x)(v,w)\right| \leq L\left|(-A)^{-\frac{1}{2}} v\right|\left|(-A)^{-\frac{1}{2}} w\right|$$

for all $v, w, x \in H$, where $L > 0$ is a positive constant.

(A4) Assumption: Initial value u_0

u_0 is a $D(A^\gamma)$ valued random variable, which satisfies

$$\mathbb{E}\,|(-A)^\gamma u_0|^4 < \infty,$$

where $\gamma > 0$ given in Assumption (A2).

Remarks

- Under Assumptions (A1)-(A4) that the SPDE (8) has a unique mild solution U_t on the time interval $[0,T]$, where U_t is the predictable stochastic process in $D(A^\gamma)$ given by (9). (Da Prato & Zabczyk (1992), Prévot & Röckner (2007))

- Since Assumption (A3) also applies to f_N, the SDE (10) also has a unique solution on $[0,T]$, which is given (implicitly) by

$$\boxed{U_t^N = e^{A_N t} u_0^N + \int_0^t e^{A_N(t-s)} f_N(U_s^N)\, ds + \int_0^t e^{A_N(t-s)}\, dW_s^N. \quad (12)}$$

- The function f is usually given as a real valued function of a real variable, but in the SPDE (8) it is considered as a function defined on H and taking values in some function space such as a subspace of H.

- The above series (11) for the cylindrical Wiener process may not converge in H, but in some space U_1 into which H can be embedded.

Our formalism allows us to consider space-time white noise (in one-dimensional domains) as well as trace class noise.

The exponential Euler scheme

$$\boxed{\begin{aligned} V_{k+1}^{N,M} &= e^{A_N h} V_k^{N,M} + A_N^{-1}\left(e^{A_N h} - I\right) f_N(V_k^{N,M}) \\ &\quad + \int_{t_k}^{t_{k+1}} e^{A_N(t_{k+1}-s)}\, dW_s^N \qquad (13) \end{aligned}}$$

with time–step $h = \dfrac{T}{M}$ for some $M \in \mathbb{N}$ and discretization times $t_k = kh$ for $k = 0, 1, \ldots, M$.

This scheme is easier to simulate than may seem on the first sight. Denoting the components of $V_k^{N,M}$ and f_N by

$$V_{k,i}^{N,M} = \left\langle e_i, V_k^{N,M} \right\rangle, \qquad f_N^i = \langle e_i, f_N \rangle, \qquad i = 1, \ldots, N,$$

we can rewrite the numerical scheme (13) as

$$\boxed{\begin{aligned} V_{k+1,1}^{N,M} &= e^{-\lambda_1 h} V_{k,1}^{N,M} + \frac{(1 - e^{-\lambda_1 h})}{\lambda_1} f_N^1(V_k^{N,M}) + \left(\frac{q_1}{2\lambda_1}(1 - e^{-2\lambda_1 h})\right)^{\frac{1}{2}} R_k^1 \\ &\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ V_{k+1,N}^{N,M} &= e^{-\lambda_N h} V_{k,N}^{N,M} + \frac{(1 - e^{-\lambda_N h})}{\lambda_N} f_N^N(V_k^{N,M}) + \left(\frac{q_N}{2\lambda_N}(1 - e^{-2\lambda_N h})\right)^{\frac{1}{2}} R_k^N, \end{aligned}}$$

where the R_k^i for $i = 1, \ldots, N$ and $k = 0, 1, \ldots, M-1$ are independent, standard normally distributed random variables.

The following theorem states the strong convergence of the exponential Euler scheme (13) and provides a rate for this strong convergence.

Theorem 2 *Suppose that Assumptions (A1-(A4) are satisfied. Then, there is a constant $C_T > 0$ such that*

$$\sup_{k=0,\dots,M} \left(\mathbb{E} \left| U_{t_k} - V_k^{N,M} \right|^2 \right)^{\frac{1}{2}} \leq C_T \left(\lambda_N^{-\gamma} + \frac{\log(M)}{M} \right) \tag{14}$$

holds for all $N, M \in \mathbb{N}$, where U_t is the solution of SDE (8), *$V_k^{N,M}$ is the numerical solution given by* (13), *$t_k = T\dfrac{k}{M}$ for $k = 0, 1, \dots, M$, and $\gamma > 0$ is the constant given in Assumption (A2).*

In fact, the exponential Euler scheme (13) converges in time with a strong order $1 - \varepsilon$ for an arbitrary small $\varepsilon > 0$ since $\log(M)$ can be estimated by M^ε, so

$$\frac{\log(M)}{M} \sim h \log \frac{1}{h} \approx h^{1-\varepsilon}$$

Importantly, the error coefficient C_T does <u>not</u> depend on the dimension N of the Ito-Galerkin SDE.

WHY? the integral $\int_{t_k}^{t_{k+1}} e^{A_N(t_{k+1}-s)}\, dW_s^N$ includes includes more information about the noise on the discretization interval

Main novelty in proof

In the literature the error component

$$E_1 = \left| \sum_{l=0}^{k} \int_{t_l}^{t_{l+1}} e^{A_N(t_{k+1}-s)} I_{l,s} (e^{A_N(s-t_l)} - I) U_{t_l}^N ds \right|$$

where

$$I_{l,s} := \int_0^1 f_N'(U_{t_l}^N + r(U_s^N - U_{t_l}^N))\, dr$$

is usually estimated as

$$\begin{aligned} E_1 &\leq \sum_{l=0}^{k} \int_{t_l}^{t_{l+1}} \left| e^{A_N(t_{k+1}-s)} I_{l,s} \left(e^{A_N(s-t_l)} - I \right) U_{t_l}^N \right| ds \\ &\leq C \sum_{l=0}^{k} \int_{t_l}^{t_{l+1}} \left| \left(e^{A_N(s-t_l)} - I \right) U_{t_l}^N \right| ds \\ &= C \sum_{l=0}^{k} \int_{t_l}^{t_{l+1}} \left| \left(e^{A_N(s-t_l)} - I \right) (-A_N)^{\gamma} \right| \left| (-A_N)^{-\gamma} U_{t_l}^N \right| ds \\ &\leq Ch^{\gamma} \sum_{l=0}^{k} \int_{t_l}^{t_{l+1}} \left| (-A_N)^{-\gamma} U_{t_l}^N \right| ds, \end{aligned}$$

which yields

$$|E_1|_{L^2(\Omega)} \leq Ch^{\gamma}.$$

In this way one can only obtain a convergence rate of γ in time, which for our example would be $\gamma = \dfrac{1}{4} - \varepsilon$ for $\varepsilon > 0$ arbitrarily small.

To obtain a higher order, we need to use the smoothening effect of the term $e^{A_N(t_{k+1}-s)}$ as we did above, which is based on the estimate

$$\left| A_N e^{A_N \tau} \right| \leq C \frac{1}{\tau}.$$

First we see that

$$E_1 \;\leq\; \sum_{l=0}^{k} \int_{t_l}^{t_{l+1}} \left| e^{A_N(t_{k+1}-s)} I_{l,s} (e^{A_N(s-t_l)} - I) U_{t_l}^N \right| \, ds,$$

where

$$\begin{aligned} \left| \int_{t_k}^{t_{k+1}} \left| e^{A_N(t_{k+1}-s)} I_{l,s} (e^{A_N(s-t_l)} - I) U_{t_l}^N \right| \, ds \right|_{L^2(\Omega)} &\leq C \left| \int_{t_k}^{t_{k+1}} \left| U_{t_l}^N \right| \, ds \right|_{L^2(\Omega)} \\ &\leq Ch \end{aligned}$$

due to Assumption (A3). Hence

$$\begin{aligned} |E_1|_{L^2(\Omega)} &\leq Ch + \sum_{l=0}^{k-1} \left| \int_{t_l}^{t_{l+1}} \left| e^{A_N(t_{k+1}-s)} I_{l,s} (e^{A_N(s-t_l)} - I) U_{t_l}^N \right| \, ds \right|_{L^2(\Omega)} \\ &\leq Ch + \sum_{l=0}^{k-1} \left| \int_{t_l}^{t_{l+1}} \left| A_N e^{A_N(t_{k+1}-s)} \right| \left| A_N^{-1} I_{l,s} (e^{A_N(s-t_l)} - I) U_{t_l}^N \right| \, ds \right|_{L^2(\Omega)} \\ &\leq Ch + C \left(\sum_{l=0}^{k-1} \left| \int_{t_l}^{t_{l+1}} (t_k - t_l)^{-1} \left| A_N^{-1} (e^{A_N(s-t_l)} - I) U_{t_l}^N \right| \, ds \right|_{L^2(\Omega)} \right) \\ &\leq Ch + C \left(\sum_{l=0}^{k-1} \left| \int_{t_l}^{t_{l+1}} (k-l)^{-1} \left| U_{t_l}^N \right| \, ds \right|_{L^2(\Omega)} \right) \\ &\leq Ch + C \left(\sum_{l=0}^{k-1} (k-l)^{-1} \left| \int_{t_l}^{t_{l+1}} \left| U_{t_l}^N \right| ds \right|_{L^2(\Omega)} \right) \end{aligned}$$

due to Assumption (A3). Finally, we obtain

$$\begin{aligned} |E_1|_{L^2(\Omega)} &\leq Ch + Ch \left(\sum_{l=0}^{k-1} (k-l)^{-1} \right) \\ &= Ch + Ch \left(\sum_{l=1}^{k} \frac{1}{l} \right) \leq Ch \left(\sum_{l=1}^{M} \frac{1}{l} \right) \leq \frac{C \log(M)}{M}, \end{aligned}$$

which is the claim for E_1.

Numerical results

We consider the semilinear stochastic heat equation (6) on the one dimensional domain $[0,1]$ with $f(u) = \frac{1}{2}u$, i.e.

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}u + \dot{W}_t \qquad (15)}$$

with the Dirichlet boundary condition and the initial value $u_0(x) = \sum_{n=1}^{\infty} n^{-0.6} \sin n\pi x$.

Since f to be linear here since we have an exact solution for comparison with the numerical solution.

Linear-implicit Euler scheme

$$E_{k+1}^{N,M} = (I - hA_N)^{-1} \left(\left[1 + \frac{1}{2}h \right] E_k^{N,M} + W_{t_{k+1}}^N - W_{t_k}^N \right)$$

Lord-Rougemont scheme

$$X_{k+1}^{N,M} = e^{A_N h} \left(\left[1 + \frac{1}{2}h \right] X_k^{N,M} + W_{t_{k+1}}^N - W_{t_k}^N \right).$$

Exponential Euler scheme

$$\begin{aligned} V_{k+1}^{N,M} &= e^{A_N h} V_k^{N,M} + \frac{1}{2} A_N^{-1} \left(e^{A_N h} - I \right) V_k^{N,M} \\ &\quad + \int_{t_k}^{t_{k+1}} e^{A_N (t_{k+1} - s)} \, dW_s^N \end{aligned}$$

These schemes all converge with order $\frac{1}{2}$ in the spatial variable, so we consider their convergence rate in time.

Fix $N = 200$ (space discretization) and then apply the above schemes with different $M = 10, 20, 25, 40, 50, 80, 100, 200, 500, 1000$ (time discretization).

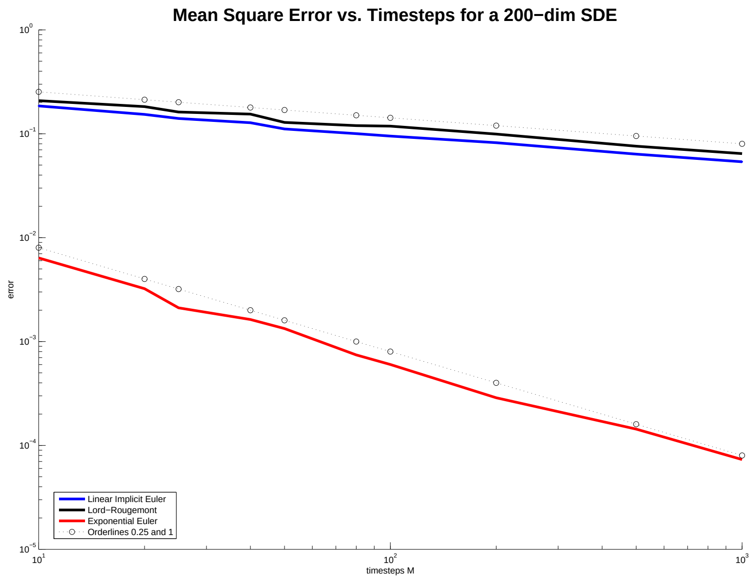

Figure 1: Mean Square Error vs. Timesteps for a 200-dim SDE as log-log plot.

The linear-implicit Euler and Lord-Rougemont schemes converge with temporal rate $\frac{1}{4}$, while the exponential Euler scheme converges with temporal rate 1.

We now consider the convergence rate with respect to their computational cost.

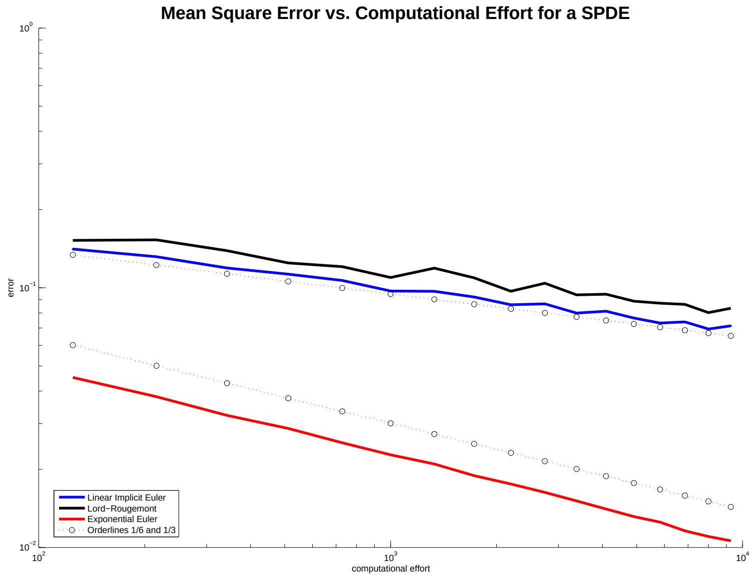

Figure 2: Mean Square Error vs. Computational Effort as log-log plot.

Here the linear-implicit Euler and Lord-Rougemont schemes clearly converge with the rate $\frac{1}{6}$, while the exponential Euler scheme converges with the rate $\frac{1}{3}$. All three schemes thus converge with their theoretically predicted order.

Shortcomings once again

Theorem 2 has several serious shortcomings:

- we need to know the eigenvalues and eigenfunctions of the operator A

<u>but</u> finite elements ???

- Assumption A2 on the nonlinearity f is very restrictive and excludes functions like

$$f(u) = \frac{u}{1+u^2}, \qquad f(u) = u - u^3$$

$\star$ This problem also arises for finite dimensional Ito SDE for which it can be overcome by using pathwise convergence rather than strong convergence

A. Jentzen, P.E. Kloeden, A. Neuenkirch, Convergence of numerical approximations of stochastic differential equations on domains: higher order convergence rates without global Lipschitz coefficients, *Numerische Mathematik* (to appear)

A. Jentzen, higher order pathwise numerical approximation of SPDEs with additive noise, *SIAM Numer. Anal.* (submitted)

$\star$ Another problem is the Fréchet differentiability of the function f when cosnidered as a mapping between function spaces.

- For Taylor expansions of solutions of SPDE in Hilbert spaces and Taylor schemes for SPDE see

A. Jentzen and P.E. Kloeden, Taylor-expansions of solutions of stochastic partial differential equations with additive noise, *Annals of Probab.* (submitted)