Enhanced diffusion in a tilted periodic potential: Universality, scaling, and the effect of disorder

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Condensed Matter Theory, University of Bielefeld

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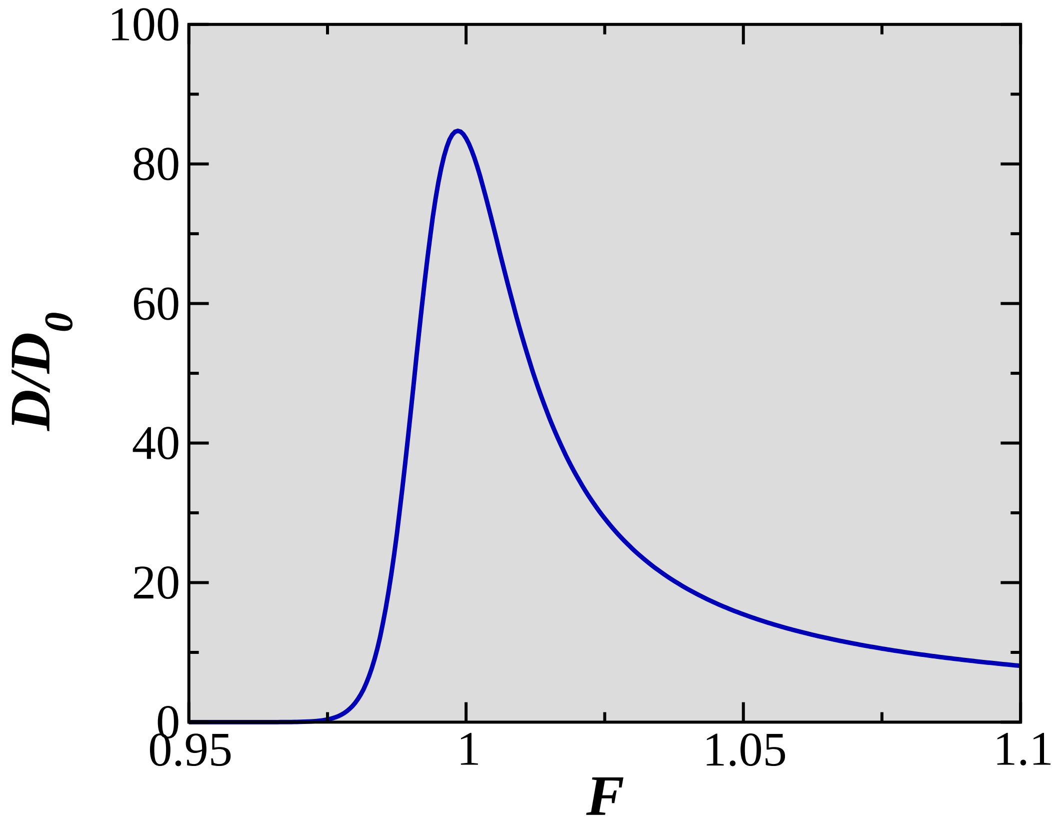

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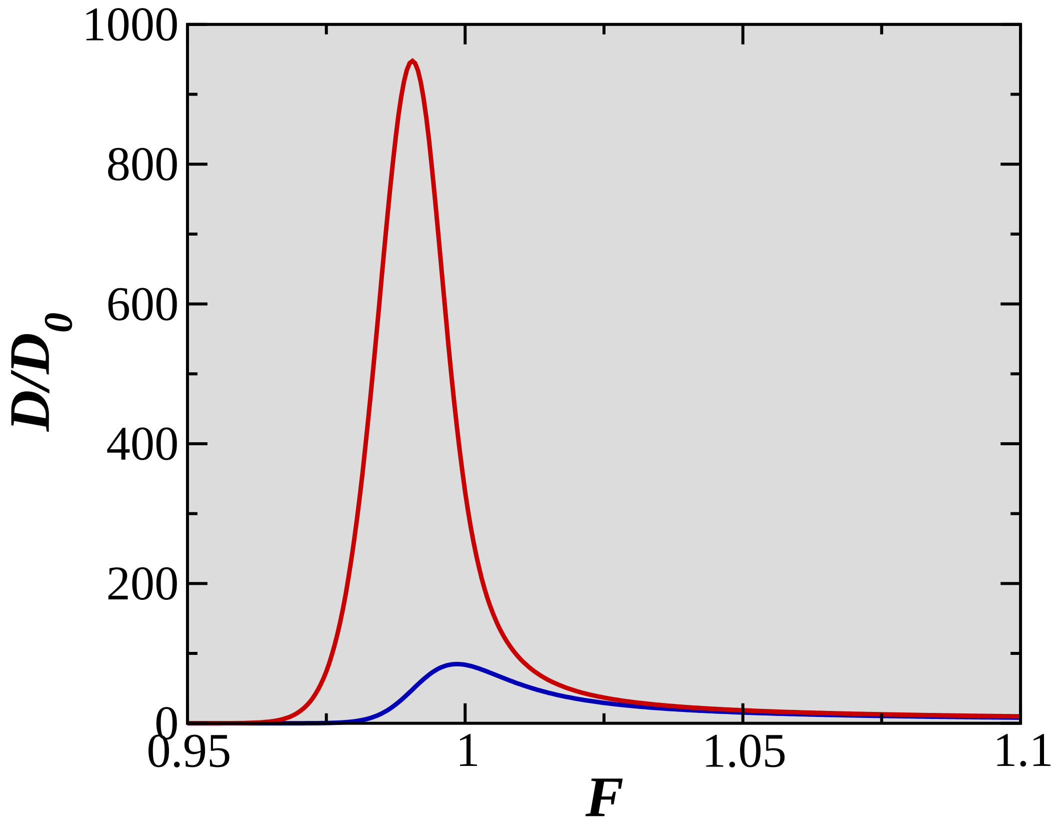

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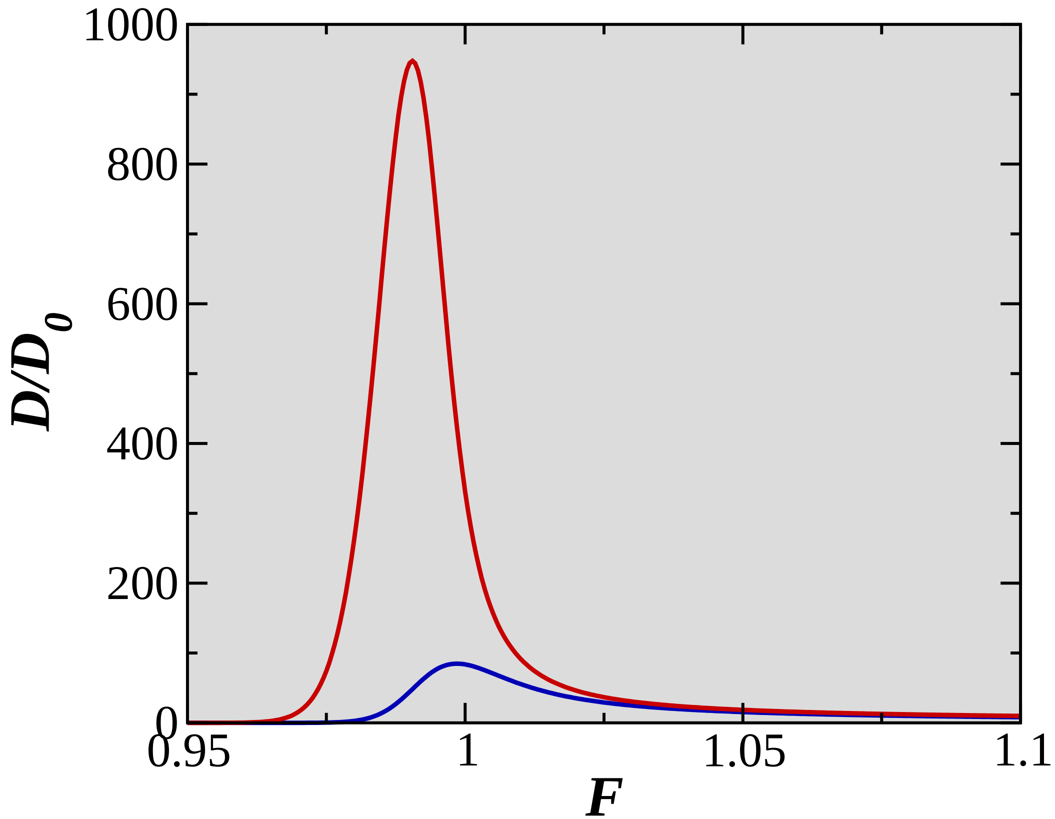

- Free diffusion
- Model
- **Diffusion enhancement**
- Two experiments
- Conclusions

Introduction: Free thermal diffusion

μm-sized particle in a thermal environment ("bath")

e.g.: pollen in water $\longrightarrow$ jittering motion [R. Brown, 1827]

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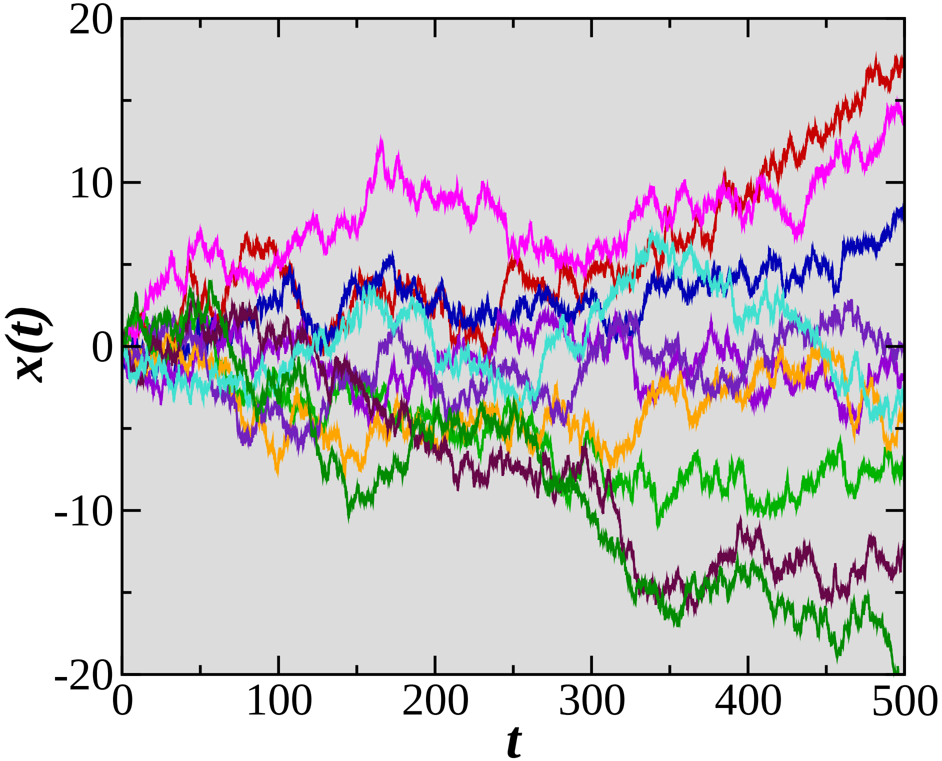

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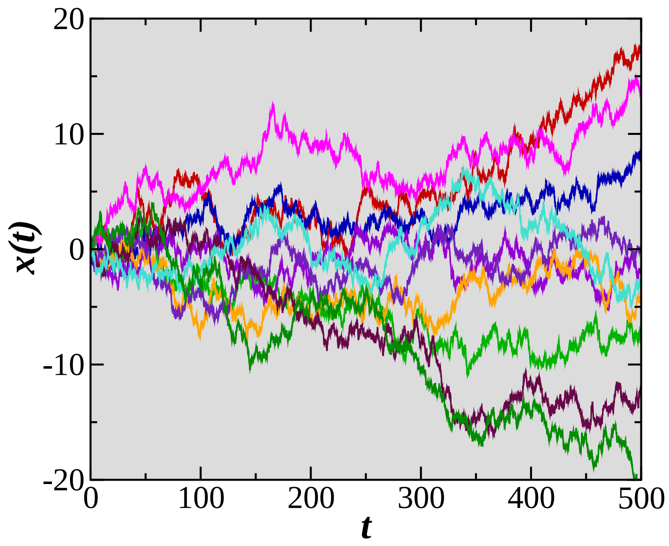
mathematical description:
[A. Einstein, 1905; M. Smoluchowski, 1906]

$$m\,\ddot{x}(t) = -\eta\,\dot{x}(t) + \sqrt{2\eta\,kT}\,\xi(t)$$



η: viscous friction coefficient
kT: thermal energy

$\xi(t)$: Gaussian white noise with
$\langle \xi(t)\rangle = 0$, $\langle \xi(t)\,\xi(s)\rangle = \delta(t-s)$

$$\Rightarrow \quad \boxed{D_0 = \lim_{t\to\infty} \frac{\langle x^2(t)\rangle}{2t} = \frac{kT}{\eta}}$$

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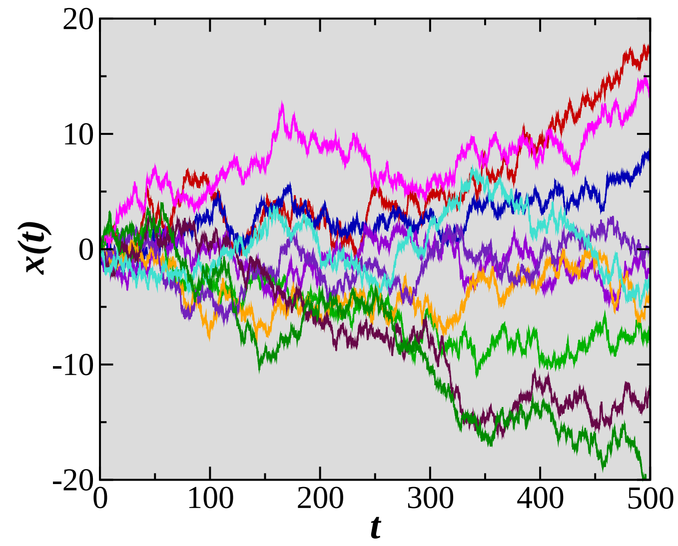
typically: inertial effects negligibly small
$\hookrightarrow$ overdamped dynamics with $m = 0$

$\Rightarrow$ $$\boxed{\eta\,\dot{x}(t) = \sqrt{2\eta\,kT}\,\xi(t)}$$



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Model: Equation of Motion

overdamped Brownian particle in a 1-dimensional potential:

$$\eta\, \dot{x}(t) = -V'(x(t)) - W'(x(t)) + \sqrt{2\eta\, kT}\,\xi(t)$$

tilted periodic potential:

$V(x) = V_0(x) - xF$
with $V_0(x+L) = V_0(x)$

random "deviations" $W(x)$:

unbiased, homogeneous Gaussian disorder with correlation

$$c(x) = \langle W(y) W(y+x) \rangle$$

η: viscous friction coefficient
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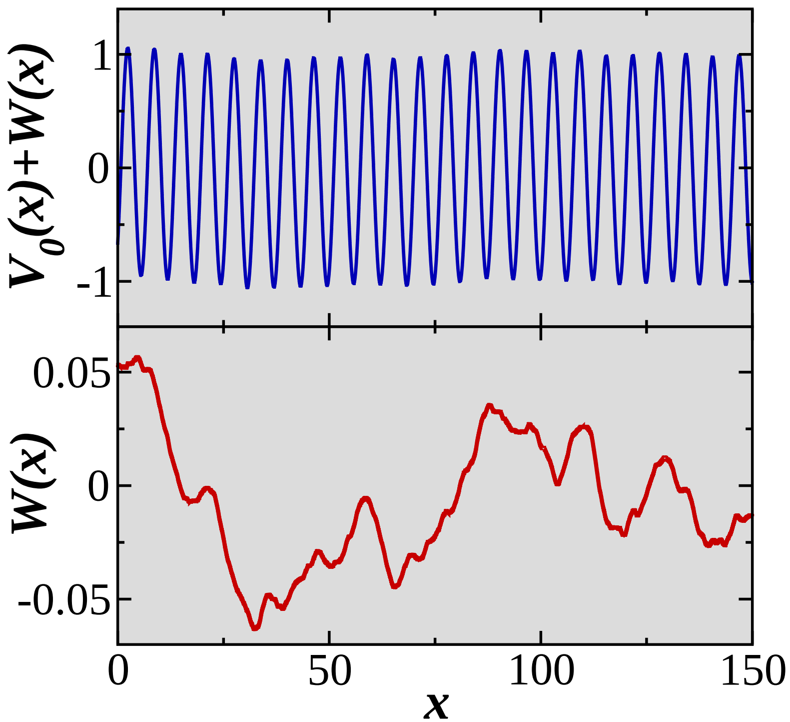


A simple Example

requirements: $W(x)$ unbiased, homogeneous, Gaussian
$W'(x)$ continuous ("random force")

$\Rightarrow$ generated via a

"spatial" critically damped harmonic oscillator:

$$\lambda^2 W''(x) = -2\lambda W'(x) - W(x) + 2\sigma\lambda^{1/2}\gamma(x)$$

$\gamma(x)$: Gaussian white noise with $\langle\gamma(x)\rangle = 0$, $\langle\gamma(x)\gamma(y)\rangle = \delta(x-y)$

$\Rightarrow$ $\boxed{W(x) \text{ Gaussian with } c(x) = \sigma^2(1 + |x|/\lambda)e^{-|x|/\lambda}}$

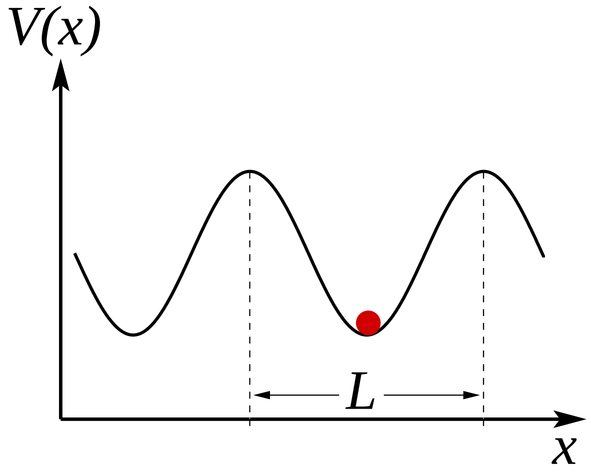
$\Rightarrow$ $\sigma^2 = c(0) = \langle W(x)^2\rangle$ "variance"

λ: correlation length

Diffusion coefficient

$$D := \lim_{t\to\infty} \frac{\langle\, [x(t) - \langle x(t)\rangle]^2 \,\rangle}{2t}$$

$$= \frac{L^2}{2} \frac{\langle\, [t(0 \to L) - \langle t(0 \to L)\rangle]^2 \,\rangle}{\langle t(0 \to L)\rangle^3}$$



exact result ($W = 0$): $$\boxed{D = D_0 \frac{B}{A^3}}$$

with $D_0 := \frac{kT}{\eta}$ **force-free diffusion coefficient**

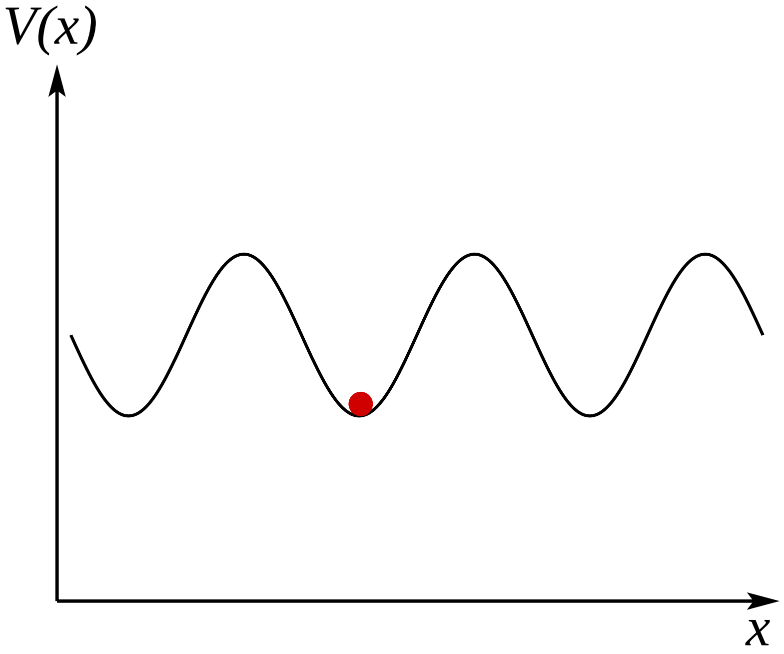
$$A := \int_0^L \frac{dx}{L} \int_0^L dy\, e^{[V(x)-V(x-y)]/kT}$$

$$B := \int_0^L \frac{dx}{L} \int_0^L dy \int_0^L dp \int_0^L dq\, e^{g/kT}$$

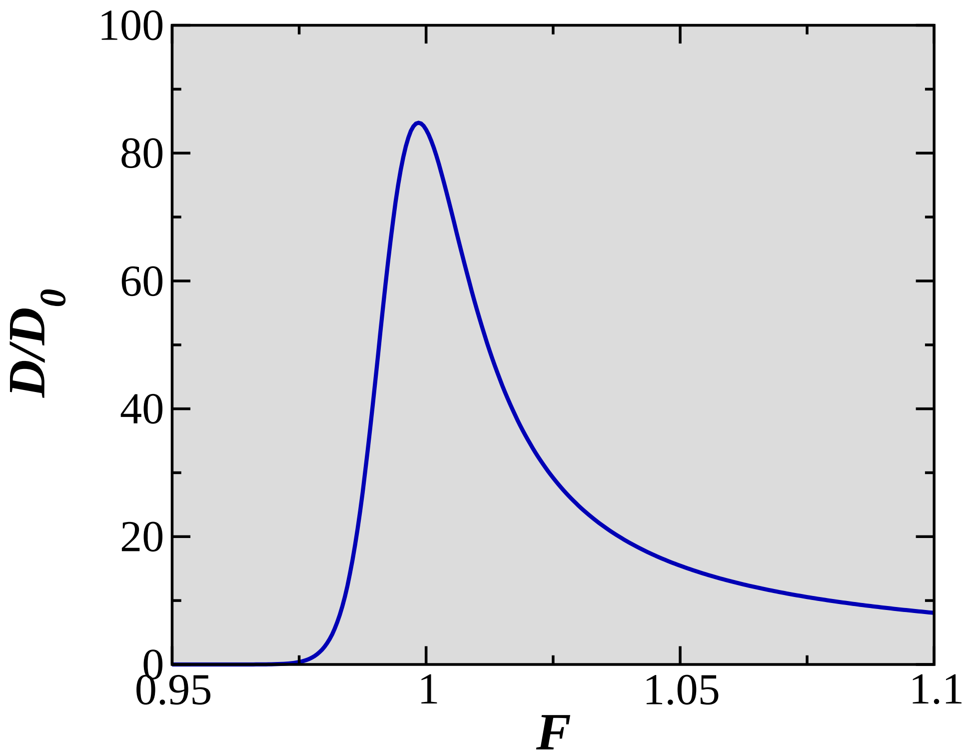
$$g := V(x) - V(x-y) - V(x-p) + V(x+q)$$

Enhancement of diffusion: $W = 0$

[P. Reimann et al., PRL **87**, 010602 (2001); PRE **65**, 031104 (2002)]



no tilt: $F = 0$



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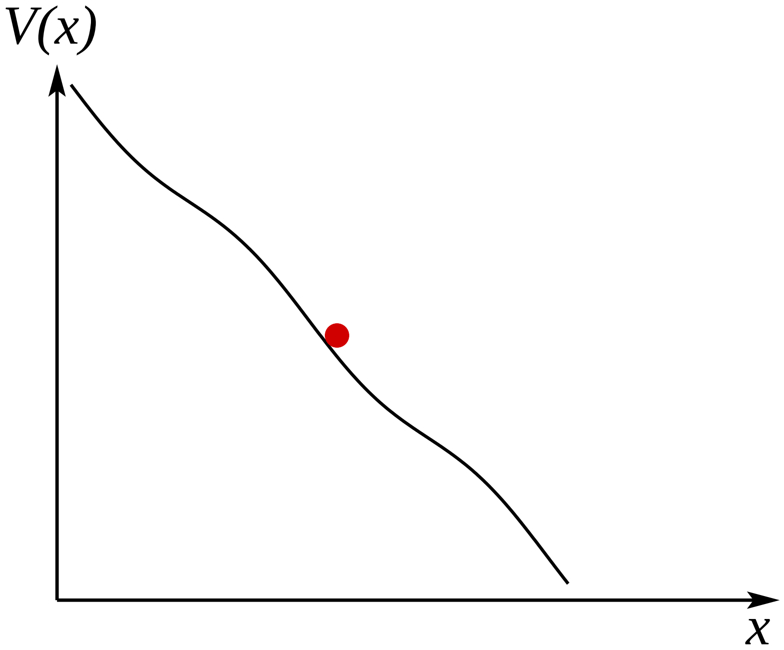
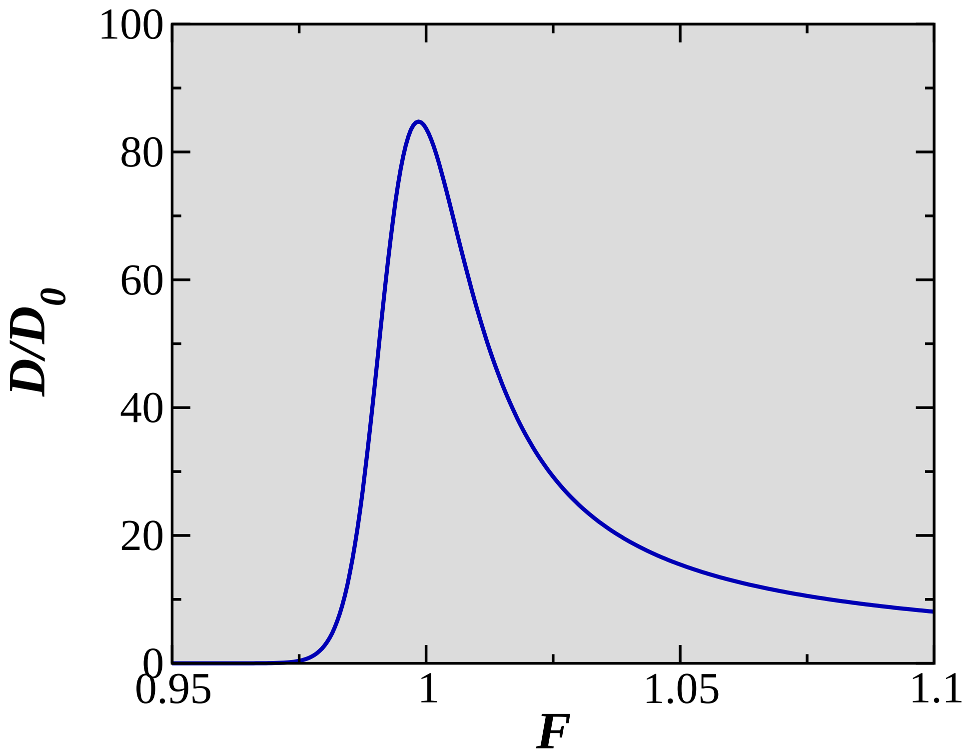


large tilt: $F \gg 1$



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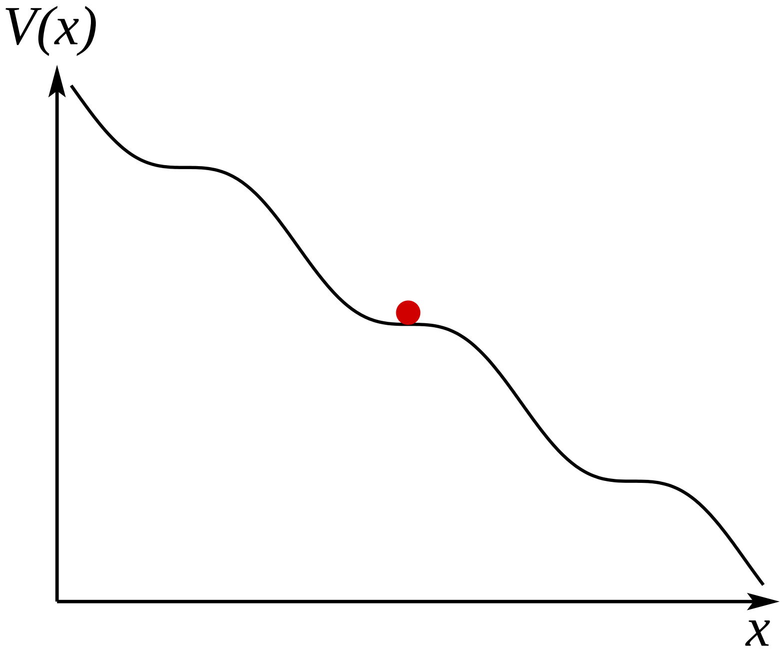
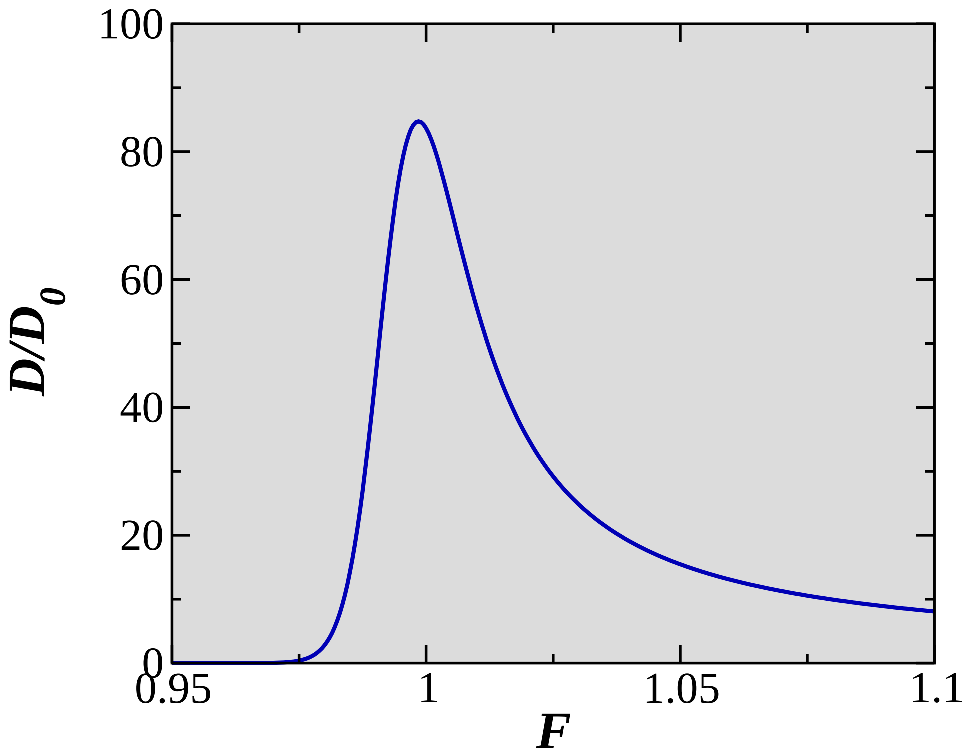


critical tilt: $F = F_c = 1$

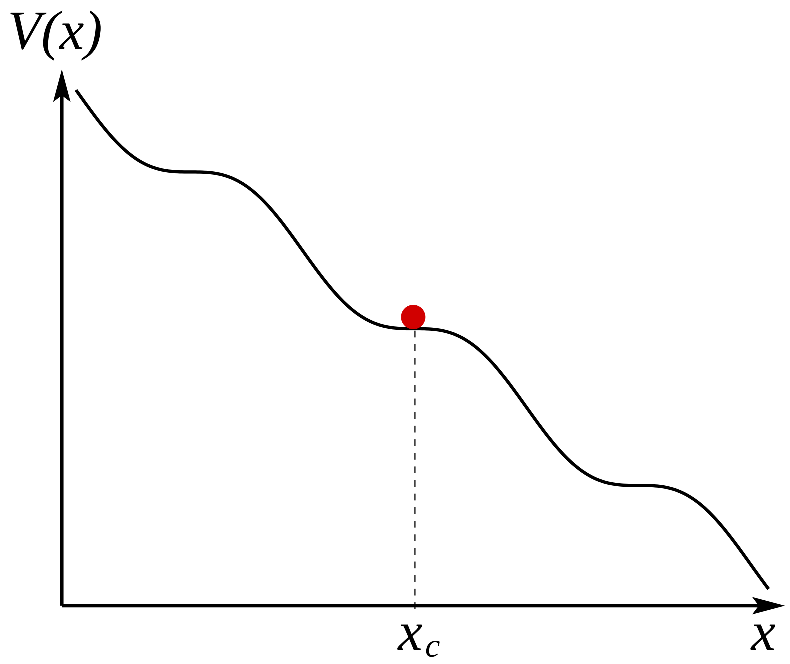


Enhancement of diffusion: $W = 0$

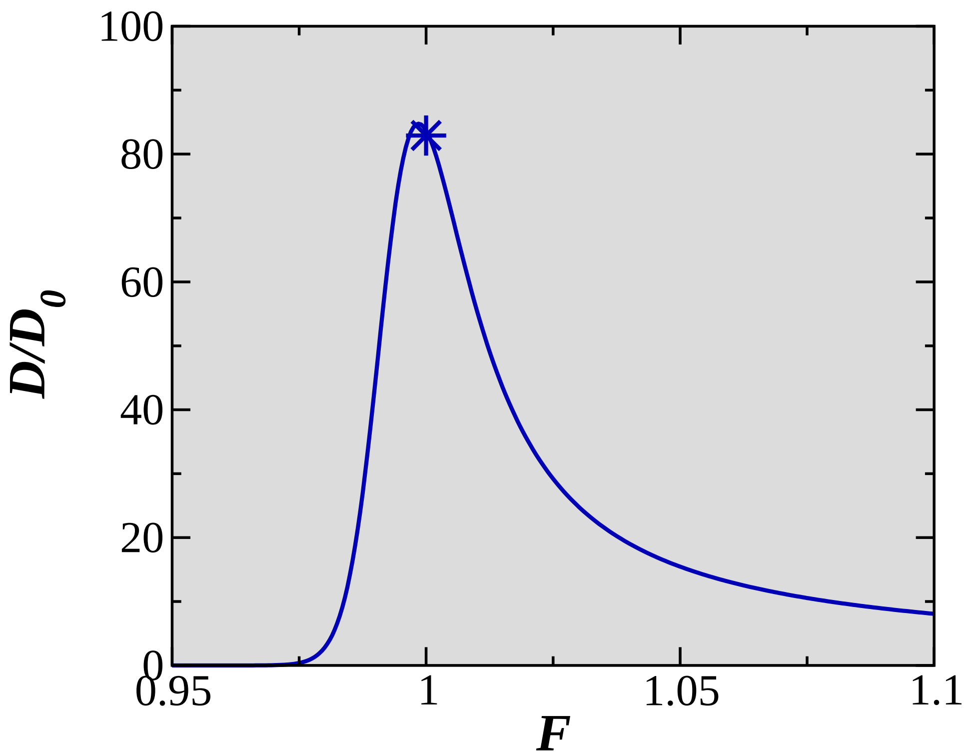
[P. Reimann et al., PRL **87**, 010602 (2001); PRE **65**, 031104 (2002)]

peak height at critical tilt F_c

$$D(F_c) \simeq 0.021\, D_0\, L^2 \left| \frac{V_0'''(x_c)}{kT} \right|^{2/3}$$



$kT = 0.001$



$$D = D_0 \frac{B}{A^3}$$

Enhancement of diffusion: $W \neq 0$

idea: $L \to NL, \quad N \to \infty \quad \Rightarrow$ average over disorder

result: $$\boxed{D = D_0 \frac{B}{A^3}}$$

with $D_0 := \dfrac{kT}{\eta}$ **force-free diffusion coefficient**

$$A \simeq \int_0^L \frac{dx}{L} \int_0^L dy\, e^{[V(x)-V(x-y)+\tilde{c}(y)]/kT}$$

$$B \simeq \int_0^L \frac{dx}{L} \int_0^L dy \int_0^L dp \int_0^L dq\, e^{[g+h]/kT}$$

$$g := V(x) - V(x-y) - V(x-p) + V(x+q)$$

$$h := \tilde{c}(y) + \tilde{c}(p) - \tilde{c}(q) - \tilde{c}(y-p) + \tilde{c}(y+q) + \tilde{c}(p+q)$$

$$\tilde{c}(x) := \frac{\sigma^2 - c(x)}{kT}\,, \quad c(x) := \langle W(y)W(y+x)\rangle\,, \quad \sigma := c(0)$$

Enhancement of diffusion: $W \neq 0$

$$D = D_0 \frac{B}{A^3} \quad \Rightarrow$$

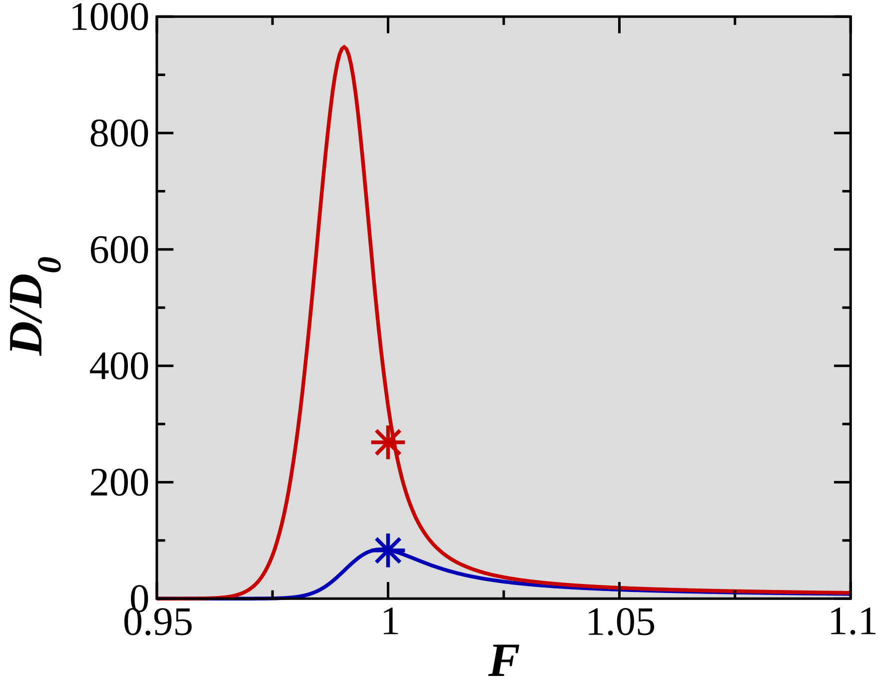
$$\begin{aligned} D(F=0) &= D(F=0)|_{W=0}\, e^{-(\sigma/kT)^2} \leq D_0 \\ D(F\to\infty) &= D_0 \\ D(F=F_c) &\simeq D(F=F_c)|_{W=0} \left(1 + 1.9\, Q\, e^{416 Q^3/3}\right) \end{aligned}$$

with $$Q := \frac{\langle W'(x)^2\rangle}{|V_0'''(x_c)|^{2/3}(kT)^{4/3}}$$

$$\langle W'(x)^2\rangle = \frac{\sigma^2}{\lambda^2}$$

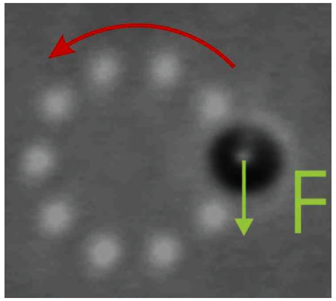
$$\sigma^2 = c(0) = \langle W(x)^2\rangle$$

$kT = 0.001$, $\sigma = 0.03$, $\lambda = L$



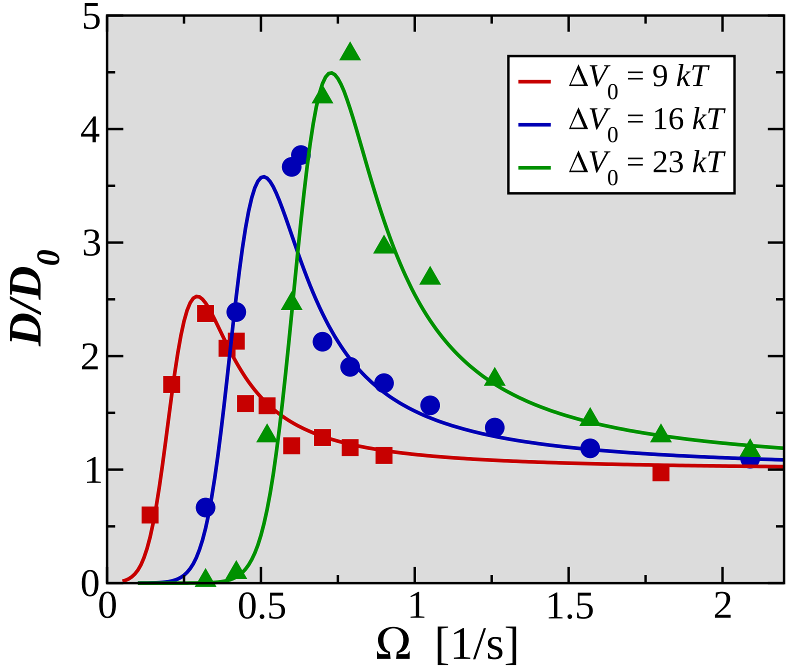
Experiment I

[M. Evstigneev et al., PRE **77**, 041107 (2008)]



particle radius: 1.5 μm

ring radius: $R = 5\,\mu$m



$$\eta \dot{x}(t) = -V_0'(x(t)) + \eta R \Omega + \sqrt{2\eta\, kT}\,\xi(t)$$

tilting force: $F = \eta R \Omega$

potential: $V_0(x) = -\frac{\Delta V_0}{2} \cos \frac{2\pi x}{L}$

$$L = 2\pi R/10 \simeq 3.14\,\mu\text{m}$$

free diffusion: $D_0 = \frac{kT}{\eta} = 0.16\,\frac{\mu\text{m}^2}{\text{s}}$

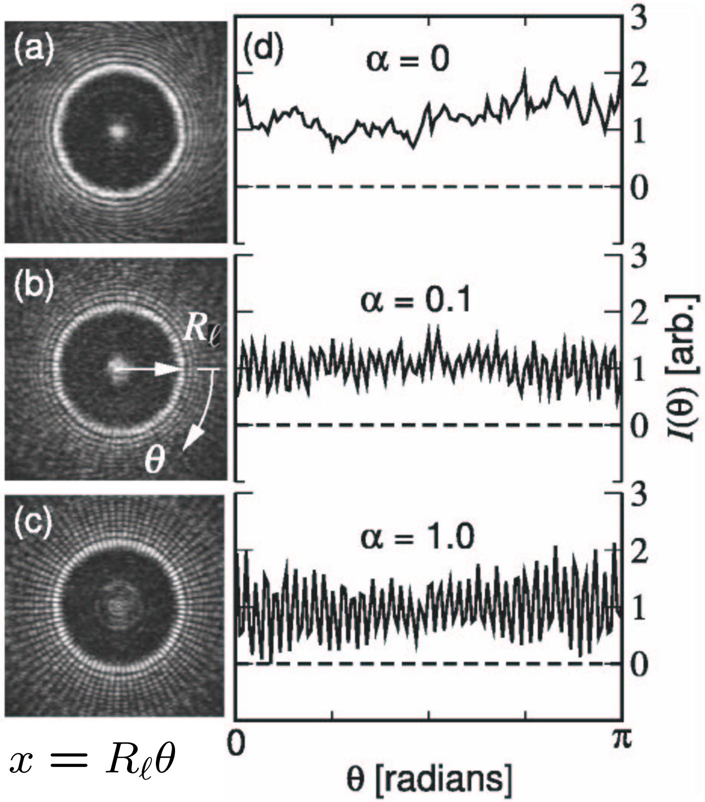
$$D = D_0 \frac{B}{A^3}$$

interpretation: $W \simeq 0$

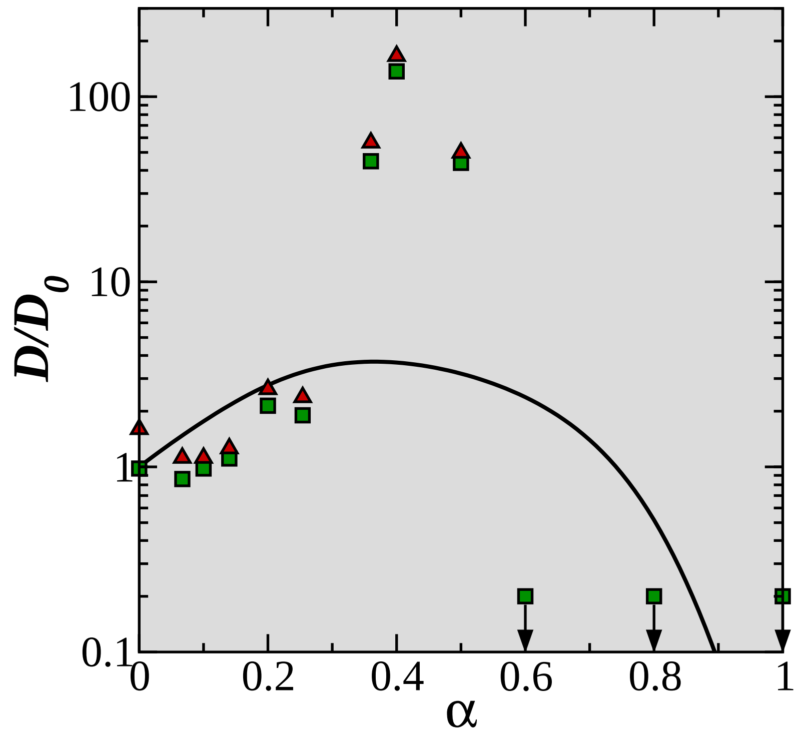
Experiment II

[S.-H. Lee and D. G. Grier, PRL **96**, 190601 (2006)]

particle radius: 1.48 μm **ring radius**: $R_\ell = 4.2\,\mu$m



$x = R_\ell \theta$



free diffusion: $D_0 = kT/\eta = 0.19\,\mu\text{m}^2/\text{s}$

$D = D_0\, B/A^3$

$$\eta \dot{x}(t) = F_0[\Phi(\alpha) + \Psi(\alpha)\cos(2\pi x/L)] + \sqrt{2\eta\, kT}\,\xi(t)$$

$$L = 0.33\,\mu\text{m}, \quad \Phi(\alpha) = (1-\alpha)/(1+\alpha), \quad \Psi(\alpha) = 2[\alpha(\epsilon^2\Phi^2 + \zeta^2)]^{1/2}/(1+\alpha)$$

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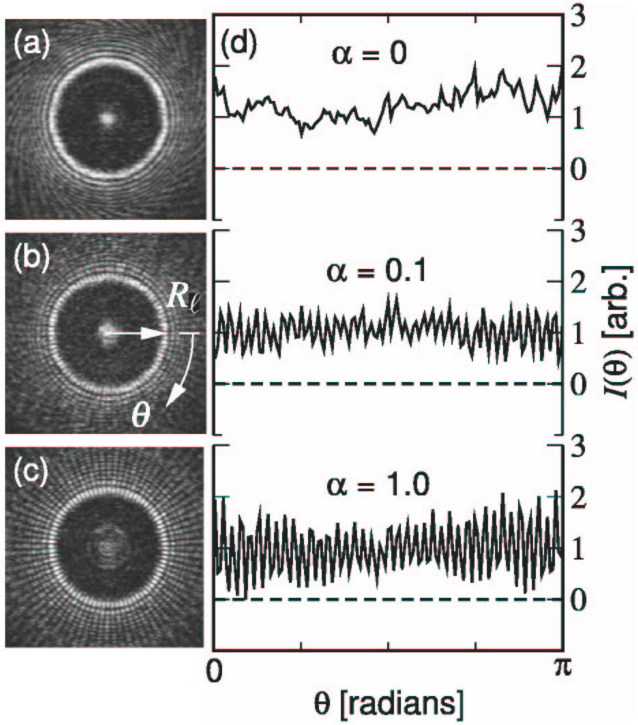

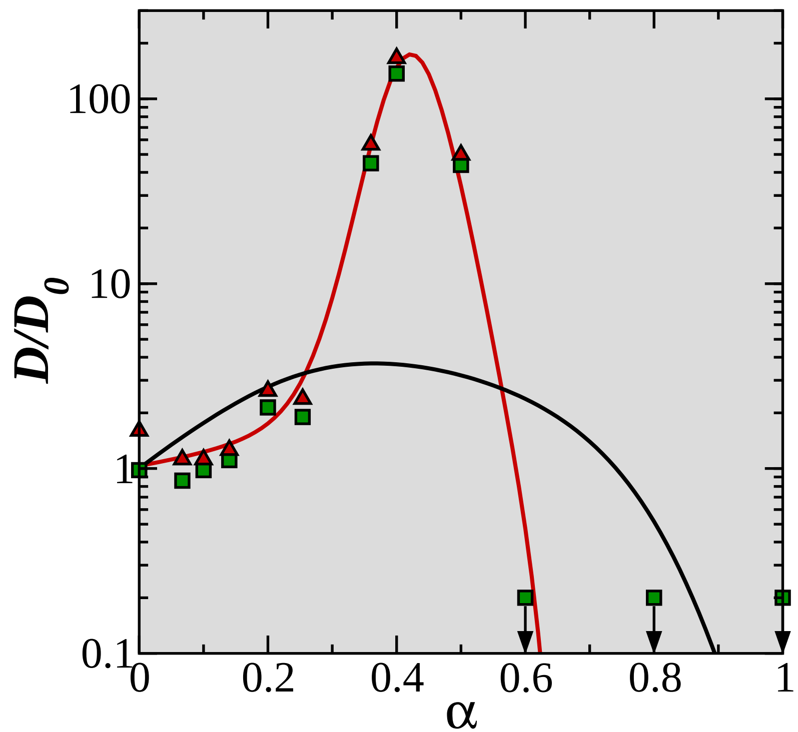

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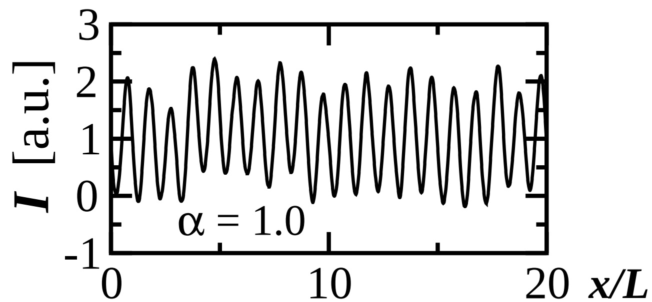
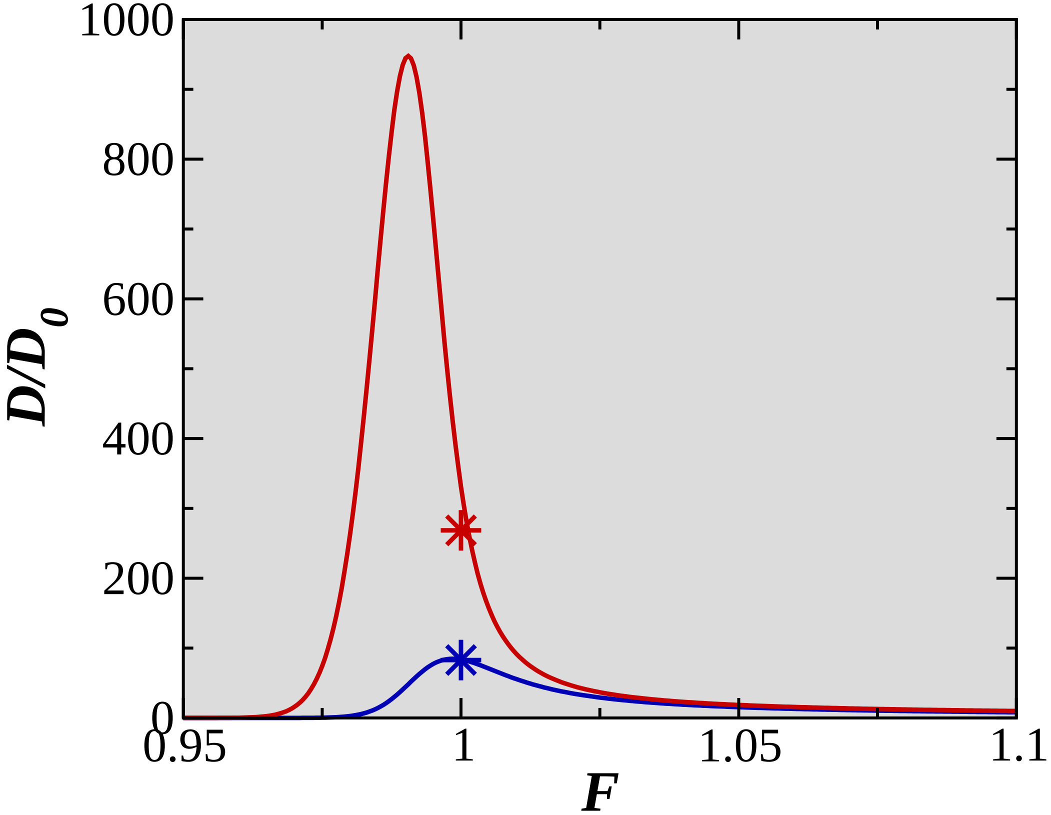
disorder: $\langle W'(x)^2 \rangle = \sigma^2/\lambda^2 = 0.01 F_0$

($\lambda = 2L,\ F_0 = 1.37$ pN, $\epsilon = 0.38,\ \zeta = 0.25$)



Summary: Weak disorder strongly improves the selective enhancement of diffusion in a tilted periodic potential

[P. Reimann and R. Eichhorn, Phys. Rev. Lett. **101**, 180601 (2008)]



- Generic model
- Diffusion enhancement
- Applications: Mixing & Purification

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