Large Deviations in Hyperbolic Billiards and Nonuniformly Hyperbolic Dynamical Systems

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Limit Theorems in Dynamical Systems

Dynamical system: (M, F, μ_0)

- State space M (smooth compact manifold)
- Discrete-time (smooth) dynamics $F : M \to M$.
- Reference measure μ_0 ($\equiv$ Lebesgue measure)

SRB measures: μ_+ is a SRB measure for (M, F, μ_0) if

- μ_+ is ergodic, i.e., for all $g \in C(M)$,

$$\frac{1}{n}\sum_{k=0}^{n-1} g \circ F^k(x) \to \mu_+(g) \quad \mu_+ \text{ a.s.}$$

- μ_+ describe the statistics of μ_0 almost every point $x \in M$

$$\frac{1}{n}\sum_{k=0}^{n-1} g \circ F^k(x) \to \mu_+(g) \quad \mu_0 \text{ a.s.}$$

For a given g and if x has initial distribution μ_+ then

$$X_n \equiv g \circ F^n , \quad n = 0, 1, 2 \cdots$$

generates an ergodic sequence of identically distributed but, in general, not independent random variables.

Under which conditions can we prove limit theorems such as central limit theorems, large deviations, etc.... for the sum

$$S_n(g) = X_0 + \cdots + X_{n-1} = \sum_{j=0}^{n-1} g \circ F^j \ ?$$

If the system is chaotic then one expects that the random variables $X_n = g \circ F^n$ are weakly dependent random variables

Chaos $\Rightarrow$ Loss of memory $\Rightarrow$ Limit Theorems

Asymptotic Variance

Assume wlog that $\mu_+(g) = 0$

Suppose that the system is mixing, i.e. decay of correlations

$$\lim_{n\to\infty} \mu_+\left((g \circ F^n)g\right) = \mu_+(g)\mu_+(g) = 0\,.$$

The asymptotic variance is

$$\begin{aligned} \sigma^2 &\equiv \lim_{n\to\infty} \mathrm{var}\left(\frac{S_n(g)}{\sqrt{n}}\right) = \lim_{n\to\infty} \mu_+\left(\frac{S_n(g)^2}{n}\right) \\ &= \mu_+(g^2) + 2\sum_{n=1}^{\infty} \mu_+\left(g\,(g \circ F^n)\right)\,. \end{aligned} \tag{1}$$

The asymptotic variance σ^2 is finite if the time correlations $\mu_+\left(g\,(g \circ F^n)\right)$ decay fast enough to be summable (= fast mixing).

Limit theorems

Central Limit Theorem: Suppose $0 < \sigma^2 < \infty$

$$\frac{S_n(g)}{\sqrt{n}} \to N(0, \sigma^2) \quad \text{(in distribution)}\,.$$

Large deviations: There exists a nonnegative convex function $I(z)$ with $I(0) = 0$ (rate function) such that for $a \in (\min g, \max g)$

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mu_+ \left\{ \frac{S_n(g)}{n} \in (a - \epsilon, a + \epsilon) \right\} = -I(a)$$

In short

$$\mu_+ \left\{ \frac{S_n(g)}{n} \approx a \right\} \sim \exp\left[-nI(a)\right]\,.$$

Moderate deviations: Choose $1/2 < \beta < 1$, i.e. intermediate scale between CLT and LDP

$$\mu_+ \left\{ \frac{S_n(g)}{n^\beta} \approx a \right\} \sim \exp\left[-n^{2\beta - 1} \frac{a^2}{2\sigma^2} \right]$$

Nonstationary large deviations

In applications to nonequilibrium statistical mechanics the SRB measure μ_+ is singular with respect to the reference (Lebesgue measure) μ_0

$$\mu_+ \perp \mu_0$$

One can also ask for non-stationary version of limit theorems, e.g.

$$\mu_0 \left\{ \frac{S_n(g)}{n} \approx a \right\} \sim \exp\left[-nJ(a)\right] \ ?$$

Are the rate functions $I(a)$ and $J(a)$ the same?

Some interest for physics, fluctuation Theorem.

Natural question for SRB measures.

Level-II large deviations

If x is distributed according to μ_+ (or μ_0) the empirical measure is defined by

$$L_n(x) \equiv \frac{1}{n} \sum_{j=0}^{n-1} \delta_{F^j(x)}$$

and is a random measure and for μ_+ (or μ_0) a.e. x

$$\lim_{n\to\infty} L_n(x) = \mu_+ \quad \text{weakly}$$

Level-II large deviations: Is there a rate function $I(\nu)$ such that

$$\mu_+ \{x\,;\, L_n(x) \approx \nu\} \sim e^{-nI(\nu)}$$

Large deviations in uniformly hyperbolic dynamical systems

Thermodynamic formalism $\Rightarrow$ large deviations estimates

(Lanford, Ruelle, Sinai, Bowen, Varadhan, Olla, Follmer, Orey, Pfister, ...) $\to$ Large deviations for Gibbs states

Anosov systems (or uniformly expanding maps) satisfy

- Large deviations for the empirical measure (Level-II)

- Nonstationary large deviations (L.S. Young, Kiefer....) with the same rate function ($I(a) = J(a)$).

Transfer operators for general weights $\Rightarrow$ large deviations (Kiefer, Baladi, Keller, Broise, etc....) works for piecewise expanding maps.

Physical motivation and examples

Hyperbolic billiards I: Equilibrium

Single particle moving freely and colliding elastically on a periodic array of strictly convex smooth obstacles in $\mathbf{R^2}$. Periodicity reduces to a system on with phase space $(\mathbf{T^2} \setminus \cup_\mathbf{i} \mathbf{\Gamma_i}) \times \mathbf{R^2}$.

Assume finite horizon: every trajectory meets an obstacle after a uniformly bounded time.

Equations of motions

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= 0 \quad + \text{ elastic reflections} \end{aligned}$$

The energy $H = \frac{p^2}{2}$ is conserved $\rightarrow$ the phase space reduces to $(\mathbf{T^2} \setminus \cup_\mathbf{i} \mathbf{\Gamma_i}) \times \mathbf{S^1}$

Theorem: The Lebesgue measure ν_0 on each energy surface is invariant, ergodic, and mixing (Sinai, Bunimovich, Chernov).

Hyperbolic billiards II: non-equilibrium.

Add an constant external electric field E and Gaussian thermostat.

$$\begin{aligned}\dot{q} &= p \\ \dot{p} &= E - \frac{E \cdot p}{p \cdot p} p \quad + \text{ elastic reflections}\end{aligned}$$

• Gaussian thermostat $\Rightarrow$ ensures that the energy $H = \frac{p^2}{2}$ is conserved.

• The system is time reversible, under $t \to -t$ and $(p, q) \to (-p, q)$.

Theorem: If E is small enough there exists a unique SRB measure $\nu_+^{(E)}$ on each energy surface which is invariant, ergodic, and mixing (Chernov, Eyink, Lebowitz, Sinai; Chernov; Wojtkowski).

Our results will be for the collision map

$$F_E : (\theta, x) \mapsto (\theta', x')$$

where x is the position of a collision on the boundary of the obstacles and θ is the angle of the incoming velocity with respect to the normal.

Discrete time dynamical system on the 2-dimensional phase space

$$M = \bigcup_i \partial\Gamma_i \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

If $E = 0$ (equilibrium) F_0 preserves the smooth measure

$$\mu_0 = \text{const} \cos(\theta)\, d\theta\, dr$$

If $E \neq 0$ (non-equilibrium) small enough F_E has a SRB measure

$$\mu_+^{(E)} \quad \text{with} \quad \mu_+^{(E)} \perp \mu_0$$

.

Entropy production rate

- Continuous-time: Let $\mu_t = \mu_0 \circ \Phi^t$ and let $H(\mu, \nu)$ be the relative entropy. Then we have

$$H(\mu_t, \mu_0) = \int_0^t \mu_s(\Sigma)\, ds\,.$$

where the entropy production Σ is

$$\Sigma = \frac{E \cdot P}{p^2} \equiv \frac{E \cdot P}{T} = \frac{\text{Work done by the force}}{\text{"Temperature"}}$$

In this context (since μ_0 is Lebesgue) we also have

$$\Sigma = \text{Phase space contraction rate}$$

- Discrete-time: For the collision map one finds

$$\Sigma = \frac{E \cdot \Delta}{T}\,, \quad \Delta = q \circ F_E - q$$

i.e., Δ is total vector displacement of the particle between two collisions.

Fluctuation Theorem

The large deviations of the entropy production σ has a universal symmetry.

$$\mu_+ \left\{ \frac{1}{n} S_n(\Sigma) \approx a \right\} \sim e^{-nI(a)}$$

with

$$I(z) - I(-z) = -z$$

the odd part of I is linear with slope $-1/2$

or

$$\frac{\mu_+ \left\{ \frac{1}{n} S_n(\Sigma) \approx \quad a \right\}}{\mu_+ \left\{ \frac{1}{n} S_N(\Sigma) \approx -a \right\}} \sim e^{ta}$$

$\Rightarrow$ One needs to prove a large deviation principle for billiard!

Goal: Prove the fluctuation theorem for "realistic" models:

$\rightarrow$ Anosov (Gallavotti-Cohen)
$\rightarrow$ "General" stochastic dynamics (Kurchan, Lebowtiz, Spohn, Maes)
$\rightarrow$ some special open classical systems (L.E. Thomas, L. R.-B.)

Limit Theorems for billiards

Assume g is Hölder continuous on M (or piecewise Hölder continuous; singularities). WLOG assume $\mu_+(g) = 0$.

$$S_n(g) = \sum_{k=0}^{n-1} g \circ F^n$$

The asymptotic variance

$$\sigma^2(g) = \lim_{n\to\infty} \frac{1}{n} \mathrm{Var}(S_n(g)) = \mu_+(g^2) + 2\sum_{n=1}^{\infty} \mu_+(g(g \circ F^n))$$

satisfies

$$0 < \sigma^2 < \infty\,, \quad \sigma^2(g) = 0 \text{ iff } g = C + h \circ F_E - h$$

Theorem (L.-S. Young, L. R.-B. 2007) Assume $\sigma^2(g) > 0$.

• Large deviations: There exists an interval (z_-, z_+) which contains $\mu_+(g) = 0$ such that for $a \in (z_-, z_+)$ we have

$$\mu_+ \left\{ \frac{S_n(g)}{n} \approx a \right\} \sim \exp\left[-nI(a)\right].$$

Moreover $I(z)$ strictly convex and real-analytic with $I''(0) = \dfrac{1}{\sigma^2}$

• Moderate deviations: Let $1/2 < \beta < 1$. Then

$$\nu \left\{ \frac{S_n(g)}{n^\beta} \approx a \right\} \sim \exp\left[-n^{2\beta-1} \frac{a^2}{2\sigma^2} \right].$$

• Central Limit Theorem: Already known: Sinai & al, Liverani, Young...

$$\nu \left\{ a \leq \frac{S_n(g)}{n^{1/2}} \leq b \right\} \to \frac{1}{\sqrt{2\pi}\sigma} \int_a^b \exp\left[-\frac{z^2}{2\sigma^2} \right] dz.$$

Remark I: We obtain large deviations estimates only in a neighborhood of the mean (z_-, z_+), and not a full large deviation principle.

The size of the neighborhood (z_-, z_+) is related to the size of g, i.e., $\max g - \min g$ and dynamical quantities $\approx$ rate of return.

I do not know whether Level-II large deviations hold for the Sinai billiard.

Remark II: Analyticity allows to obtain various refinements of the limit theorems (prefactors), e.g. for non-lattice g

$$\lim_{n\to\infty} J_n \nu \left(\frac{S_n(g)}{n} \geq z \right) = 1$$

with

$$J_n = \theta \sqrt{e''(\theta) 2\pi n} e^{nI(z)}$$

where $I(z)$ and $e(\theta)$ are related by Legendre transform.

The same holds for the central limit theorem... sharp estimates.

All the refinement are obtained by applying standard probabilistic techniques.

Remark III: Many other limits theorems for billiards and nonuniformly hyperbolic dynamical systems have been proved recently (Chernov, Dolgopyat, Szasz, Varju, Melbourne, Nicol, ...).

Remark IV: We do not know whether nonstationary large deviation hold.

Young towers

Our theorem is proved using Young towers introduced by Lai-Sang Young in 1995. The towers are a symbolic representation of non-uniformly hyperbolic dynamical systems.

Special type of Markov partition with countably many states, based on ideas of renewal theory: choose a set $\Lambda \subset M$ and construct a partition of $\Lambda \approx \cup_i \Lambda_i$ where Λ_i is a stable subset which "returns" ($\equiv$ full intersection) after time R_i. This gives a Markov extension. Finally quotient out the stable manifolds.

Consequence: our large deviation results apply to

- Billiards
- Quadratic maps
- Piecewise hyperbolic maps
- Hénon-type maps
- Rank-one chaos (Qiudong Wang and L.S. Young) Some periodically kicked limit cycles and certain periodically forced non-linear oscillators with friction.

Tower Ingredients

- Measure space (Δ_0, m) and a map $f : \Delta_0 \to \Delta_0$ (noninvertible)

- Return time $R : \Delta_0 \to \mathbf{N}$.

  Assume exponential tail: $m\{R \geq n\} \leq De^{-\gamma n}$ (need for large deviations)

  Assume aperiodicity: g.c.d.$\{R(x)\} = 1$ (need for mixing)

- Tower $=$ suspension of f under the return time R

$\underbrace{\Delta_l \equiv \{x \in \Delta_0 \, ; \, R(x) \geq l+1\}}_{\text{l-th floor}}$ and $\underbrace{\Delta \equiv \sqcup_{l \geq 0} \Delta_l}_{\text{tower}}$ (disjoint union)

Dynamics $F : \Delta \to \Delta$ $\qquad F(x,l) = \begin{cases} (x, l+1) & R(x) > l+1 \\ (f(x), 0) & R(x) = l+1 \end{cases}$

- Markov partition $\Delta_l = \Delta_{l,1} \cup \cdots \Delta_{l,j_l}$ with $j_l < \infty$.

  F maps Δ_{lj} onto a collection of $\Delta_{l+1,k}$'s plus possibly Δ_0.

  The Markov partition is generating (i.e. each point has a unique coding).

- Dynamical distance:

$s(x,y) = \inf\{n, F^i(x)$ and $F^i(y)$ belong to the same $\Delta_{l,k}$, $0 \leq i \leq n\}$

For $\beta < 1$ let $d_\beta(x,y) = \beta^{s(x,y)}$

- Distortion estimates: Let JF the Jacobian of F with respect to m.

$$\left| \frac{JF(x)}{JF(y)} - 1 \right| \leq C d_\beta(x,y)$$

Remark: If $JF = $ const on each Δ_{lj} then we have a Markov chain on a countable state space.

Transfer operators and large deviations

Think of m as the (image of) Lebesgue measure on unstable manifolds. The (image of the) SRB measure has then the form

$$\nu = h dm \,, \quad h \in L^1(m) \,.$$

The transfer operator $\mathcal{L}_0$ is the adjoint of $U\psi = \psi \circ F$

$$\int \varphi \, \psi \circ F \, dm = \int \mathcal{L}_0(\varphi)\psi \, dm$$

$$\mathcal{L}_0\varphi(x) = \sum_{y \,:\, F(y)=x} \frac{1}{JF(y)} \varphi(y)$$

$\nu = h dm$ F-invariant iff $\mathcal{L}_0 h = h$

Moment generating function and large deviations

Consider the moment generating function

$$\mu_+\left(\exp\left[\theta S_n(g)\right]\right)$$

for the random variable $S_n(g) = g + g \circ F + \cdots + g \circ F^{n-1}$.

If

$$e(\theta) \equiv \lim_{n\to\infty} \frac{1}{n} \log \mu_+\left(\exp\left[\theta S_n(g)\right]\right)$$

exists and is smooth (at least C^1) then we have large deviations with

$$I(z) = \sup_{\theta}(\theta z - e(\theta)), \qquad \text{Legendre Transform}.$$

(Gartner-Ellis Theorem)

Moment generating functions and transfer operators

To study the large deviations for $S_n(g)$ consider the generalized transfer operator

$$\mathcal{L}_g \varphi(x) = \sum_{y\,:\,F(y)=x} \frac{e^{g(y)}}{JF(y)} \varphi(y)$$

Then we have

$$\begin{aligned} \mu_+ \left(\exp\left[\theta S_n(g)\right] \right) &= m \left(\exp\left[\theta S_n(g)\right] h \right) \\ &= m \left(\mathcal{L}_0^n \left[\exp\left[\theta S_n(g)\right] h \right] \right) \\ &= m \left(\mathcal{L}_{\theta g}^n (h) \right) \end{aligned}$$

$\Rightarrow$ Large deviations follow from spectral properties of $\mathcal{L}_{\theta g}$

Spectral properties of transfer operators

Suppose $\mathcal{L}_{\theta g}$ is quasi-compact on some Banach space $X \ni h$, i.e. the essential spectral radius strictly smaller than the spectral radius.

By a Perron-Frobenius argument $\mathcal{L}_{\theta g}$ a maximal eigenvalue $\exp[e(\theta)]$ and a spectral gap (aperiodicity) and thus

$$e(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \nu \left(\exp\left[\theta S_n(g)\right]\right)$$

By analytic perturbation theory $e(\theta)$ is real-analytic and then standard probabilistic techniques implies

$$\mu_+ \left\{ x\,;\, \frac{S_n(g)}{n} \approx z \right\} \sim e^{-nI(z)}$$

$$I(z) = \text{Legendre transform of } e(\theta)$$

as well as moderate deviations, central limit theorem, and so on...

Choice of Banach space

Recall $m\{R \geq n\} \leq De^{-\gamma n}$. Choose $\gamma_1 < \gamma$ and set

$$v(x) = e^{\gamma_1 l} \quad x \in \Delta_l$$

Banach space

$$X = \{\varphi : X \to \mathbf{C} ; \|\varphi\|_v \equiv \|\varphi\|_{v,\sup} + \|\varphi\|_{v,Lip} < \infty\}$$

with

$$\varphi_{v,\sup} = \sup_{l,j} \sup_{x \in \Delta_{l,j}} |\varphi(x)| e^{\gamma_1 l}$$

$$\varphi_{v,Lip} = \sup_{l,j} \sup_{x,y \in \Delta_{lj}} \frac{|\varphi(x) - \varphi(y)|}{d_\beta(x,y)} e^{\gamma_1 l}$$

Banach space of weigthed Lipschitz functions

Spectral analysis

Lasota York estimate: For g bounded Lipschitz

$$\|\mathcal{L}_g^n(\varphi)\|_v \leq \|\mathcal{L}_g^n(1)\|_{v,\mathrm{sup}} \left(\beta^n \|\varphi\|_v + C\|\varphi\|_{v,\mathrm{sup}}\right)$$

Pressure

$$P(g) = \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_g^n(1)\|_{v,\mathrm{sup}} \, .$$

Pressure at infinity: Control on the high floors of the towers!

$$P_*(g) = \lim_{n\to\infty} \frac{1}{n} \log \| \inf_{k\geq 0} \mathcal{L}_g^n(1)^{>k} \|_{v,\mathrm{sup}} \, .$$

($\varphi^{>k} = \varphi$ for $x \in \Delta_l$ with $l > k$ and 0 otherwise)

Theorem:

The spectral radius of $\mathcal{L}_g$ is $e^{P(g)}$.

The essential spectral radius of $\mathcal{L}_g$ is $\max\{e^{P_*(g)}, \beta e^{P(g)}\}$

$\Rightarrow \mathcal{L}_g$ is quasicompact if $P_*(g) < P(g)$.

Theorem: $P_*(g) < P(g)$ if $(\max g - \min g) < \gamma$.

Theorem: If $P_*(g) < P(g)$ then $\exp(P(g))$ is a (simple) eigenvalue and no other eigenvalue on the circle $\{|z| = \exp(P(g))\}$.

Conclusion: The moment generating function

$$e(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \nu \left(\exp\left[\theta S_n(g)\right]\right)$$

exists and is analytic if $|\theta| \leq \gamma/(\max g - \min g)$.

$\square$

Fluctuation theorem

Combine

- Time-reversal i, $i(p, q) = (-p, q)$
- Entropy production =phase space contraction

$$\Sigma = -\log JF^s - \log JF^u$$

- The SRB measure is "the equilibrium state" for the potential $-\log JF^u$ (use the Markov extension).
- The large deviation principle.