Spatial discretization of dynamical systems

Peter Kloeden
Goethe-Universität, Frankfurt am Main

Joint work with

Phil Diamond (Brisbane), Peter Imkeller (Berlin),

Jamie Mustard (Geelong), Alexei Pokrovskii (Cork)

Spatial discretization

Consider a continuous mapping $f : X \to X$ on a compact metric space (X, d).

The difference equation

$$\boxed{x_{n+1} = f(x_n)} \tag{1}$$

generates a discrete time dynamical system on X.

Consider a finite subset X_h of X with grid fineness

$$\boxed{\Delta_h := \sup_{x \in X} \inf_{x_h \in X_h} d\left(x, x_h\right)}$$

Examples

- $X = [0,1]$, $X_h = 2^N$-bit computer numbers in $[0,1]$
- $X = [0,1]$, $X_h = \left\{ \dfrac{j}{2^N} : j = 0, 1, \ldots, N \right\}$ N-dyadic numbers

Consider a "projection" $P_h : X \to X_h$, e.g. round-off operator

The mapping $f_h := P_h \circ f : X_h \to X_h$ generates a discrete time dynamical system on X_h through the difference equation

$$\boxed{x_{n+1}^{(h)} = f_h\left(x_n^{(h)}\right)} \tag{2}$$

What is the relationship between the dynamical behaviour of the original dynamical system (1) and the spatially discretized system (2) as

$$\Delta_h \to 0\ ?$$

Plan

- the effect of spatial discretization on attractors

- the effect of spatial discretization on chaos

- the approximation of Lebesgue measure preserving maps on a torus by permutations

- approximation by Markov chains of invariant measures of spatial discretized

    i) deterministic difference equations

    ii) random difference equations

Spatial discretization of attractors

P. Diamond and P. E. Kloeden,
Spatial discretization of mappings, *J. Computers Math. Applns.* **26** (1993), 85-94.

P. E. Kloeden and J. Lorenz,
Stable attracting sets in dynamical systems and in their one-step discretizations,
SIAM J. Numer. Analysis **23** (1986), 986-995.

Assume that

- $f : X \to X$ is Lipschitz with constant $K > 0$

- the projection $P_h : X \to X_h$ satisfies for a constant $M > 0$

$$\boxed{d\,(P_h(x), x) \leq Mh}$$

Theorem 1

Suppose that a nonempty compact subset L of X is uniformly asymptotically stable (UAS) for the dynamical system f on X.

Then there exists a nonempty compact subset L_h of X_h which is UAS for the dynamical system $f_h := P_h \circ f$ on X_h such that the Hausdorff distance

$$H(L_h, L) \to 0 \quad \text{as} \quad h \to 0+$$

Sketch of proof

The UAS of the set L for the system f implies that there exists a

Lyapunov function $\qquad V : X \to \mathbb{R}^+,$

which is Lipschitz continuous, and a constant $0 < q < 1$ such that

$$V(f(x)) \leq q\,V(x), \qquad \forall x \in X.$$

Then the discretized system satisfies the key inequality

$$\boxed{V\left(f_h(x_h)\right) \leq q\,V(x_h) + KMh} \qquad \forall\, x_h \in X_h$$

Define

$$L_h := \left\{ x_h \in X_h \,:\, V\left(x_h\right) \leq \frac{2KMh}{1-q} \right\},$$

which is a nonempty, compact subset of X_h for all $h > 0$.

The key inequality and other properties of the Lyapunov function V imply that L_h is UAS for f_h on X_h and satisfies the convergence asserted in the theorem.

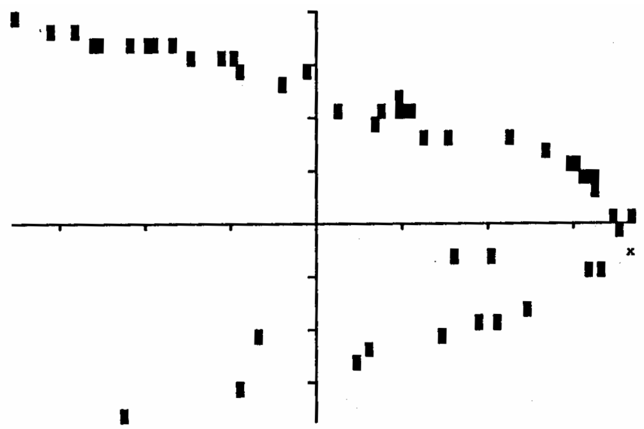

Figure 1. Λ_h : $h = 0.025$.

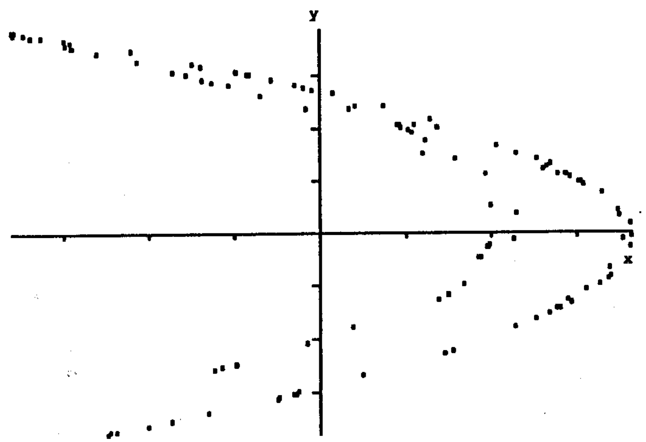

Figure 3. Λ_h : $h = 0.0005$.

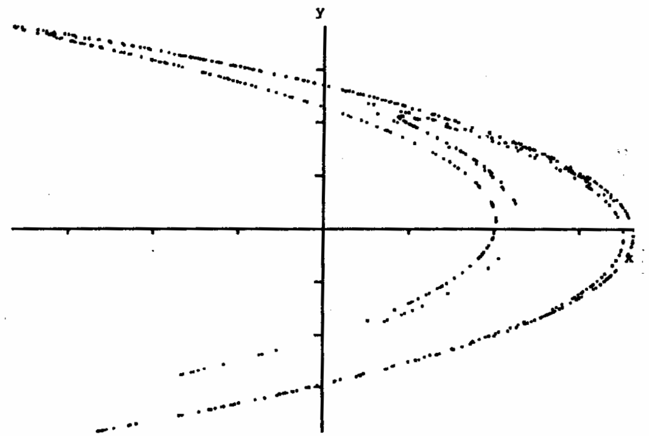

Figure 2. Λ_h : $h = 0.005$.

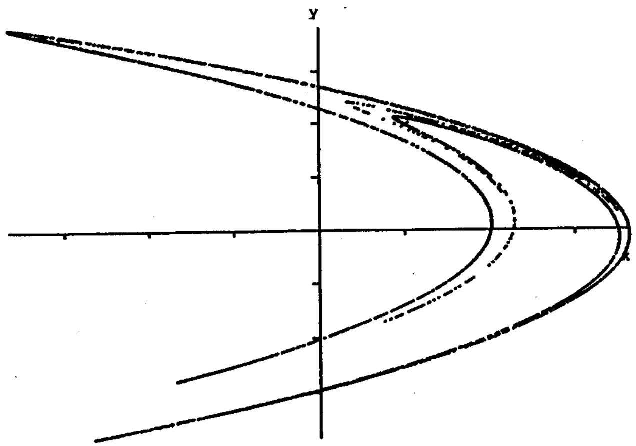

Figure 4. Λ_h : $h =$ double precision.

Fig. 1 consists of stable cycles of periods 4, 11 and 33

Fig. 3 consists of stable cycles of periods 30 and 78

Complications

• a fixed point $f(\bar{x}) = \bar{x} \in X$ need not belong to X_h

• if such a fixed point $\bar{x} \in X_h$, then it need not be a fixed point of f_h.

• f_h may have spurious cycles in X_h, i.e. periodic solutions which do not correspond to periodic solutions of f.

In fact, the dynamics of f_h on X_h is always eventually periodic

Moreover, the convergence $H(L_h, L) \to 0$ as $h \to 0$ is deceptive

• the attracting set L_h of f_h may contains transients as well as limit points and cycles

• it is better to consider the omega set of limiting values

$$L_h^* := \bigcap_{j \geq 1} \overline{\bigcup_{n \geq 1} f_h^j(L_h)},$$

i.e. the global attractor, which may be a proper subset of L_h.

Without additional assumptions about the dynamics of f on L such as hyperbolicity, we only have the weaker convergence in the Hausdorff semi-distance

$$H^*(L_h^*, L) := \max_{x_h \in L_h^*} d(x_h, L) \to 0 \quad \text{as} \quad h \to 0+$$

the effect can be extreme

Example Consider the extended tent mapping $f : [0, 2] \to [0, 2]$ defined by

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \le x \le \frac{1}{2} \\ 2(1-x) & \text{if} \quad \frac{1}{2} \le x \le 1 \\ 0 & \text{if} \quad 1 \le x \le 2 \end{cases}$$

which has the chaotic attractor $L = [0, 1]$. Consider the N-dyadics

$$X_h := \left\{ \frac{j}{2^N}, 1 + \frac{j}{2^N} \ : j = 0, 1, \ldots, N \right\}, \qquad h = 2^{-N}.$$

Since $f : X_h \to X_h$, here we take $f_h \equiv f$.

$$\boxed{f_h^N(x_h) = 0, \quad \forall\, x_h \in X_h} \qquad \Longrightarrow \qquad L_h^* = \{0\}$$

the chaos has collapsed onto trivial behaviour

This collapsing effect is not exceptional

Theorem 2
For any continuous $f : X \to X$ *and any cycle* $\{c_1, \ldots, c_p\}$ *of* f *there exists a finite subset* X_h *of* X *which contains* $\{c_1, \ldots, c_p\}$ *and a mapping* $f_h : X_h \to X_h$ *for* $h \to 0$ *such that the dynamics of* f_h *collapses on* $\{c_1, \ldots, c_p\}$.

P. Diamond, P.E. Kloeden und A. Pokrovskii,
Cycles of spatial discretizations of shadowing dynamical systems,
Mathematische Nachrichten **171** (1995), 95–110.

Invariant measures

- allow us to circumvent some of the above difficulties with attractors and cycles

- are <u>robuster</u> for approximation and comparison

> *A measure μ on X is called <u>f-invariant</u> if*
>
> $$\mu(B) = \mu\left(f^{-1}(B)\right), \quad \forall\, B \in \mathcal{B}(X),$$

for the Borel subsets $\mathcal{B}(X)$ of X, where

$$f^{-1}(B) := \{x \in X \,:\, f(x) \in B\}$$

Can we always approximate an invariant measure μ of f on X by an invariant measure μ_h of f_h on X_h? <u>how?</u>

SPECIAL CASE: mappings on a torus

Consider

- a d-dimensional torus $\mathbb{T}^d$, where $d \geq 1$,
- a <u>measurable</u> mapping $f : \mathbb{T}^d \to \mathbb{T}^d$;
- a uniform $\frac{1}{N}$ partition $\mathbb{T}^d_N$ of $\mathbb{T}^d$.

How should we construct a mapping f_N on $\mathbb{T}^d_N$ to approximate f?

P.E. Kloeden and J. Mustard,
Construction of permutations approximating Lebesgue measure preserving dynamical systems under spatial discretization.
J. Bifurcation & Chaos **7** (1997), 401–406.

Theorem 3
Suppose that the Lebesgue measure on $\mathbb{T}^d$ *is* f*-invariant. Then there exists a permutation* $P_N(f)$ *on* $\mathbb{T}^d_N$ *with*

$$H^*\left(\mathrm{Gr}(P_N(f)), \mathrm{Gr}(f)\right) \leq \frac{1}{N}$$

where H^* is the Hausdorff semi-distance on $\mathbb{T}^d \times \mathbb{T}^d$ and $\mathrm{Gr}(f)$ is the graph of f defined by

$$\mathrm{Gr}(f) := \left\{(x, y) \in \mathbb{T}^d \times \mathbb{T}^d \,:\, y = f(x)\right\}$$

Comments

- f can be non-injective here, i.e. not 1 to 1

- the inverse of the theorem holds if f is <u>continuous</u>

- Peter Lax has an theorem about permutations approximating area-preserving diffeomorphisms

Outline of proof

- <u>enumerate</u> $\mathbb{T}_N^d = \{x_1, \ldots, x_M\}$, where $M = N^d$

- <u>define</u> the $\frac{1}{N}$-band about the graph $\mathrm{Gr}(f)$ of f, i.e.

$$S_N(f) := \left\{ (x, y) \in \mathbb{T}_N^d \times \mathbb{T}_N^d \,:\, \mathrm{dist}\left((x, y), \mathrm{Gr}(f)\right) \leq \frac{1}{N} \right\}$$

The following problems are equivalent by the f-invariance of the Lebesgue measure and a combinatorial theorem of Frobenius and König,

(1) *construct a permutation* $P_N(f)$ *on* $\mathbb{T}_N^d$ *with* $Gr(P_N(f)) \subseteq S_N(f)$*.*

(2) *choose a diagonal (possibly permuted) without zeros of the* $M \times M$ *matrix* $A_N(f) = [a_{i,j}]$ *defined by*

$$a_{i,j} = \begin{cases} 1 & \text{if } (x_i, x_j) \in S_N(f) \\ 0 & \text{otherwise} \end{cases}$$

reformulate the problem as an optimal assignment LP problem

GENERAL CASE: using Markov chains

Consider a finite subset $X_N = \{x_1^{(N)}, \ldots, x_N^{(N)}\}$ of a compact metric space (X, d) with fineness parameter

$$h_N := \Delta_N := \sup_{x \in X} \inf_{x_j^{(N)} \in X_N} d\left(x, x_j^{(N)}\right) \to 0 \quad \text{as } N \to \infty$$

How do we construct an approximation f_N on X_N of a function $f : X \to X$?

The choice is usually not unique: there may be several nearest grid points to an $f(x_j^{(N)}) \notin X_N$.

There are two ways to handle the problem:

1) <u>setvalued</u>: use a setvalued mapping

$$F_N(x_j^{(N)}) := \left\{ \text{nearests points in } X_N \text{ to } f(x_j^{(N)}) \right\}$$

and then consider the setvalued dynamical system $x_{n+1} \in F_N(x_n)$ on X_N.

2) <u>stochastic</u>: use a Markov chain P_N on X_N with transition probabilities

$$p_{i,j}^{(N)} = \begin{cases} > 0 & \text{if } x_i^{(N)} \text{ in a neighbourhod of } f(x_j^{(N)}) \\ 0 & \text{otherwise} \end{cases}$$

Distances

1) between a Markov chain P_N on $X_N \subset X$ and a mapping $f : X \to X$

$$D(P_N, f) := \max_{1 \le i \le N} \sum_{j=1}^{N} p_{i,j}^{(N)} \operatorname{dist}\left(\left(x_i^{(N)}, x_j^{(N)}\right), \mathrm{Gr}(f)\right)$$

2) between a probability vector p_N on X_N and a probability measure μ on X

Prokhorov metric $\qquad \rho(\mu_N, \mu)$

where μ_N is the extension of p_N to a measure on X.

Let $f : X \to X$ be <u>Borel measurable</u> and consider the generalized inverse

$$\widetilde{f^{-1}}(B) := \left\{ x \in X \ : \ \exists\, y \in \overline{B} \text{ with } (x, y) \in \overline{\mathrm{Gr}(f)} \right\}$$

A Borel measure μ on X is called <u>f-semi-invariant</u> if

$$\boxed{\mu(B) \leq \mu\left(\widetilde{f^{-1}}(B)\right), \qquad \forall\, B \in \mathcal{B}(X)}$$

f <u>continuous</u> $\quad \Longrightarrow \quad$ f-semi-invariant $\quad \equiv \quad$ f-invariant

Theorem 4

A probability measure μ *on* X *is* f*-semi-invariant if and only if it is* *stochastically approachable*, *i.e. for each* N *there exist*

1) a grid X_N *with fineness* $\Delta_N \to 0$ *as* $N \to \infty$

2) a Markov chain P_N *on* X_N

3) probability measure μ_N *on* X *corresponding to an equilibrium probability vector* $\bar{p}_N$ *of* P_N *on* X_N, *such that*

$$D(P_N, f) \to 0, \quad \rho(\bar{\mu}_N, \mu) \to 0 \qquad \text{as} \quad N \to \infty$$

P. Diamond, P.E. Kloeden and A. Pokrovskii,
Interval stochastic matrices, a combinatorial lemma, and the computation of invariant measures, *J. Dynamics & Diff. Eqns.* **7** (1995), 341–364.

Key idea in the proof: interval stochastic matrices

An $N \times N$ matrix $C = [c_{i,j}]$ with nonnegative components is called

$$\left.\begin{array}{c}\text{substochastic}\\[2ex] \text{stochastic}\\[2ex] \text{superstochastic}\end{array}\right\} \text{ if } \sum_{j=1}^{N} c_{i,j} \quad \left\{\begin{array}{cc}\leq & 1\\[2ex] = & 1\\[2ex] \geq & 1\end{array}\right. \qquad \text{for } i = 1, \ldots, N.$$

Let $A = [a_{i,j}]$ be substochastic and $B = [b_{i,j}]$ be superstochastic. Then

$$\widehat{AB} := \{P \text{ stochastic } : a_{i,j} \leq p_{i,j} \leq b_{i,j}, \quad \forall\, i, j = 1, \ldots, N\}$$

is called an interval stochastic matrix with boundaries A and B.

- The <u>(j, I)-flow</u> of an interval stochastic matrix $\widehat{AB}$ is defined by

$$H_j\left(I, \widehat{AB}\right) := \min\left\{\sum_{i\in I} b_{i,j}, 1 - \sum_{i\notin I} a_{i,j}\right\},$$

where $j \in \subset \{1, \ldots, N\} \subset \{1, \ldots, N\}$, $I \subset \{1, \ldots, N\}$

- A probability vector p_N on X_N is called <u>$\widehat{AB}$-semi-invariant</u> if the inequalities

$$\sum_{j=1}^{N} p_j H_j\left(I, \widehat{AB}\right) \geq \sum_{j=1}^{N} p_j$$

for every subset $I \subset \{1, \ldots, N\}$.

Lemma
A probability vector p_N on X_N is $\widehat{AB}$-semi-invariant if and only $p_N = p_N P_N$ for some $P_N \in \widehat{AB}$

In the proof of Theorem 4 we use

$$a_{i,j} \equiv 0, \qquad b_{i,j} = \begin{cases} 1 & \text{if } \operatorname{dist}\left((x,y), \operatorname{Gr}(f)\right) \le \frac{1}{N} \\ 0 & \text{otherwise} \end{cases}$$

i.e. we consider only those $\left(x_i^{(N)}, x_j^{(N)}\right) \in S_N(f)$, a $\frac{1}{N}$-neighbourhood of $\operatorname{Gr}(f)$.

$$\Longrightarrow \qquad H_j\left(I, \widehat{AB}\right) = \begin{cases} 1 & \text{if } b_{i,j} = 1 \text{ for some } i \in I \\ 0 & \text{otherwise} \end{cases}$$

Moreover, a probability vector p_N on X_N is $\widehat{AB}$-semi-invariant if and only

$$\sum_{j \in J(I)}^{N} p_j \geq \sum_{j \in I}^{N} p_j$$

for all $I \subset \{1, \ldots, N\}$, where

$$J(I) := \{j \,:\, b_{i,j} = 1 \text{ for some } i \in I\}$$

Convergence follows from this choice of matrix components

Other technical details include weak convergence of measures, etc

Random difference equations

- probability space $(\Omega, \mathcal{F}, \mathbb{P})$, ergodic process $\theta : \Omega \to \Omega$

- compact metric space (X, d), measurable mapping $f : X \times \Omega \to X$

<u>random difference equation</u> $\boxed{x_{n+1} = f(x_n, \theta^n(\omega))}$

$\Longrightarrow$ skew product $(x, \omega) \mapsto F(x, \omega) := \begin{pmatrix} f(x, \omega) \\ \theta(\omega) \end{pmatrix}$

$\Longrightarrow$ invariant measure μ on $X \times \Omega$ $\boxed{\mu = F^* \mu}$

BUT we can only discretize the state space X, i.e. use a grid

$$X_N = \{x_1^{(N)}, \ldots, x_N^{(N)}\} \quad \text{with} \quad h_N \to 0 \quad \text{as } N \to \infty$$

We can decompose the invariant measure $\mu = F^*\mu$ as

$$\mu(B, \omega) \;=\; \mu_\omega(B)\, \mathbb{P}(d\omega) \qquad\qquad \forall\, B \in \mathcal{B}(X)$$

where the measures μ_ω on X are $\underline{\theta\text{-invariant w.r.t. } f}$, i.e.

$$\boxed{\mu_{\theta(\omega)}(B) \;=\; \mu_\omega\left(f^{-1}(B, \omega)\right), \quad \forall\, B \in \mathcal{B}(X),\, \omega \in \Omega}$$

On the deterministic grid X_N we now consider

- random Markov chains $\{P_N(\omega), \omega \in \Omega\}$
- random probability vectors $\{p_N(\omega), \omega \in \Omega\}$

$$\boxed{p_{N,n+1}(\theta^{n+1}(\omega)) = p_{N,n}(\theta^n(\omega))P_N(\theta^n(\omega))} \quad \forall \quad n \in \mathbb{Z}, \quad \omega \in \Omega$$

equilibrium probability vector $\boxed{\bar{p}_N(\theta(\omega)) = \bar{p}_N(\omega)P_N(\omega)}$

$\Longrightarrow$ random measure $\mu_{N,\omega}$ on X

Theorem 5

A random probability measure $\{\mu_\omega, \omega \in \Omega\}$ is θ-semi-invariant w.r.t. f on X if and only if it is randomly stochastically approachable, i.e. for each N there exist

1) a grid X_N with fineness $\Delta_N \to 0$ as $N \to \infty$
2) a random Markov chain $\{P_N(\omega), \omega \in \Omega\}$ on X_N
3) random probability measure $\{\mu_{N,\omega}, \omega \in \Omega\}$ on X corresponding to a random equilibrium probability vectors $\{\bar{p}_N(\omega), \omega \in \Omega\}$ of the $\{P_N(\omega), \omega \in \Omega\}$ on X_N with the expected convergences.

$$\mathbb{E}D\left(P_N(\omega), f(\cdot,\omega)\right) \to 0 \qquad \mathbb{E}\rho\left(\mu_{N,\omega}, \mu\right) \to 0$$

P. Imkeller and P.E. Kloeden,
On the computation of invariant measures in random dynamical systems,
Stochastics & Dynamics **3** (2003), 247–265.