Stochastic Flows and Signed Measure-Valued SPDEs

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1. Stochastic Flows and (Positive) Measure-V. SPDEs

1.1 Stochastic Flows of SODEs

$(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ stochastic basis, P complete.

$\tilde{F}: \mathbb{R}^d \times \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^d$ - measurable, adapted

$\tilde{J}: \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \Omega \to \mathcal{M}_{d\times d}$ " "

($d\times d$ - matrices)

$w(dp,dt) = (w_1(dp,dt), \ldots, w_d(dp,dt))^T$

$\mathbb{R}^d$-valued standard Gaussian space-time white noise - $w_j(dp,dt)$ i.i.d. ....

$$(1.1) \quad \begin{cases} dr = \tilde{F}(r,t)dt + \int \tilde{J}(r,p,t)\, w(dp,dt) \\ r(0) = q \in \mathbb{R}^d . \end{cases}$$

$\varrho(r^1 - r^2) := |r^1 - r^2| \wedge 1, \ \tilde{\varrho} \in \{1, \ldots, 3\}$

Assume (local) Lipschitz & linear growth

$$(L)\ \begin{cases} \sum_{k,\ell=1}^{d} \int \{ \tilde{J}_{k\ell}(r^1, p, t\wedge \bar{\tau}_n) - \tilde{J}_{k\ell}(r^2, p, t\wedge\bar{\tau}_n) \}^2 dp \\ \quad \le K_n^2 \tilde{g}^2(r^1 - r^2), \quad \bar{\tau}_n \uparrow \infty \text{ a.s., as } K_n \uparrow \infty \\ \quad \ldots \ldots (F) \\ \quad \ldots \ldots \end{cases}$$

Facts: (K. (2008), ...)

(i) $\forall q \in \mathbb{R}^d$ $\exists!$ solution $r(\cdot, q) \in C([0,\infty); \mathbb{R}^d)$ a.s.

(ii) $\exists\ \bar{r}: \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$

(a) $\forall t$ $\bar{r}(t,\cdot,\cdot)$ $\mathcal{F}_t \otimes \mathcal{B}^d - \mathcal{B}^d$ - measurable

(b) $\bar{r}(\cdot, \omega, q) \equiv r(\cdot, \omega, q)$ $\lambda^d \otimes P$ - a.e.

(iii) $P\{\omega : |r_0^1(\omega) - r_0^2(\omega)| = 0\} = 0$ implies

$\Rightarrow P\{\bigcup_{t\ge 0} \{\omega : |\bar{r}(t,\omega, r_0^1(\omega)) - \bar{r}(t,\omega, r_0^2(\omega))| = 0\} = 0$ ∎

Let $m \in \mathbb{N}$, $j = (j_1, \ldots, j_d) \in (\mathbb{N} \cup \{0\})^d$, $|j| = j_1 + \cdots + j_d$.

Suppose $\forall T > 0$:

$$\max_{k,\ell \le d} \operatorname*{ess\,sup}_{\omega,\, t \le T} \Big\{ \max_{|j| \le m} |||\partial^j \tilde{F}_\ell(t)||| + \max_{|j| \le m+1} ||| \int \partial_r^j \tilde{J}_{k\ell}(\cdot, p, t)^2 dp ||| \Big\}$$

$$< \infty$$

where $|||f||| := \sup_{q \in \mathbb{R}^d} |f(q)|$, $f: \mathbb{R}^d \to \mathbb{R}$

If $H(m)$ holds, then for j, $|j| \leq m-1$

$(\partial_q^j \bar{F})(\cdot,\cdot,\cdot) \in L_{0,\mathcal{F}}(\Omega; C(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d))$

(measur., $\bar{F}(\cdot \wedge t, \cdot, \cdot)$ adapted ..)

consider: $(\bar{F}(t,\omega,a), \bar{F}(t,\omega,b))$

set: $D_{2d} := \{(q^1, \dots, q^d, q^{d+1}, \dots, q^{2d}) : (q^1, \dots, q^d) = (q^{d+1}, \dots, q^{2d})\}$

$\Omega_{CH} := \{\omega : (\bar{F}(\cdot,\omega,\cdot), \bar{F}(\cdot,\omega,\cdot)) \in C(\mathbb{R}_+ \times \mathbb{R}^{2d}; \mathbb{R}^{2d})\}$

__Proposition 1.1__

Assume $H(m)$, $m=1$. Set

$\Omega_t := \{\omega : (\bar{F}(t,\omega,a), \bar{F}(t,\omega,b)) \in D_{2d}^c \ \forall (a,b) \in D_{2d}^c\} \cap \Omega_{CH}$.

Then

(i) $\Omega_t \in \mathcal{F}_t$

(ii) $P(\Omega_t) = 1$

Proof: Partition D_{2d}^c into $2d$ disjoint sets $S_i^\pm$, $i=1,\dots,d$; approximate $S_i^\pm$ by closed sets $S_{i,n}^\pm$. By continuity of $(\bar F(t_1;a), \bar F(t_1;b))$ in (a,b) it suffices to consider $(\tilde a,\tilde b) \in Q^{2d} \cap \left(\bigcup_{i=1}^{d} \bigcup_{+,-} S_i^\pm\right)$. Apply fact (iii). ■

Corollary 1.2

Let $S^+, S^- \in \mathcal{B}^d \otimes \mathcal{F}_0$ and assume the ω-sections $A(\omega)$, $B(\omega)$ of $S^\pm$ satisfy $A(\omega) \times B(\omega) \subset D_{2d}^c$ a.s.

Then $\forall t > 0$

$$(\bar F(t,\omega, A(\omega)), \bar F(t,\omega, B(\omega))) \in D_{2d}^c \text{ a.s.}$$

1.2 Flows of Measures and SPDEs

$\mathbb{M} := \{\mu : \mu \text{ finite Borel measure on } \mathbb{R}^d\}$

$\mathbb{B}_{L,\infty} := \{f \in C_b(\mathbb{R}^d; \mathbb{R}) : \|f\|_{L,\infty} < \infty\}$.

$\|f\|_{L,\infty} := |||f||| \vee \|f\|_L$

$$\|f\|_L := \sup_{0<|r-q|\le 1} \frac{|f(r)-f(q)|}{\rho(r-q)}$$

$$\gamma(\mu-\nu) := \sup_{\|f\|_{L,\infty}\le 1} \left| \int f(r)(\mu-\nu)(dr) \right|$$

$\Rightarrow$ $(\mathbb{M}, \gamma)$ complete separable metric space.

Suppose: $\mathcal{X}_0 : \Omega \to \mathbb{M}$ $\mathcal{F}_0$-measurable

Set

(1.2) $$\mathcal{Y}(t,\omega) := \int \delta_{r(t,\omega,q)}\, \mathcal{X}_0(dq,\omega)$$

Abbreviate

$$m(r(t,q),dt) := \int \mathcal{J}(r(t,q),p,t)\, w(dp,dt)$$

$$\hat{D}_{\mathcal{X}_0}(r(t,q))\,dt := [m_{k\ell}(r(t,q),dt), m_\ell(r(t,q),dt)]$$

Proposition 1.3

Suppose (L). Then

(i) $\mathcal{Y}(\cdot) \in C_{\mathcal{F}}([0,\infty); M)$ a.s.

(ii) $\mathcal{Y}(\cdot)$ is a (unique) (PDE-) weak solution of the SPDE:

$$(1.3)\quad \begin{cases} d\mathcal{Y} = \frac{1}{2}\sum_{k,l=1}^{d} \partial^2_{kl}\left(\tilde{D}_{kl}(\cdot,t)\,\mathcal{Y}\right)dt \\ \qquad -\nabla\cdot\left(\mathcal{Y}\tilde{F}(\cdot,t)\right)dt \\ \qquad -\nabla\cdot\left(\mathcal{Y}\int \tilde{\mathcal{J}}(\cdot,p,t)\,w(dp,dt)\right) \\ \mathcal{Y}(0) = \mathcal{X}_0 . \end{cases}$$

(iii) Suppose in addition to (L) $H(m)$, $m \geq 1$.

If $\mathcal{X}_{0,1}, \mathcal{X}_{0,2}$: $\mathrm{supp}(\mathcal{X}_{0,1}) \cap \mathrm{supp}(\mathcal{X}_{0,2}) = \emptyset$ a.s.

$\Rightarrow$

$\forall t \;\; \mathrm{supp}(\mathcal{Y}(t,\mathcal{X}_{0,1})) \cap \mathrm{supp}(\mathcal{Y}(t,\mathcal{X}_{0,2})) = \emptyset$ a.s.

Proof: (i) follows from the continuity of $r(t,\omega,q)$

(ii) " " Itô's formula

(iii) " " Corollary 1.2

1.3 Interacting SODEs & Quasilinear SPDEs

$F: \mathbb{R}^d \times \mathbb{M} \times \mathbb{R}_+ \longrightarrow \mathbb{R}^d$

(measurable)

$\mathcal{J}_\varepsilon: \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M} \times \mathbb{R}_+ \longrightarrow \mathbb{M}_{d\times d}$

(local) Lipschitz & linear growth

$$(L\,I)\begin{cases} \sum_{k,\ell=1}^{d} \int \left(\mathcal{J}_{\varepsilon,k\ell}(r^1, p, \mu_1, t) - \mathcal{J}_{\varepsilon,k\ell}(r^2, p, \mu_2, t)\right)^2 dp \\ \le K_N^2 \left\{ \gamma^2(\mu_1) \vee \gamma^2(\mu_2)\, \varrho^2(r^1 - r^2) + \gamma^2(\mu_1 - \mu_2) \right\} \\ \cdots \\ \cdots \end{cases}$$

$$(\ast)\begin{cases} dr = F(r, \mathcal{X}, t)\,dt + \overbrace{\int \mathcal{J}_\varepsilon(r, p, \mathcal{X}, t)\, w(dp, dt)}^{=:\, m(r, \mathcal{X}, dt)} \\ \qquad r(0) = q \in \operatorname{supp}(\mathcal{X}_0) \\ \mathcal{X}(t) := \mathcal{X}(t, \mathcal{X}_0) := \int \delta_{r(t, \mathcal{X}, q)}\, \mathcal{X}_0(dq) \end{cases}$$

where $\mathcal{X}(\cdot)$ is the solution of the quasilinear SPDE

$$(\ast\ast)\begin{cases} d\mathcal{X} = \left\{ \frac{1}{2} \sum_{k,\ell=1}^{d} \partial^2_{k\ell} \left(\mathcal{X}\, D_{k\ell}(\cdot, \mathcal{X})\right) - \nabla \cdot \left(\mathcal{X} F(\cdot, \mathcal{X})\right) \right\} dt \\ \qquad - \nabla \cdot \left(\mathcal{X} \int \mathcal{J}_\varepsilon(\cdot, p, \mathcal{X}, t)\, w(dp, dt) \right) \\ \mathcal{X}(0) = \mathcal{X}_0. \end{cases}$$

Theorem 1.4

$\exists\ (F(t,x,q),\ \mathcal{X}(t,x_0))$ solving (1.4)/(1.5).

Proof: W.l.o.g. assume global Lipschitz, lin growth and by mass conservation,

$\mathcal{X}_0(\omega)(\mathbb{R}^d) = \gamma(\mathcal{X}_0(\omega)) \overset{!}{=} \gamma(\mathcal{X}(t,\mathcal{X}_0(\omega))) \le \sigma < \infty$ a.s.

Choose $\tilde{\mathcal{Y}} \in C_{\mathcal{F}}([0,\infty);\mathcal{M})$ a.s., $\gamma(\tilde{\mathcal{Y}}(t,\omega)) \le \sigma$ a.s.

Set
$$\left.\begin{aligned} \tilde{F}(r,\omega,t) &:= F(r,\tilde{\mathcal{Y}}(t,\omega),t) \\ \tilde{\mathcal{J}}(r,p,\omega,t) &:= \mathcal{J}_\varepsilon(r,p,\tilde{\mathcal{Y}}(t,\omega),t) \end{aligned}\right\} \Rightarrow \text{cond. (1.1) hold}$$

$\mathcal{Y}_0(t,\omega) := \mathcal{X}_0(\omega) \qquad (=: \tilde{\mathcal{Y}}(t,\omega))$

$\Rightarrow \exists!\ F(t,\omega,\mathcal{Y}_0(\omega),t)$ solving (1.4)

$\Rightarrow$ (iteratively) $\exists!\ \mathcal{Y}_n(\cdot)$

$$\mathcal{Y}_n(t,\omega) = \int \delta_{F(t,\omega,\mathcal{Y}_{n-1}(\omega),q)}\,\mathcal{X}_0(dq,\omega)$$

and

$\mathcal{Y}_n(t)$ solves bilinear SPDE (1.5)

with
$$\tilde{F}(r,\omega,t) := F(r,\mathcal{Y}_{n-1}(\omega,t),t)$$
$$\tilde{\mathcal{J}}(r,\omega,t) := \mathcal{J}_\varepsilon(r,p,\mathcal{Y}_{n-1}(\omega,t),t)$$

By Lipschitz assumptions (etc.) $\forall n, m$

$$E \sup_{0 \le t \le T} \gamma^2(y_n(t) - y_m(t))$$

$$\le C_T \int_0^T E \gamma^2(y_{n-1}(s) - y_{m-1}(s))^2 \, ds$$

$\Rightarrow$ (contraction mapping principle)

$\exists !\ x(\cdot) \in E_{\mathcal{F}}([0,\infty); M)$ a.s.

(i) $\gamma(x(t,\omega)) \le c \quad \forall t$ a.s.

(ii) $E \sup_{0 \le t \le T} \gamma^2(y_n(t) - x(t)) \to 0$, as $n \to \infty$.

Set $\bar{x}(t) := \int \delta_{F(t, x, q)} x_0(dq)$

Then

$$E \sup_{0 \le t \le T} \gamma^2(y_n(t) - \bar{x}(t))$$
$$\le C_T \int_0^T E \gamma^2(y_{n-1}(s) - x(s)) \, ds \xrightarrow{(ii)} 0$$

$\Rightarrow \quad \bar{x}(\cdot) = x(\cdot)$ a.s.

and $x(t)$ solves (1.5).

Smoothness and uniqueness for (1.5) under $H(m)$, $m \ge 3$ ($m \ge 5$) if $X_0 \in H_0 := L_2(\mathbb{R}^d; dr)$

If $x(t) = X(t) dr$ ...

$\ldots$

2. Stochastic Flows & Signed Measure-v. SPDEs

$\mathbb{M}_s := \{\mu = \mu^+ - \mu^-, \ \mu^\pm \in \mathbb{M}, \text{ Hahn-Jordan}\}$

γ can be extended to a metric on $\mathbb{M}_s$ but $(\mathbb{M}_s, \gamma)$ is incomplete – Ex. $\mu_n := \sum_{k=1}^{n} (-1)^k \delta_{\frac{1}{k}}$ in $\mathbb{R}$ ..., Cauchy no limit

Therefore, we first analyze flows of SODEs and SPDEs with measure components in $\mathbb{M}_+ \times \mathbb{M}_-$ $(\ni (\mu^+, \mu^-))$

$\hat{\gamma}(\mu, \nu) := \gamma(\mu^+ - \nu^+) + \gamma(\mu^- - \nu^-)$.

All previous calculations carry over.

Starting in (1.5) with $\mathcal{X}_0 := \mathcal{X}_0^+ - \mathcal{X}_0^-$ we obtain $\mathcal{X}(t) = \mathcal{X}^+(t) - \mathcal{X}^-(t)$.

__Theorem 2.1__

If $\mathrm{supp}(\mathcal{X}_0^+) \cap \mathrm{supp}(\mathcal{X}_0^-) = \emptyset$ a.s. then $\forall t$ $\mathrm{supp}(\mathcal{X}^+(t)) \cap \mathrm{supp}(\mathcal{X}^-(t)) = \emptyset$ a.s. (i.e. $(\mathcal{X}^+(t), \mathcal{X}^-(t))$ is the Hahn-Jordan decomposition of $\mathcal{X}(t)$)

__Proof__: Proposition 1.3, (iii). ∎

3. 2D- Fluid Mechanics

macroscopic PDE for the vorticity of a 2D fluid

(3.1) $\partial_t X(r,t) + \nu\Delta X(r,t) - \nabla\cdot\int(\nabla^\perp g)(r-q)X(q,t)\,dq$

$g(|r|) = \frac{1}{2\pi}\ln(|r|)$, $\nabla^\perp := (-\partial_2, \partial_1)^T$ $(=u(r,t))$

smooth approx. of $g \to g_\delta$

$K_\delta(r) := \nabla^\perp g_\delta(|r|)$

$\partial_t X(r,t) = \nu\Delta X(r,t)dt + \nabla\cdot\int K_\delta(r-q)X(q,t)\,dq$

point vortices

 counterclockwise, clockwise

intensity a_i

let $F(r,\mathcal{X}) := \int K_\delta(r-\tilde{r})\mathcal{X}(d\tilde{r})$

$\Gamma_\varepsilon(r,p) \in M_{d\times d}$ - components $\to$ δ-functions as $\varepsilon\downarrow 0$

Assume Lipschitz conditions on Γ_ε, F

(3.2) $\begin{cases} dr(t) = F(r,\mathcal{X})dt + \sqrt{2\nu}\int\Gamma_\varepsilon(r,p)\,w(dp,dt) \\ \mathcal{X}(t) = \int \delta_{r(t,\mathcal{X},q)}\,\mathcal{X}_0(dq) \end{cases}$

$$d\mathcal{X} = \nu\Delta\mathcal{X}(t)dt - \nabla\cdot(\mathcal{X}F(\cdot,\mathcal{X}))dt - \nabla\cdot\left(\mathcal{X}\int\Gamma_\varepsilon(\cdot,p)\,w(dp,dt)\right)$$

$\mathcal{X}(0) = \mathcal{X}_0$

Stochastic Navier-Stokes Equation

[illegible]