Finite element approximation of the stochastic wave equation

Mihály Kovács

Department of Mathematics and Statistics
University of Otago, Dunedin, New Zealand

RDS'09 Bielefeld, 18-20 Nov 2009

Outline

$$\begin{cases} u_{tt} - \Delta u = \dot{W}, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0,\ u_t(0) = u_1. \end{cases}$$

- Abstract framework
- Finite element approximation
- Strong convergence
- Weak convergence

Co-workers

Stig Larsson (Chalmers)
Fardin Saedpanah (Chalmers)
Fredrik Lindgren (Chalmers)

Semigroup approach

Linear SPDE with additive noise:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), & t > 0 \\ X(0) = X_0 \end{cases}$$

- $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$, filtered probability space
- H, U Hilbert spaces
- $W(t)$, Q-Wiener process on U with respect to $\{\mathcal{F}_t\}_{t\geq 0}$
- $B : U \to H$, bounded linear operator
- $X(t)$, H-valued stochastic process
- $E(t) = e^{-tA}$, C_0-semigroup of bounded linear operators on H
- X_0 is $\mathcal{F}_0$-measurable H-valued random variable

Weak solution

A weak solution satisfies the integral equation: for all $\eta \in D(A^*)$

$$\langle X(t), \eta \rangle + \int_0^t \langle X(s), A^*\eta \rangle \, \mathrm{d}s = \langle X_0, \eta \rangle + \int_0^t \langle \eta, B \, \mathrm{d}W(s) \rangle$$

The unique weak solution is given by (mild solution)

$$X(t) = E(t)X_0 + \int_0^t E(t-s)B \, \mathrm{d}W(s)$$

Must give a rigorous meaning to $W(t)$ and define the stochastic integral.

Q-Wiener process

- covariance operator $Q : U \to U$, self-adjoint, positive definite, bounded, linear operator
- $Qe_j = \gamma_j e_j, \quad \gamma_j > 0, \quad \{e_j\}_{j=1}^{\infty}$ ON basis
- $\beta_j(t)$, independent identically distributed, real-valued, Brownian motions
- $W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j$

Two important cases:

- $\mathrm{Tr}(Q) < \infty$. $\quad W(t)$ converges in $L_2(\Omega, U)$:
$$\mathbf{E}\Big\| \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j \Big\|^2 = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)^2\big) = t \sum_{j=1}^{\infty} \gamma_j = t\,\mathrm{Tr}(Q) < \infty$$
- $Q = I$, "white noise". $\quad W(t)$ is not U-valued, since $\mathrm{Tr}(I) = \infty$, but converges in a weaker sense.

Q-Wiener process

If $\mathrm{Tr}(Q) < \infty$:

- $W(0) = 0$
- continuous paths $t \mapsto W(t)$
- independent increments
- Gaussian law $\mathbf{P} \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q), \quad s \leq t$

$\{W(t)\}_{t \geq 0}$ generates a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ so that it becomes a square integrable U-valued martingale.

We can integrate with respect to W: $\displaystyle \int_0^t B(s)\,\mathrm{d}W(s).$

The integral can be defined also when $\mathrm{Tr}(Q) = \infty$.

Stochastic integral

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

- We can define the stochastic integral (deterministic integrand) $\int_0^t B(s)\,\mathrm{d}W(s)$ if $\int_0^t \|B(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s < \infty$.
- Isometry property: $\mathbf{E}\Big\| \int_0^t B(s)\,\mathrm{d}W(s)\Big\|^2 = \int_0^t \|B(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s$

Hilbert-Schmidt operator B:

$$\|B\|_{\mathrm{HS}}^2 = \sum_{l=1}^{\infty} \|B\varphi_l\|^2 < \infty, \qquad \{\varphi_l\} \text{ arbitrary ON basis in } U$$

Da Prato and Zabczyk, *Stochastic Equations in Infinite Dimensions*
C. Prévôt and M. Röckner, *A Consise Course on Stochastic Partial Differential Equations*

The stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi,t) - \Delta u(\xi,t) = \dot W(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0,\ \dfrac{\partial u}{\partial t}(\xi,0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

$$\Lambda = -\Delta, \quad D(\Lambda) = \dot H^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

$$\dot H^\beta = D(\Lambda^{\beta/2}), \quad |v|_\beta = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^\infty \lambda_j^\beta (v,\phi_j)^2\Big)^{1/2}, \quad \beta \in \mathbf{R}$$

$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}u_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W, \quad X = \begin{bmatrix} u \\ u_t \end{bmatrix},\ A = -\begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$H = \dot H^0 \times \dot H^{-1}, \quad D(A) = \dot H^1 \times \dot H^0, \quad U = \dot H^0 = L_2(\mathcal{D})$$

Abstract framework

Let $u_0 = 0,\ u_1 = 0$ for simplicity.

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \quad t > 0 \\ X(0) = 0 \end{cases}$$

- $X(t)$, $H = \dot{H}^0 \times \dot{H}^{-1}$-valued stochastic process
- $W(t)$, $U = \dot{H}^0$-valued Q-Wiener process w.r.t $\{\mathcal{F}_t\}_{t\geq 0}$

- $E(t) = e^{-tA} = \begin{bmatrix} \cos(t\Lambda^{1/2}) & \Lambda^{-1/2}\sin(t\Lambda^{1/2}) \\ -\Lambda^{1/2}\sin(t\Lambda^{1/2}) & \cos(t\Lambda^{1/2}) \end{bmatrix}$, C_0-semigroup

$$\cos(t\Lambda^{1/2})v = \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})(v,\phi_j)\phi_j \quad \big(\lambda_j, \phi_j \text{ are the eigenpairs of } \Lambda\big)$$

Regularity

Theorem. If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then there exists a unique weak solution

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \int_0^t E(t-s)B\,\mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda^{-1/2}\sin\left((t-s)\Lambda^{1/2}\right)\mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda^{1/2}\right)\mathrm{d}W(s) \end{bmatrix}$$

and

$$\|X(t)\|_{L_2(\Omega,\dot{H}^{\beta}\times\dot{H}^{\beta-1})} \leq C(t)\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}.$$

Two cases:

- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ iff $d = 1,\ \beta < 1/2$.

The finite element method

Spatial discretization

- triangulations $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- finite element spaces $\{S_h\}_{0<h<1}$
- $S_h \subset \dot{H}^1 = H_0^1(\mathcal{D})$ continuous piecewise linear functions
- $\Lambda_h : S_h \to S_h,$ discrete Laplacian, $\quad (\Lambda_h \psi, \chi) = (\nabla\psi, \nabla\chi), \ \forall \chi \in S_h$
- $P_h : \dot{H}^0 \to S_h,$ orthogonal projection, $\quad (P_h f, \chi) = (f, \chi), \ \forall \chi \in S_h$
- $A_h = \begin{bmatrix} 0 & I \\ -\Lambda_h & 0 \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ P_h \end{bmatrix}$
- $\begin{cases} \mathrm{d}X_h(t) + A_h X_h(t)\,\mathrm{d}t = B_h\,\mathrm{d}W(t), & t > 0 \\ X_h(0) = 0 \end{cases}$
- $E_h(t) = e^{-tA_h} = \begin{bmatrix} \cos(t\Lambda_h^{1/2}) & \Lambda_h^{-1/2}\sin(t\Lambda_h^{1/2}) \\ -\Lambda_h^{1/2}\sin(t\Lambda_h^{1/2}) & \cos(t\Lambda_h^{1/2}) \end{bmatrix}$

The finite element method (continued)

The weak solution is:

$$X_h(t) = \begin{bmatrix} X_{h,1}(t) \\ X_{h,2}(t) \end{bmatrix}$$

$$= \int_0^t E_h(t-s) B_h \, \mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda_h^{-1/2} \sin\left((t-s)\Lambda_h^{1/2}\right) P_h \, \mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda_h^{1/2}\right) P_h \, \mathrm{d}W(s) \end{bmatrix}$$

where

$$\cos(t\Lambda_h^{1/2})v = \sum_{j=1}^{N_h} \cos\left(t\sqrt{\lambda_{h,j}}\right)(v, \phi_{h,j})\phi_{h,j}$$

$\lambda_{h,j}$, $\phi_{h,j}$ are the eigenpairs of Λ_h

Strong and weak error

- Strong error:

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} = (\mathbf{E}\|X_h(t) - X(t)\|_H^2)^{1/2}$$

- Weak error:

$$\mathbf{E}G(X_h(T)) - \mathbf{E}G(X(T))$$

for $G : H \to \mathbb{R}$.

Error estimates for the deterministic problem

$$\begin{cases} v_{tt}(t) + \Lambda v(t) = 0, \ t > 0 \\ v(0) = 0, \ v_t(0) = f \end{cases} \qquad \Rightarrow v(t) = \Lambda^{-1/2} \sin\left(t\Lambda^{1/2}\right) f$$

$$\begin{cases} v_{h,tt}(t) + \Lambda_h v_h(t) = 0, \ t > 0 \\ v_h(0) = 0, \ v_{h,t}(0) = P_h f \end{cases} \qquad \Rightarrow v_h(t) = \Lambda_h^{-1/2} \sin\left(t\Lambda_h^{1/2}\right) P_h f$$

We have, for $K_h(t) = \Lambda_h^{-1/2} \sin\left(t\Lambda_h^{1/2}\right) P_h - \Lambda^{-1/2} \sin\left(t\Lambda^{1/2}\right)$

$\|K_h(t)f\| \leq C(t)h^2\|f\|_{\dot{H}^2}$ "initial regularity of order 3"

$\|K_h(t)f\| \leq 2\|f\|_{\dot{H}^{-1}}$ "initial regularity of order 0" (stability)

$\|K_h(t)f\| \leq C(t)h^{\frac{2}{3}\beta}\|f\|_{\dot{H}^{\beta-1}}, \quad 0 \leq \beta \leq 3$

$\beta - 1$ can not be replaced by $\beta - 1 - \epsilon$ for $\epsilon > 0$ (J. Rauch 1985)

Strong convergence in L_2 norm

Theorem. If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0, 3]$, then

$$\|X_{h,1}(t) - X_1(t)\|_{L_2(\Omega,\dot{H}^0)} \leq C(t)\, h^{\frac{2}{3}\beta} \big\|\Lambda^{(\beta-1)/2}Q^{1/2}\big\|_{\mathrm{HS}}$$

Higher order FEM: $\quad O(h^{\frac{r}{r+1}\beta}), \quad \beta \in [0, r+1].$

Proof. $\{f_k\}$ an arbitrary ON basis in $\dot{H}^0$

$$\begin{aligned}
\|X_{h,1}(t) - X_1(t)\|^2_{L_2(\Omega,\dot{H}^0)} &= \mathbf{E}\big(\|X_{h,1}(t) - X_1(t)\|^2\big) \\
&= \mathbf{E}\Big(\Big\| \int_0^t K_h(t-s)\,\mathrm{d}W(s)\Big\|^2\Big) \\
\{\text{Isometry}\} &= \int_0^t \big\|K_h(s)Q^{1/2}\big\|^2_{\mathrm{HS}}\,\mathrm{d}s = \int_0^t \sum_{k=1}^{\infty} \big\|K_h(s)Q^{1/2}f_k\big\|^2\,\mathrm{d}s \\
&\leq C(t)h^{\frac{4}{3}\beta} \sum_{k=1}^{\infty} \big\|Q^{1/2}f_k\big\|^2_{\dot{H}^{\beta-1}} = C(t)h^{\frac{4}{3}\beta}\big\|\Lambda^{(\beta-1)/2}Q^{1/2}\big\|^2_{\mathrm{HS}}
\end{aligned}$$

Kovács, Larsson and Saedpanah, preprint 2009

Strong convergence in L_2 norm (special cases)

Two cases:

- If $\|Q^{1/2}\|_{\text{HS}}^2 = \text{Tr}(Q) < \infty$, then $\beta = 1$.
  $\|X_{h,1}(t) - X_1(t)\|_{L_2(\Omega,\dot{H}^0)} \leq C(t)h^{2/3}$
- If $Q = I$, then $\|\Lambda^{(\beta-1)/2}\|_{\text{HS}} < \infty$ iff $d = 1,\ 0 \leq \beta < 1/2$.
  $\|X_{h,1}(t) - X_1(t)\|_{L_2(\Omega,\dot{H}^0)} \leq C(t)h^s, \quad s < 1/3$

Earlier works in one dimension ($d = 1,\ Q = I$):

- L. Quer-Sardanyons & M. Sanz-Solé 2006: spatially semidiscrete difference scheme $O(h^s),\ s < 1/3$
- J.B. Walsh 2006: a fully discrete leapfrog method $O(h^{1/2})$

Proof technique: Green's function.

- We extend Quer-Sardanyons & Sanz-Solé.
- We explain the discrepancy with Walsh.

Comparison with the heat equation

Regularity is the same for both the heat and wave equations:

$$\|X_1(t)\|_{L_2(\Omega,\dot{H}^\beta)} \leq C\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}$$

Strong convergence in L_2-norm:

$$\|X_{h,1}(t) - X_1(t)\|_{L_2(\Omega,\dot{H}^0)} \leq C\, h^{\frac{2}{3}\beta}\, \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} \qquad \text{(wave equation)}$$

$$\|X_h(t) - X(t)\|_{L_2(\Omega,\dot{H}^0)} \leq C\, h^{\beta}\, \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} \qquad \text{(heat equation)}$$

Weak error representation: preliminaries

Consider

$$\mathrm{d}Y(t) = E(T-t)B\,\mathrm{d}W(t),\ t \in (0,T];\ Y(0) = E(T)X_0,$$

with weak solution

$$Y(t) = E(T)X_0 + \int_0^t E(T-s)B\,\mathrm{d}W(s).$$

Similarly, consider

$$\mathrm{d}Y_h(t) = E_h(T-t)B\,\mathrm{d}W(t),\ t \in (0,T];\ Y_h(0) = E_h(T)X_{0,h},$$

with weak solution

$$Y_h(t) = E_h(T)X_{0,h} + \int_0^t E_h(T-s)B_h\,\mathrm{d}W(s).$$

Note: $X(T) = Y(T)$, $X_h(T) = Y_h(T)$. No drift term in eq. for Y and Y_h.

Weak error representation: preliminaries

Auxiliary problem:

$$\mathrm{d}Z(t) = E(T-t)B\,\mathrm{d}W(t),\ t\in(\tau,T];\ Z(\tau)=\xi,$$

where ξ is a $\mathcal{F}_\tau$-measurable random variable.

Unique weak solution: $Z(t,\tau,\xi) = \xi + \int_\tau^t E(T-s)B\,\mathrm{d}W(s)$

Define $u : H\times[0,T]\to\mathbb{R}$ by

$$u(x,t) = \mathbf{E}\left(G(Z(T,t,x))\right).$$

If $G\in C_{\mathrm{b}}^2(H,\mathbb{R})$, then u is a solution to Kolmogorov's equation

$$u_t(x,t) + \tfrac{1}{2}\operatorname{Tr}\left(u_{xx}(x,t)E(T-t)BQ[E(T-t)B]^*\right) = 0,\quad t\in[0,T),\ x\in H,$$
$$u(x,T) = G(x),$$

Weak error representation: preliminaries

Nuclear operators on H:

$T \in \mathcal{L}_1(H)$
if there are sequences $\{a_j\}, \{b_j\} \subset H$ with $\sum_{j=1}^{\infty} \|a_j\| \|b_j\| < \infty$ and such that

$$Tx = \sum_{j=1}^{\infty} \langle x, b_j \rangle a_j, \quad \forall x \in H.$$

Banach space with norm: $\|T\|_{\mathrm{Tr}} = \inf \Big\{ \sum_{j=1}^{\infty} \|a_j\| \|b_j\| : Tx = \sum_{j=1}^{\infty} \langle x, b_j \rangle a_j \Big\}.$

For $T \in \mathcal{L}_1(H)$: $\mathrm{Tr}(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$ with $\{e_k\}_{k=1}^{\infty}$ ONB of H

Connection with HS: $\|T\|_{\mathrm{HS}}^2 = \mathrm{Tr}(T^* T) = \|T^* T\|_{\mathrm{Tr}}$

Weak error representation

THEOREM. If

$$\mathrm{Tr}\left(\int_0^T E(t)BQ[E(t)B]^*\,\mathrm{d}t\right) < \infty$$

and $G \in C_{\mathrm{b}}^2(H,\mathbb{R})$, then the weak error $e_h(T) = \mathbf{E}G(X_h(T)) - \mathbf{E}G(X(T))$ has the representation

$$\begin{aligned} e_h(T) &= \mathbf{E}\left(u(Y_h(0),0) - u(Y(0),0)\right) \\ &\quad + \frac{1}{2}\mathbf{E}\int_0^T \mathrm{Tr}\Big(u_{xx}(Y_h(t),t) \\ &\qquad \times [E_h(T-t)B_h + E(T-t)B]Q[E_h(T-t)B_h - E(T-t)B]^*\Big)\,\mathrm{d}t \end{aligned}$$

Weak error representation: proof

If ξ is $\mathcal{F}_t$ measurable, then $u(\xi,t) = \mathbf{E}\Big(G(Z(T,t,\xi))\Big|\mathcal{F}_t\Big)$. Therefore, by the law of double expectation,

$$\mathbf{E}\Big(u(\xi,t)\Big) = \mathbf{E}\Big(\mathbf{E}\Big(G(Z(T,t,\xi))\Big|\mathcal{F}_t\Big)\Big) = \mathbf{E}\Big(G(Z(T,t,\xi))\Big).$$

Thus, with $\xi = Y(0)$ and $t = 0$,

$$\mathbf{E}\Big(u(Y(0),0)\Big) = \mathbf{E}\Big(G(Z(T,0,Y(0)))\Big) = \mathbf{E}\Big(G(Y(T))\Big) = \mathbf{E}\Big(G(X(T))\Big)$$

and, with $\xi = Y_h(T)$ and $t = T$,

$$\mathbf{E}\Big(u(Y_h(T),T)\Big) = \mathbf{E}\Big(G(Z(T,T,Y_h(T)))\Big) = \mathbf{E}\Big(G(Y_h(T))\Big) = \mathbf{E}\Big(G(X_h(T))\Big).$$

Hence,

$$\begin{aligned} e_h(T) &= \mathbf{E}\Big(G(X_h(T)) - G(X(T))\Big) = \mathbf{E}\Big(u(Y_h(T),T) - u(Y(0),0)\Big) \\ &= \mathbf{E}\Big(u(Y_h(0),0) - u(Y(0),0)\Big) + \mathbf{E}\Big(u(Y_h(T),T) - u(Y_h(0),0)\Big). \end{aligned}$$

Weak error representation: proof

Using Itô's formula for $u(Y_h(t), t)$ and Kolmogorov's equation

$$
\begin{aligned}
&\mathbf{E}\Big(u(Y_h(T), T) - u(Y_h(0), 0)\Big) \\
&= \mathbf{E} \int_0^T u_t(Y_h(t), t) \\
&\quad + \frac{1}{2} \operatorname{Tr} \Big(u_{xx}(Y_h(t), t)[E_h(T-t)B_h]Q[E_h(T-t)B_h]^*\Big) \, \mathrm{d}t \\
&= \frac{1}{2} \mathbf{E} \int_0^t \operatorname{Tr} \Big(u_{xx}(Y_h(t), t) \\
&\qquad \times [E_h(T-t)B_h]Q[E_h(T-t)B_h]^* - [E(T-t)B]Q[E(T-t)B]^*\Big) \, \mathrm{d}t.
\end{aligned}
$$

The proof can be finished by algebraic manipulation and playing around with traces.

Applications: Wave equation

THEOREM: Let $g \in C_{\mathrm{b}}^2(\dot{H}^0, \mathbb{R})$ and assume that $\|\Lambda^{\beta-\frac{1}{2}} Q \Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}} < \infty$ for some $\beta \in [0, \frac{r+1}{2}]$. Then, there are $C > 0$, $h_0 > 0$, depending on g, X_0, Q, and T but not on h, such that for $h \leq h_0$,

$$\left| \mathbf{E}\big(g(X_{h,1}(T)) - g(X_1(T))\big) \right| \leq C h^{\frac{r}{r+1} 2\beta}.$$

The proof uses the weak error representation theorem with $G(X) := g(P_1 X) = g(X_1)$ and the deterministic error estimate

$$\|K_h(t)\| := \|\Lambda_h^{-1/2} \sin(t\Lambda_h^{1/2}) P_h v - \Lambda^{-1/2} \sin(t\Lambda^{1/2}) v\| \leq C(T) h^{\frac{r}{r+1} s} |v|_{s-1},$$

$$t \in [0, T], \; s \in [0, r+1].$$

Wave equation: sketch of proof

Set $G(X) := g(X_1)$.

$$(u_x(Y(t),t),\Phi) = \mathbf{E}(\langle g'(P_1 Z(Y(t),t,T)), P_1\Phi\rangle|\mathcal{F}_t)$$

and

$$(u_{xx}(Y(t),t)\Phi,\Psi) = \mathbf{E}(\langle g''(P_1 Z(Y(t),t,T))P_1\Phi, P_1\Psi\rangle|\mathcal{F}_t).$$

Recall, the abstract weak error representation:

$$\begin{aligned} e_h(T) &= \mathbf{E}\big(u(Y_h(0),0) - u(Y(0),0)\big) \\ &\quad + \frac{1}{2}\mathbf{E}\int_0^T \mathrm{Tr}\,\Big(u_{xx}(Y_h(t),t) \\ &\quad\quad \times [E_h(T-t)B_h - E(T-t)B]Q[E_h(T-t)B_h + E(T-t)B]^*\Big)\,\mathrm{d}t. \end{aligned}$$

Wave equation: sketch of proof

The first term:

$$|\mathbf{E}\big(u(Y_h(0),0)-u(Y(0),0)\big)| \leq C \sup_{x\in \dot{H}^0} \|g'(x)\| C h^{\frac{2r}{r+1}\beta}\mathbf{E}|||X_0|||_{2\beta}.$$

The second term, using $G(X) = g(X_1)$:

$$\mathbf{E}\Big(\operatorname{Tr}\big(u_{xx}(Y_h(t),t)[E_h(T-t)B_h + E(T-t)B]Q[E_h(T-t)B_h - E(T-t)B]^*\big)\Big)$$
$$= \mathbf{E}\Big(\operatorname{Tr}\big(K_h(T-t)Q[\Lambda_h^{-\frac{1}{2}}S_h(T-t)P_h + \Lambda^{-\frac{1}{2}}S(T-t)]g''(P_1 Z(Y(t),t,T)\big)\Big)$$

Wave equation: remark

Recall strong convergence rate: $O(h^{\frac{r}{r+1}\beta})$

under the assumption $\|\Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$.

It can be shown that

$$\|\Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \leq \|\Lambda^{\beta-\frac{1}{2}} Q \Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}}$$

with equality when A and Q have a common basis of eigenvectors,

in particular, when $Q = I$.

If $Q = I$, then $d = 1$, $\beta < \frac{1}{2}$ and the weak rate is almost $O(h^{\frac{r}{r+1}})$.

Related results

This weak convergence analysis is from:

Kovács, Larsson and Lindgren, preprint 2009

In this paper we also study:

- stochastic heat equation

  $dX + \Lambda X \, dt = \, dW$

  strong rate: $O(h^{\beta})$ Yan 2004
  weak rate: $O(h^{2\beta}|\log(h)|)$ Debussche and Printems 2009 (fully discrete)
  Geissert, Kovács and Larsson 2009

  nonlinear equation in 1-D (only time-stepping): Debussche 2008, to appear

- linearized stochastic Cahn-Hilliard equation

  $dX + \Lambda^2 X \, dt = dW$
  $dX_h + \Lambda_h^2 X_h \, dt = P_h dW$

Weak error for the leapfrog scheme: Hausenblas preprint 2009