Impact of degenerate noise on a change of stability

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Introduction

- Stochastic PDEs near a change of stability (Bifurcation)
- Dominant modes evolve on a *slow* time-scale
- Stable modes decay on a *fast* time-scale
- Evolution of dominant modes given by Amplitude eq.
- Formal derivation well known, e.g. [Cross, Hohenberg, '93]

Deterministic Results

- PDEs on bounded domains:
  Invariant manifolds – center manifold
  Flow given by an ODE on the manifold
- PDEs on unbounded domains:
  Many rigorous results for (cf. Guido Schneider et.al)
  Amplitude or Modulation Equations

Our Setting:

- Stochastic PDEs on *bounded* domains
- Amplitude equation is a (stochastic) ODE
- No center manifold theory for SPDEs available

Stabilization due to Noise

AIM:

- Rigorous error estimates for amplitude equations
- Understand effect of noise transported by the nonlinearity
- Here only two examples: Swift-Hohenberg, Burgers

Interesting result:

Additive noise may lead to stabilization

(or a shift of bifurcation)

in the transient behaviour of dominant modes

Stabilization due to Noise

Well known phenomenon due to multiplicative Noise. For example:

By Itô noise, due to Itô-Stratonovic correction, or Stratonovic noise due to averaging over stable and unstable directions

- **For SDE:** [Arnold, Crauel, Wihstutz '83], [Pardoux, Wihstutz '88 '92].....
- **For SPDE:** [Kwiecinska '99], [Caraballo, Mao et.al. '01], [Cerrai '05], [Caraballo, Kloeden, Schmallfuß '06]....

By Rotation: [Baxendale et.al.'93], [Crauel et.al.'07].....

Only very few examples due to additive noise:

Blow up through a small tube: e.g. [Scheutzow et.al.'93]...

Numerical Examples

Burgers-type equation

Example: Burgers-type equation

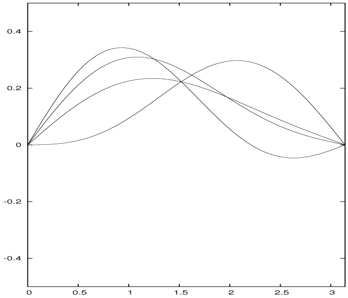
$$\partial_t u = (\partial_x^2 + 1)u + \frac{1}{100}u + u\partial_x u + \frac{\sigma}{10}\xi \qquad \text{(B)}$$

- $u(t,x) \in \mathbb{R}$, $t > 0$, $x \in [0,\pi]$
- Dirichlet boundary conditions $(u(t,0) = u(t,\pi) = 0)$
- $\xi(t,x) = \partial_t \beta(t) \sin(2x)$ – highly degenerate noise

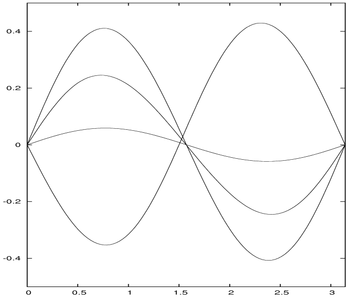
Observation:

- 0 is is stabilized (sin destabilized) by large noise (see [Roberts '03])
- BUT: Large noise acting on $\sin(2x)$!

Snapshots of solutions at different times



$\sigma = 2$



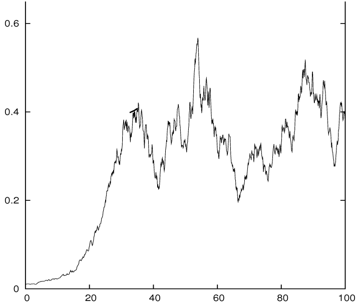
$\sigma = 10$

First Fouriermode

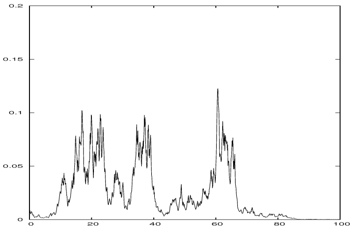
stabilization of first mode due to larger noise

typical realization of $T \mapsto u_1(T)$

$$u(t,x) = u_1(\epsilon^2 t)\sin(x) + u_2(\epsilon^2 t)\sin(2x) + \dots$$



$\sigma = 2$



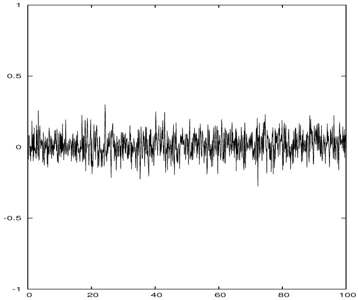
$\sigma = 10$

Second Fouriermode

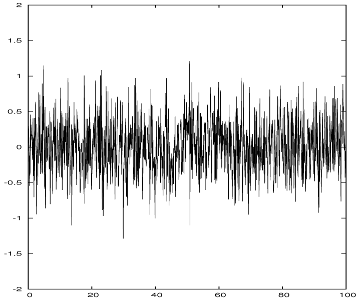
There is only noise on the second mode.

typical realization of $T \mapsto u_2(T)$

$$u(t,x) = u_1(\epsilon^2 t)\sin(x) + u_2(\epsilon^2 t)\sin(2x) + \dots$$



$\sigma = 2$



$\sigma = 10$

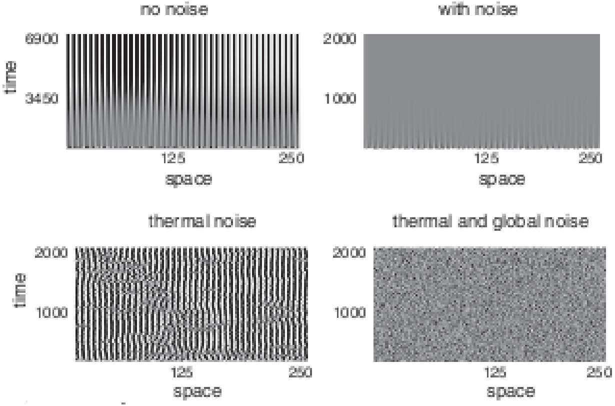
2^{nd} Example

Swift-Hohenberg

Swift-Hohenberg

[Hutt, Schimansky-Geier et.al. 07, 08]

Space-independent (gobal) noise destroys modulated pattern in 1D-Swift-Hohenberg-Eq.



Swift-Hohenberg Equation

Example: Swift-Hohenberg Equation

$$\partial_t u = -(1+\partial_x^2)^2 u + \tfrac{1}{20} u - u^3 + \tfrac{\sigma}{10}\partial_t \beta(t) \qquad \text{(SH)}$$

- $u(t,x) \in \mathbb{R}$, $t > 0$, $x \in [0, 2\pi]$
- periodic boundary conditions

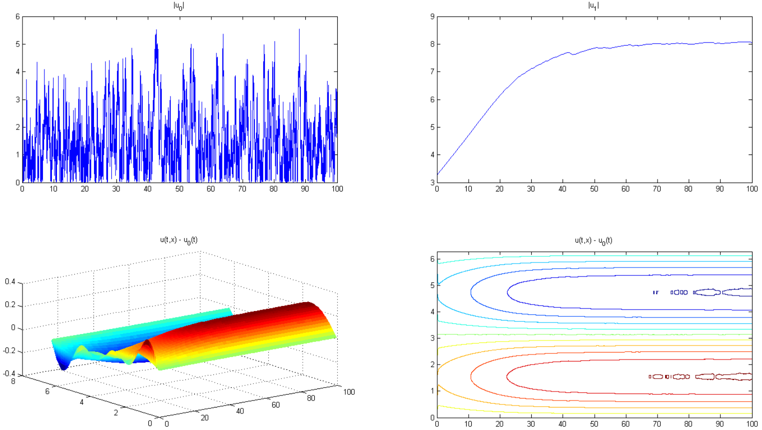
Observation:

- 0 is is stabilized by large noise
- Noise only in time, constant in space

Swift-Hohenberg

(Nolde/Wöhrl)

$\sigma = 0.5$

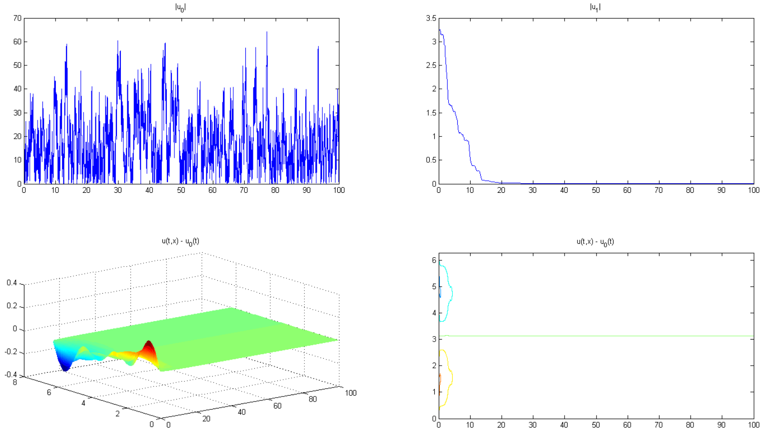


Semiimplicit spectral Galerkin-method using fft in Matlab

Swift-Hohenberg

(Nolde/Wöhrl)

$\sigma = 5$



Semiimplicit spectral Galerkin-method using fft in Matlab

Results for Swift-Hohenberg Equation

The SPDE

$$\partial_t u = -Lu + \nu\epsilon^2 u - u^3 + \sigma\epsilon\partial_t\beta \qquad \text{(SH)}$$

- $L = (1 + \partial_x^2)^2$
- periodic boundary conditions on $[0, 2\pi]$
- $\text{span}\{\sin, \cos\}$ – dominant pattern
- β is a real-valued Brownian motion
- $\sigma\epsilon \ll 1$ – noise strength
- $|\nu|\epsilon^2 \ll 1$ – distance from bifurcation

The Ansatz

$$\partial_t u = -Lu + \nu\epsilon^2 u - u^3 + \sigma\epsilon\partial_t\beta \quad \text{(SH)}$$

Ansatz:

OU-process $Z(t) = \int_0^t e^{-(t-\tau)} d\beta(\tau)$

$$u(t,x) = \epsilon Z(t) + \epsilon a_1(\epsilon^2 t)\sin(x) + \epsilon a_2(\epsilon^2 t)\cos(x) + \mathcal{O}(\epsilon^2)$$

Result: Amplitude Equation [DB, Mohammed, '10]

$$\partial_T a = (\nu - \tfrac{3}{2}\sigma^2)a - \tfrac{3}{4}a|a|^2 \quad \text{(A)}$$

Interesting Facts

$$\partial_T a = (\nu - \tfrac{3}{2}\sigma^2)a - \tfrac{3}{4}a|a|^2 \qquad \text{(A)}$$

- Amplitude equation is deterministic
- Noise leads to a stabilizing deterministic correction

New contribution to the amplitude equation:

$$3a \cdot \sigma^2(\epsilon\partial_T\tilde{\beta})^2 \ ,$$

where $\tilde{\beta}(T) = \epsilon\beta(T\epsilon^{-2})$
is a Brownian motion on the slow time-scale $T = \epsilon^2 t$

Noise^2?

What is noise^2?

Instead of $\epsilon\partial_T\tilde{\beta}$ consider a fast Ornstein-Uhlenbeck-process

$$Z(T\epsilon^{-2}) = Z_\epsilon(T) = \epsilon^{-1}\int_0^T e^{-(T-s)\epsilon^{-2}} d\tilde{\beta}(s) \approx \epsilon\partial_T\tilde{\beta}(T) \ ,$$

where $\tilde{\beta}(T) = \epsilon\sigma\beta(\epsilon^{-2}T)$ is a rescaled Brownian motion

[DB, Hairer, Pavliotis, '07] [DB, Mohammed '08] Averaging with error bounds

$dX = \mathcal{O}(\epsilon^{-r})dt + \mathcal{O}(\epsilon^{-r})d\tilde{\beta}$ and $X(0) = \mathcal{O}(\epsilon^{-r})$, $r > 0$, then

$$\int_0^T X(s)Z_\epsilon(s)^2 ds = \tfrac{1}{2}\int_0^T X(s)ds + \mathcal{O}(\epsilon^{1-2r})$$

Averaging

other averaging results

For odd powers:

$$\int_0^T X(s)Z_\epsilon(s)ds = \mathcal{O}(\epsilon^{1-r})$$

$$\int_0^T X(s)Z_\epsilon(s)^3 ds = \mathcal{O}(\epsilon^{1-3r})$$

and so on ...

Note: $X = \mathcal{O}(f_\epsilon)$
if $\forall\ p > 1,\ T > 0$ there is a $C > 0$ such that

$$\mathbb{E} \sup_{[0,T]} |X|^p \leq C f_\epsilon^p$$

The Theorem

$$\partial_t u = -Lu + \nu\epsilon^2 u - u^3 + \sigma\epsilon\partial_t\beta \qquad \text{(SH)}$$

$$\partial_T a = (\nu - \tfrac{3}{2}\sigma^2)a - \tfrac{3}{4}a|a|^2 \qquad \text{(A)}$$

Theorem – Approximation [DB, Mohammed '10]

u is solution of (SH) in H^1 – a is solution of (A)
$u(0) = \epsilon\, a(0) \cdot (\sin, \cos) + \epsilon^2 \psi_0$ with $\psi_0 \perp \sin, \cos$.

Then for κ, T_0, $p > 0$ there is $C > 0$ such that

$$\mathbb{P}\Big(\sup_{t\in[0,T_0/\epsilon^2]} \|u(t) - \epsilon v(\epsilon^2 t)\|_{H^1} > \epsilon^{2-\kappa} \Big) < C\epsilon^p + \mathbb{P}(\|u(0)\|_{H^1} > \epsilon^{1-\kappa/2})$$

with $v(T) = Z_\epsilon(T) + e^{T\epsilon^{-2}L}\psi_0 + a(T)\cdot(\sin, \cos)$.

Remark: Theorem holds in a much more general setting.

Numerical justification

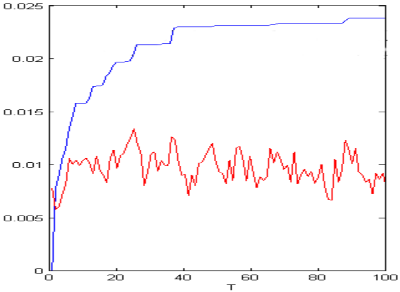
(Wöhrl)

Numerical approximation of

$$\color{red}{r(T)} = \|\epsilon^{-1}u(T\epsilon^{-2}) - Z_\epsilon(T) - a(T)\cdot(\sin,\cos)\|_\infty$$

$$\text{and} \quad \color{blue}{\sup_{s\in[0,T]} r(s)}$$

$\nu = 1$, $\sigma = \sqrt{3/5}$, $\epsilon = 1/100$,
mean over a few (< 100) realizations



Outlook / Other Results

- Higher order corrections

  [DB, Mohammed, '10]

  Problem: Relies on Martingal approximation/representation results, $\dim(\mathcal{N}) = 1$

- Modulated pattern, unbounded domain (cf. Hutt et.al.)

  j.w.w. Wael Mohammed, Konrad Klepel

- different kind of noise (e.g. Levy-noise)

  j.w.w. Erika Hausenblas

Burgers-type equation

Two cases:

1. full noise
2. degenerate noise

An Equation of Burgers type

Equation of Burgers type

$$\partial_t u = (\partial_x^2 + 1)u + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \epsilon^2 \partial_t W \qquad \text{(B)}$$

- $u(t,x) \in \mathbb{R},\ t > 0,\ x \in [0,\pi]$
- Dirichlet boundary conditions $\qquad (u(t,0) = u(t,\pi) = 0))$
- $\epsilon^2 \ll 1$ – noise strength
- $\nu\epsilon^2 u$ – linear (in)stability
- $\partial_t W(t,x)$ – Gaussian white noise

The Linear Operator

The Linear Operator:

$$L = -\partial_x^2 - 1 \qquad \text{Dirichlet b.c. on } [0,\pi]$$

Dominant mode/pattern: $\mathcal{N} = \mathrm{span}\{\sin\}$ – kernel of L

Noise / Wiener Process

$$W(t,x) = \sum_{k=1}^{\infty} \sigma_k \beta_k(t) \sin(kx)$$

$\sigma_k \in \mathbb{R}, \quad |\sigma_k| \leq C, \quad \{\beta_k\}_{k\in\mathbb{N}}$ i.i.d. Brownian motions

Remark: For space-time white noise $\sigma_k = 1 \ \forall k$.

The Amplitude Equation

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \epsilon^2 \partial_t W \qquad \text{(B)}$$

Ansatz:

$$u(t,x) = \epsilon a(\epsilon^2 t)\sin(x) + \mathcal{O}(\epsilon^2)$$

Result: Amplitude Equation [B. '05] [B., Mohammed '08]

$$\partial_T a = \nu a - \tfrac{1}{12}a^3 + \partial_T \beta, \qquad \text{(A)}$$

where $\beta(T) = \epsilon\sigma_1\beta_1(\epsilon^{-2}T)$ rescaled noise in $\mathcal{N}$.

The Theorem

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \epsilon^2 \partial_t W \qquad \text{(B)}$$

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 + \partial_T \beta \qquad \text{(A)}$$

Theorem – Approximation [B. '05] [B., Mohammed '08]

u is solution of (B) – a is solution of (A)
$u(0) = \epsilon a(0) \sin + \epsilon^2 \psi_0$ with $\psi_0 \perp \sin$ and $a(0), \psi_0 = \mathcal{O}(1)$.

Then for κ, T_0, $p > 0$, $\alpha < \frac{1}{2}$ there is $C > 0$ such that

$$\mathbb{P}\Big(\sup_{t \in [0, T_0 \epsilon^{-2}]} \| u(t) - \epsilon a(t\epsilon^2) \sin \|_{H^\alpha} > \epsilon^{2-\kappa} \Big) < C\epsilon^p .$$

Remark: Theorem holds in a much more general setting.

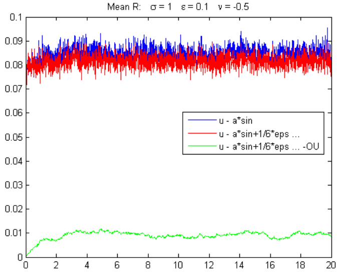
Numerical verification

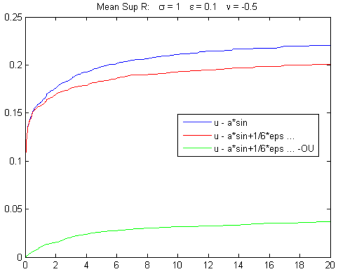
(Nolde)

$\epsilon = 0.1$, $\sigma = 1$, $\nu = -0.5$

$$R(T) = \|\epsilon^{-1} u(T\epsilon^{-2}) - a(T)\sin\|_\infty$$

Approximation of $\mathbb{E}R(T)$ (left) and $\mathbb{E}\sup_{[0,T]} R$ (right)





Next order corrections $\frac{1}{6}a^2(T)$ and a fast Ornstein Uhlenbeck process Z_ϵ (noise) taken into account.

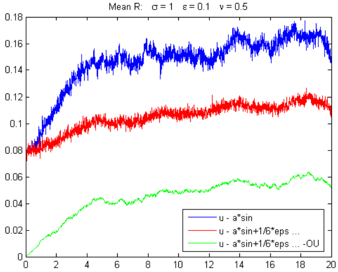
Numerical verification

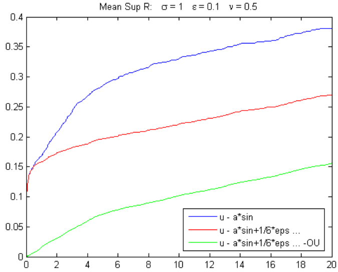
(Nolde)

$\epsilon = 0.1$, $\sigma = 1$, $\nu = 0.5$

$$R(T) = \|\epsilon^{-1}u(T\epsilon^{-2}) - a(T)\sin\|_\infty$$

Approximation of $\mathbb{E}R(T)$ (left) and $\mathbb{E}\sup_{[0,T]} R$ (right)





Impact of the Noise

Recall:

Dominant modes driven only by noise acting directly on $\mathcal{N}$.

No impact of $\beta_2, \ \beta_3, \ \ldots$

Recall:

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 + \partial_T \beta \ , \qquad \text{(A)}$$

where $\beta(T) = \epsilon \sigma_1 \beta_1(\epsilon^{-2} T)$ rescaled noise in $\mathcal{N}$.

Does degenerate noise effects the dominant mode?

How does this lead to the Stabilization?

Stabilisation due to Additive Noise – Setting

One extreme Example:

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \sigma\epsilon\phi$$

- Dirichlet boundary conditions on $[0, \pi]$
- $\mathcal{N} = \text{span}\{\sin\}$ – One dominating mode
- $\phi(t, x) = \partial_t \beta_2(t) \sin(2x)$ – Noise only on 2nd mode

Larger noise, otherwise first result still applies

Stabilisation due to Additive Noise – Result

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \sigma\epsilon\phi \qquad \text{(B2)}$$

Amplitude Equation [DB, Hairer, Pavliotis, 07]

$$da = (\nu - \tfrac{\sigma^2}{88})a\,dT - \tfrac{1}{12}a^3 dT + \tfrac{\sigma}{6} a \circ d\tilde{\beta}_2 \qquad \text{(A2)}$$

in Stratonovic sense, with $\tilde{\beta}_2(T) = \epsilon\beta_2(\epsilon^{-2}T)$.

- For $\nu \in (0, \sigma^2/88)$

  Stabilisation of 0 $\longleftrightarrow$ Destabilisation of sin
- Technical problem:

$$u(t) - \epsilon a(\epsilon^2 t)\sin \approx \tfrac{\epsilon^2\sigma}{3} \underbrace{\partial_T \tilde{\beta}_2(T)}_{\text{white noise}} \sin(2\cdot) + \mathcal{O}(\epsilon^2)$$

Formal Motivation

New contribution to the amplitude equation from the quadratic term

- $a \cdot \partial_T \beta_2$ — ($1^{\text{st}} \times 2^{\text{nd}}$ Mode) – Itô or Stratonovic?
- $a \cdot (\epsilon \partial_T \tilde{\beta}_2)^2$ — involves 3^{rd} mode

Noise^2 defined as before, using averaging and fast OU-process

Stabilisation due to Additive noise – Theorem

Theorem [DB, Hairer, Pavliotis '07]

Let u be a continuous $H_0^1([0,\pi])$-valued solution of (B2) with $u(0) = \epsilon a(0)\sin + \epsilon\psi_0$,

where $\psi_0 \perp \sin$ and $a(0), \psi_0 = \mathcal{O}(1)$.

Let a be a solution of (A2) and define

$$R(t) = e^{-Lt}\psi_0 + \sigma \int_0^t e^{-3(t-s)} d\beta_2(s) \sin(2\,\cdot),$$

then for all $\kappa, p, T_0 > 0$ there is a constant C such that

$$\mathbb{P}\Big(\sup_{t\in[0,T_0\epsilon^{-2}]} \|u(t) - \epsilon a(\epsilon^2 t)\sin - \epsilon R(t)\|_{H^1} > \epsilon^{3/2-\kappa}\Big) \leq C\epsilon^p.$$

More Noise – Near White Noise

What about more noise?

More Noise – Near White Noise

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \sigma\epsilon\partial_t W \qquad \text{(B3)}$$

with $W(t,x) = \sum_{k=2}^{\infty} \beta_k(t) \sin(kx)$ (near white noise)

Amplitude Equation

There is a Brownian motion B and constants $(\nu_0,\ \sigma_a,\ \sigma_b)$ s. t.

$$da = \nu_0 a\, dT - \tfrac{1}{12} a^3 dT + \sqrt{\sigma_a a^2 + \sigma_b}\, dB\ . \qquad \text{(A3)}$$

Multiplicative AND Additive Noise!

Additive noise arises from noise^2 times independent noise.

Relies on martingale approximation result (one-dimensional $\mathcal{N}$)
Error estimate depends on estimate for quadratic variations.

Martingale Approximation

$$M(T) = \int_0^T h(a, Z_\epsilon) d\beta_k \text{ with } Z_\epsilon(t) = \frac{1}{\epsilon} \int_0^t e^{-(t-s)\lambda\epsilon^{-2}} d\beta_l(s).$$

Lemma [DB, Hairer, Pavliotis, '07]

M continuous martingale with quadratic variation f

g arbitrary adapted increasing process with $g(0) = 0$.

Then, with respect to an enlarged filtration, there exists a continuous martingale $\tilde{M}(t)$ with quadratic variation g such that, for every $\gamma < 1/2$ there exists a constant C with

$$\mathbb{E} \sup_{[0,T_0]} |M - \tilde{M}|^p \leq C\, \mathbb{E} \sup_{[0,T_0]} |f - g|^{p/2} + C \big(\mathbb{E} g(T_0)^{2p}\big)^{1/4} \big(\mathbb{E} \sup_{[0,T_0]} |f - g|^p\big)^\gamma .$$

Summary

- SPDEs near a change of stability
- Transient dynamics via amplitude equations
- Stabilisation due to additive noise
- Effect of noise on dominant modes
- Noise transported by nonlinearity between Fourier-modes

Outlook

Further results / Work in Progress

- Attractivity results usually straightforward
- Large Domains – Modulated Pattern
  [DB, Hairer, Pavliotis, '07], [DB, Klepel, Mohammed]
- Higher order corrections
  [Roberts, Wei Wang, '09], [DB, Mohammed '10]
- Levy noise [DB, Hausenblas]
- Local Shape of random invariant manifolds
  [DB, Wei Wang, '09]
- Approximation of invariant measures
  for Swift-Hohenberg [DB, Hairer, '04],
  [DB, Hairer, Pavliotis, '07]
  for Burgers open

More Noise – Theorem

[DB, Hairer, Pavliotis, 07]

For $\alpha \in [0, \frac{1}{2})$ let u be a cont. $H_0^\alpha([0,\pi])$-valued sol. of (B3) with $u(0) = \epsilon a(0)\sin + \epsilon\psi_0$,

where $\psi_0 \perp \sin$ and $a(0), \psi_0 = \mathcal{O}(1)$.

Let a be a solution of (A3) and define

$$R(t) = e^{-tL}\psi_0 + \int_0^t e^{-(t-s)L} dW(s)\,.$$

Then for all $\kappa, p, T_0 > 0$ there is a constant $C > 0$ such that

$$\mathbb{P}\Big(\sup_{t\in[0,T_0\epsilon^{-2}]} \|u(t) - \epsilon a(\epsilon^2 t)\sin - \epsilon R(t)\|_{H^\alpha} > \epsilon^{\frac{5}{4}-\kappa}\Big) \leq C\epsilon^p.$$