Hunting French Ducks in a Noisy Environment

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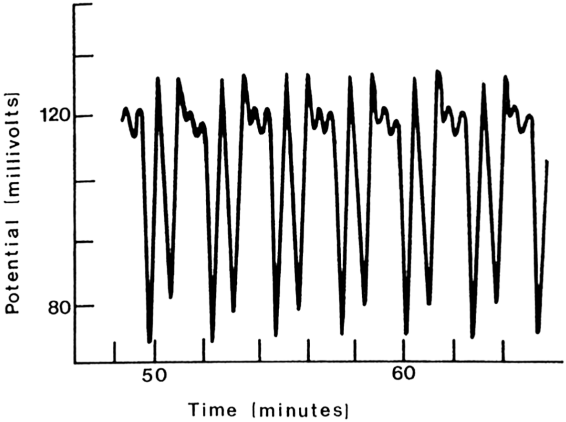
Outline

Mixed-Mode Oscillations (MMOs)

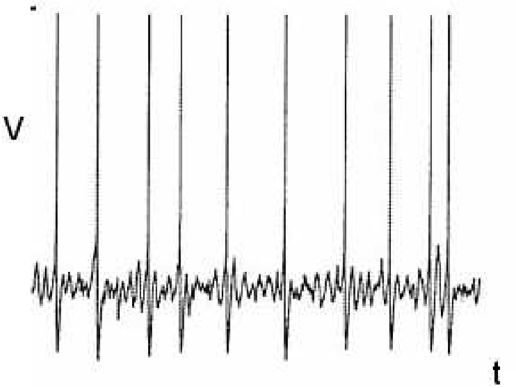
Belousov-Zhabotinsky reaction (Hudson, Hart and Marinko 1979):



Notation: $\ldots L_{j-1}^{s_{j-1}} L_j^{s_j} L_{j+1}^{s_{j+1}} \ldots$ here: $L^s = 2^2$.

Mixed-Mode Oscillations (MMOs)

Layer II Stellate Cells (Dickson et al. 2000):



Q: What is the mechanism for the small-amplitude oscillations?

Fast-Slow Systems

A general **fast-slow system** is a special ODE:

$$\begin{aligned} \frac{dx}{dt} &= x' &= f(x,y) \\ \frac{dy}{dt} &= y' &= \epsilon g(x,y) \end{aligned}$$

where $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ and $0 < \epsilon \ll 1$.

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where $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ and $0 < \epsilon \ll 1$.

On the **slow time scale** $s = \epsilon t$ we get:

$$\begin{aligned} \epsilon\frac{dx}{ds} &= \epsilon\dot{x} &= f(x,y) \\ \frac{dy}{ds} &= \dot{y} &= g(x,y) \end{aligned}$$

The Singular Limits

$$\begin{aligned} x' &= f(x,y) \\ y' &= \epsilon g(x,y) \end{aligned}$$

$\overset{\epsilon=0}{\Longrightarrow}$

fast subsystem

$$\begin{aligned} x' &= f(x,y) \\ y' &= 0 \end{aligned}$$

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Idea: Combine the two systems to analyze the case $0 < \epsilon \ll 1$.

Define the **critical manifold**

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C_0 is **normally hyperbolic** at $P \in C_0$ if

$$(D_x f)(P) \qquad \text{has no eigenvalues } \lambda_j \text{ with zero real parts.}$$

- C_0 is **attracting** if $\lambda_j < 0$ for all j.
- C_0 is **repelling** if there exists $\lambda_j > 0$.

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Theorem (Fenichel's Theorem, 1979)

A normally hyperbolic critical manifold C_0 perturbs ($0 < \epsilon \ll 1$) to a **slow manifold** *C_ϵ. C_ϵ is an $O(\epsilon)$-distance away from C_0 and the slow subsystem flow approximates the flow on C_ϵ.*

An Example - The Planar Fold

$$\begin{aligned} \epsilon \dot{x} &= y - x^2 \\ \dot{y} &= \mu - x \end{aligned}$$

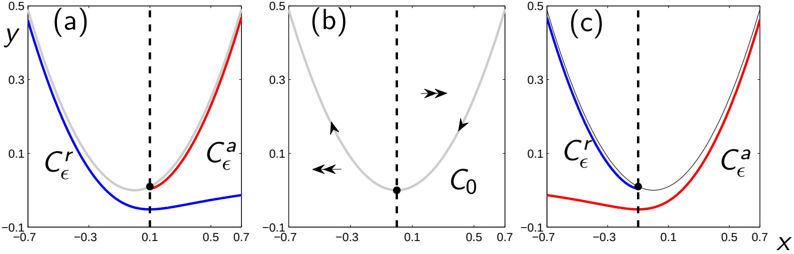

Figure: $\epsilon = 0.05$. (a) $\mu = 0.1$ (b) $\mu = 0$ (c) $\mu = -0.1$.

Folded Singularties in $\mathbb{R}^3$

Consider the following normal form:

$$\begin{aligned} \epsilon \dot{x} &= y - x^2, \\ \dot{y} &= -(\mu + 1)x - z, \\ \dot{z} &= \frac{\mu}{2}, \end{aligned}$$

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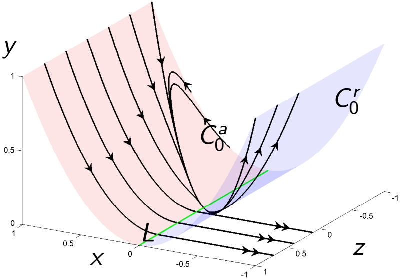
The critical manifold decomposes as:

$$C_0 = \{(x, y, z) \in \mathbb{R}^3 : y = x^2\} = C_0^a \cup L \cup C_0^r$$



Let's calculate the slow flow

$$0 = y - x^2, \qquad \Rightarrow \quad \dot{y} = 2x\dot{x}.$$

Therefore the slow subsystem is

$$\begin{aligned} 2x\dot{x} &= -(\mu + 1)x - z, \\ \dot{z} &= \frac{\mu}{2}. \end{aligned}$$

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Set $s \to 2x\ s$; the **desingularized slow subsystem** is

$$\begin{aligned} \dot{x} &= -(\mu+1)x - z, \\ \dot{z} &= \mu x. \end{aligned}$$

Equilibrium $(x, z) = (0, 0)$ for desingularized slow flow.
Eigenvalues are

$$(\lambda_s, \lambda_w) := (-1, -\mu).$$

The origin $(0, 0)$ is a **folded node** for $\mu \in (0, 1)$.

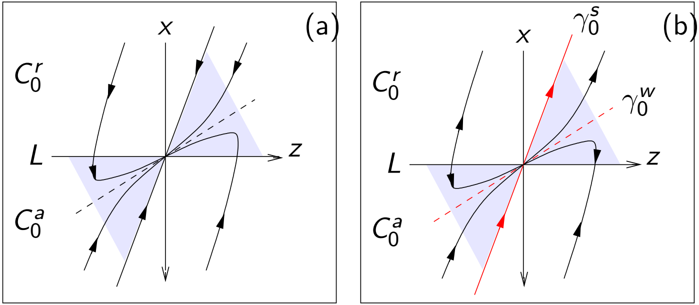

Figure: Strong singular canard γ_0^s; weak singular canard γ_0^w.

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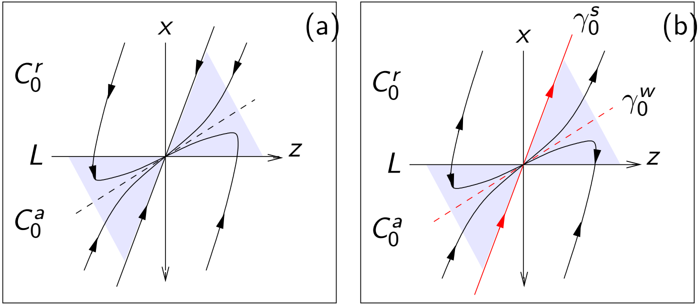

Figure: Strong singular canard γ_0^s; weak singular canard γ_0^w.

Definition: A **maximal canard** is an orbit in $C_\epsilon^a \cap C_\epsilon^r$.

Theorem (Benoît 1990; Szmolyan/Krupa/Wechselberger 2000)

For $\epsilon > 0$ sufficiently small the singular strong canards $\gamma_0^{s,w}$ perturb to maximal canards $\gamma_\epsilon^{s,w}$. Suppose $k \in \mathbb{N}$ and

$$2k + 1 < \mu^{-1} < 2k + 3 \qquad \text{and} \qquad \mu^{-1} \neq 2(k+1).$$

there are k other maximal canards that rotate around γ_ϵ^{w}.

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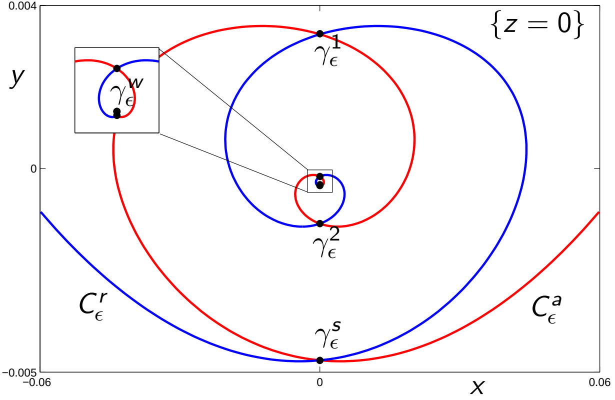

Geometric Desingularization (or Blow-Up)

Recall the normal form

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and apply

$$(x, y, z, s) = \left(\sqrt{\epsilon}\bar{x}, \epsilon\bar{y}, \sqrt{\epsilon}\bar{z}, \sqrt{\epsilon}\bar{s}\right)$$

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This yields (dropping overbars for convenience)

$$\begin{aligned} \dot{x} &= y - x^2, \\ \dot{y} &= -(\mu+1)x - z, \\ \dot{z} &= \frac{\mu}{2}. \end{aligned}$$

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The linearized variational equation around the weak canard γ_0^w is

$$\mu \frac{du}{dz} = \underbrace{\begin{pmatrix} 4z & 2 \\ -2(\mu+1) & 0 \end{pmatrix}}_{=:A(z)} u = A(z)u.$$

Eigenvalues are $2z \pm i\omega(z) \Rightarrow$ contraction $(z<0)$ + rotation!?

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Lemma (Canard Spacing)

On $\{z = 0\}$ the distance of the k-th maximal canard to γ_0^w is

$$\mathcal{O}(e^{-c_0(2k+1)^2\mu})$$

Stochastic Folded Nodes

Consider the normal form

$$\begin{aligned} dx_s &= \frac{1}{\epsilon}(y_s - x_s^2)ds + \frac{\sigma}{\sqrt{\epsilon}} dW_s^{(1)}, \\ dy_s &= [-(\mu+1)x_s - z_s]\, ds + \sigma' dW_s^{(2)}, \\ dz_s &= \frac{\mu}{2} ds. \end{aligned}$$

Main Idea: Control sample paths near deterministic solution.

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Strategy:

1. Geometric desingularization (Blow-Up).
2. Linearization around deterministic solution.
3. Covariance evolution provides tubular neighbourhoods.
4. Stay inside tubes for $-1 < z < \sqrt{\mu}$.
5. Need to control nonlinearity and diffusion.

Blow-up (rescale) the normal form as before

$$(x, y, z, s) = \left(\sqrt{\epsilon}\bar{x}, \epsilon\bar{y}, \sqrt{\epsilon}\bar{z}, \sqrt{\epsilon}\bar{s}\right)$$

then (dropping overbars for convenience)

$$\begin{aligned} dx_s &= (y_s - x_s^2)ds + \tfrac{\sigma}{\epsilon^{3/4}} dW_s^{(1)}, \\ dy_s &= \left[-(\mu + 1)x_s - z_s\right] ds + \tfrac{\sigma'}{\epsilon^{3/4}} dW_s^{(2)}, \\ dz_s &= \tfrac{\mu}{2} ds. \end{aligned}$$

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We also re-scale the noise level parameters and set

$$(\epsilon^{3/4}\sigma, \epsilon^{3/4}\sigma') =: (\bar{\sigma}, \bar{\sigma}')$$

Observe: Can use s or z as "time" variable.

The Stochastic Variational Equation

Focusing on $(x_z, y_z) = (x_z^{\text{det}} + \xi_z, y_z^{\text{det}} + \eta_z)$ we get

$$\begin{aligned} d\xi_z &= \frac{2}{\mu}(\eta_z - \xi_z^2 - 2x_z^{\text{det}}\xi_z)dz + \frac{\sqrt{2}\sigma}{\sqrt{\mu}}dW_z^{(1)}, \\ d\eta_z &= -\frac{2}{\mu}(\mu + 1)\xi_z dz + \frac{\sqrt{2}\sigma'}{\sqrt{\mu}}dW_z^{(2)}. \end{aligned}$$

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Proposition

Linearize the variational equation; set $V(z) := \sigma^{-2}Cov(z)$ then

$$\begin{aligned} \dot{v}_{11} &= -8x^{\rm det}(z)v_{11} + 4v_{12} + 2, \\ \dot{v}_{22} &= -4(\mu+1)v_{12} + 2(\sigma'/\sigma)^2, \\ \dot{v}_{12} &= -2(\mu+1)v_{11} + 2v_{22} - 4x^{\rm det}(z)v_{12}. \end{aligned}$$

Note $v_{12} = Cov(\xi_z, \eta_z) = Cov(\eta_z, \xi_z) = v_{21}$.

Neighbourhoods of Deterministic Solutions

Let $(x(z), y(z)) =: w(z)$ be a deterministic solution. Define a **tube-shaped**-neighbourhood

$$\begin{aligned}\mathcal{B}(r) &= \{(x, y, z) : z_0 \leq z \leq \sqrt{\mu}, \\ &\quad [(x, y) - w(z)] \cdot V(z)^{-1}[(x, y) - w(z)] < r^2\} .\end{aligned}$$

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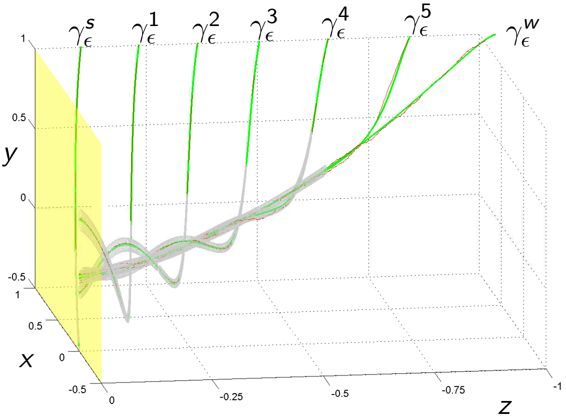
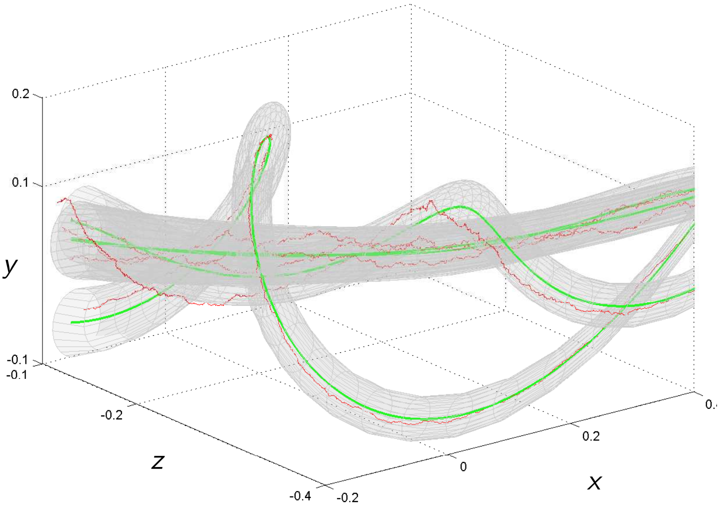
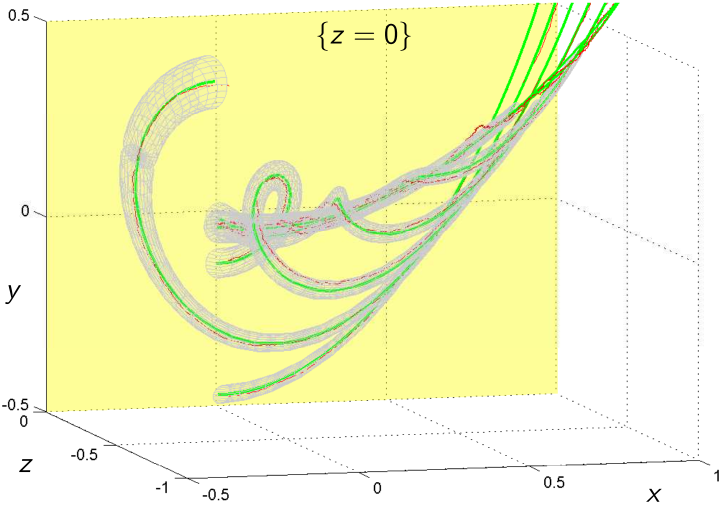


Zoom



Front View



Covariance Estimates

Theorem (Covariance Tubes)

On the section $\{z = 0\} = \{\bar{z} = 0\}$ *we have (as* $\mu \to 0$*):*

$$v_1 = \mathcal{O}(1/\sqrt{\mu}), \quad v_2 = \mathcal{O}(1/\sqrt{\mu}), \quad v_3 = \mathcal{O}(1), \quad (v_1 - v_2) = \mathcal{O}(1)$$

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Sketch of Proof.

1. Change coordinates in the variational equation.
2. Use a symmetry to reduce it to a planar system.
3. A complex eigenvalue pair crosses the imaginary axis at $z = 0$.
4. View the planar system as a fast subsystem with slow time z.
5. Apply the delayed Hopf bifurcation theory.

□

The Nonlinear Variational SDE

It turns out that in suitable coordinates we have to deal with

$$d\zeta_z = \frac{1}{\mu}\big[A(z)\zeta_z + b(\zeta_z, z)\big]\,dz + \frac{\sigma}{\sqrt{\mu}}F(z)dW_z,$$

where $\zeta_z = (\xi_z, \eta_z)$ and $A(z)$ is now given by

$$A(z) = \begin{pmatrix} -2x(z) & \omega_2(z) \\ -\omega_2(z) & -2x(z) \end{pmatrix}.$$

Staying inside $\mathcal{B}(r)$ ...

Theorem (Staying inside Covariance Tubes)

There exists a function $K(z, z_0)$ *such that for* $\kappa = 1 - \mathcal{O}(\cdot)$

$$\mathbb{P}\left\{\tau_{\mathcal{B}(r)} < z\right\} \leq K(z, z_0) \exp\left\{-\kappa \frac{r^2}{2\sigma^2}\right\}$$

holds for all z *such that* $z_0 \leq z \leq \sqrt{\mu}$*.*

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Sketch of Proof.

1. Consider a short time interval $[z_1, z_2]$.
2. Consider the fundemental solution $U(z, u)$ for $\mu\dot{\zeta} = A(z)\zeta$.
3. Set $\Upsilon_u := U(z, u)\zeta_u$ and observe $\Upsilon_u = \Upsilon_u^0 + \Upsilon_u^1$

$$\begin{aligned} \Upsilon_u^0 &= \frac{\sigma}{\sqrt{\mu}} \int_{z_0}^{u} U(z, v) F(v) dW_v, \\ \Upsilon_u^1 &= \frac{1}{\mu} \int_{z_0}^{u} U(z, v) b(\zeta_v, v) dv. \end{aligned}$$

Sketch of Proof (continued).

4. Υ_u^0 is a Gaussian martingale.
5. Doob's submartingale inequality, let $M_u := \|Q(z_1, z_2)\Upsilon_u^0\|$

$$\mathbb{P}\left\{\sup_{z_1 \le u \le z_2} e^{M_u^2} \ge e^{r^2}\right\} \le \frac{1}{e^{r^2}} \mathbb{E}\left[e^{M_{z_2}}\right] \le (\cdots)\mathcal{O}\left(e^{-\frac{r^2}{2\sigma^2}}\right)$$

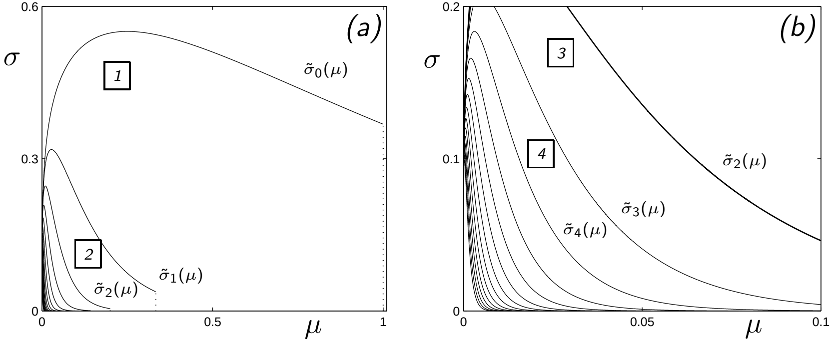
where $Q(z_1, z_2)$ is defined via the covariance matrix V.

6. From last step we bound $\mathbb{P}\left\{\sup_{z_1 \le u \le z_2} M_u \ge r\right\}$.
7. Estimate $\|Q(z_1, z_2)\Upsilon_u^1\|$ directly and show that it is small.
8. We find that escape during a short time is highly unlikely.
9. Piece previous result together for a “nice” partition of $[z_0, z]$.

□

Theorem (Noise, Canards and SAOs)

Depending on noise intensity $\tilde{\sigma}$ and bifurcation parameter μ the "noisy interactions" of canards are:

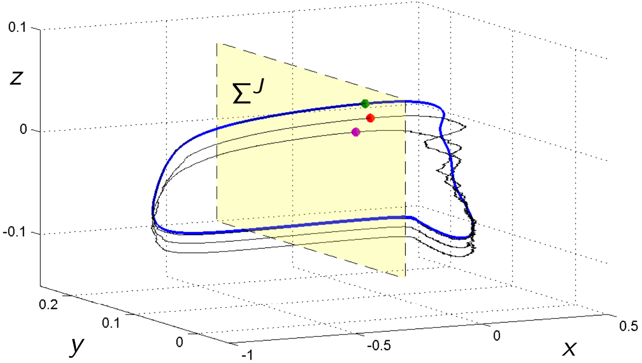


$$\tilde{\sigma}_k(\mu) = \mu^{1/4} e^{-(2k+1)^2 \mu}$$

Further Result - Early Jumps

For $z > \sqrt{\mu}$ beyond the folded node, SDE paths jump early.

$$
\begin{aligned}
dx &= \tfrac{1}{\epsilon}(y - x^2 - x^3)ds + \tfrac{\sigma}{\sqrt{\epsilon}} dW_s^{(1)} \ , \\
dy &= \left[-(\mu+1)x - z\right] ds + \sigma' dW_s^{(2)} \ , \\
dz &= \left[\tfrac{\mu}{2} + ax + bx^2\right] ds \ .
\end{aligned}
$$



Theorem (Escape of Sample Paths)

In blow-up coordinates, consider $z > \sqrt{\mu}$ *and let* $\mathcal{D}$ *be a tube around* γ^w *that grows like* $\mathcal{O}(\sqrt{z})$*. Then the probability that a sample path stays in* $\mathcal{D}$ *becomes small as soon as*

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Sketch of Proof.

1a. **Diffusion-dominated escape** from small set near γ^w.

1b. Subdivide again, need Markov property to re-start.

2a. **Drift-dominated escape** from $\mathcal{D}$.

2b. Change to polar coordinates.

2c. Use averaging to consider radius SDE.

2d. Show that drift dominates diffusion.

□

Back to Mixed-Mode Oscillations...

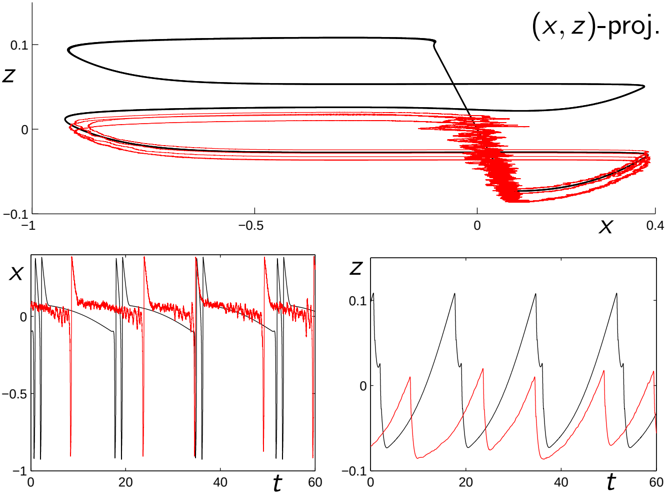

Figure: 3D model system again, different parameters ...

Conclusions

Overview I:

- Fast-slow systems can have intricate singularities.
- The SAOs of MMOs are often caused by these mechanisms.
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Overview I:

- Fast-slow systems can have intricate singularities.
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- Deterministic scenario is often unrealistic (biophysics!).

Overview II:

- Metastable sample paths for SDEs are natural extension.
- Variational equations around solutions play a key role.
- Use Doob's inequality to control sample paths.
- Early jumps after passage through folded node region.
- Intricate dependencies between σ, μ and ϵ.

Main Reference:

(1) N. Berglund, B. Gentz, C. Kuehn, *Hunting French ducks in a noisy environment*, in preparation.

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(3) N. Berglund, B. Gentz, *Noise-Induced Phenomena in Slow-Fast Dynamical Systems*, Springer, 2006.

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Other References:

(2) M. Desroches, B. Krauskopf, J. Guckenheimer, C. Kuehn, H. Osinga & M. Wechselberger, *Mixed Mode Oscillations with Multiple Time Scales*, in revision.

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Thank you for your attention.