Uniqueness and blow–up for dissipative stochastic PDEs

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Summary

The Navier–Stokes equations

The motivations at the basis of this lecture is to understand problems like

$$\dot{u} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + \dot{W},$$
$$\operatorname{div} u = 0.$$

and in general the (**possible?**) non–uniqueness of the statistics for a class of dissipative stochastic PDEs, which includes models with:

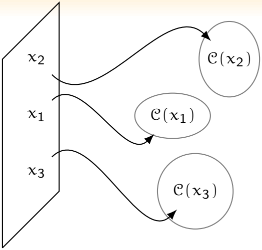
- formal balance of energy (whatever it is!),
- existence of weak solutions,
- smoothness for short times.

For instance (in dimension $d = 1$),

$$\dot{h} + \Delta^2 h + \Delta|\nabla h|^2 = \dot{W}.$$

[flandoli: st. flour lecture notes 2010]

Markovian framework: the strategy



To bypass the non global well–posedness of the problem we consider a special class of solutions, which constitute a Markov process.

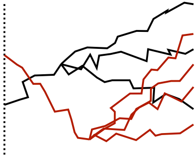
- Consider the set $\mathscr{C}(x)$ of all solutions starting at x, for each x,
- prove a "set" version of the Markov property,
- find a "selection" by variational methods
- continuity and strong mixing of each Markov process.

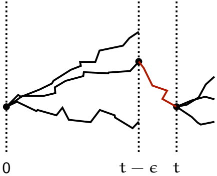
[krylov, stroock–varadhan, flandoli–mr]

Short time coupling with a smooth process

The control of error for continuity follows from the existence of local "strong" solutions.

For smooth i. c. there is a small random time τ_∞ such that up to τ_∞ all solutions coincide.
Essentially, any two solutions have the same distribution on the event $\{\tau_\infty > t\}$.





The real picture is that the "uniqueness of strong solutions" argument is applied at the very last moment only, thanks to the Markov property.

Consequences

On the long time behaviour: [mr]

- every Markov solution is uniquely ergodic: $P(t, x, \cdot) \longrightarrow \mu_\infty$,
- convergence to μ_∞ is exponentially fast,

$$\|P(t, x, \cdot) - \mu_\infty\|_{TV} \leqslant c_1(1 + |x_0|^\gamma)e^{-c_2 t},$$

- all invariant measures are **equivalent**.

On uniqueness: [flandoli–mr, mr]

- If for some initial condition there is a regular solution on a deterministic time interval $\Rightarrow$ well–posedness,
- if for some initial condition uniqueness in law holds on a time interval $\Rightarrow$ **uniqueness in law** holds for all i. c.,
- If all invariant measures coincide $\Rightarrow$ uniqueness in law.

Extensions: [mr–xu]

- A finite number of noise modes can be zero,
- exponential decay of the noise coefficients.

The Kolmogorov equation approach

An alternative approach allows to define a Markov semigroup and an associated martingale solution by solving the Kolmogorov equation of the diffusion

$$\begin{cases} \frac{\partial V}{\partial t} = \frac{1}{2}\operatorname{Tr}(SS^\star DV) + \langle Au + B(u,u), DV\rangle - K\|Au\|^2 V \\ V(0) = \varphi, \end{cases}$$

with an additional potential, and recover the original solution via a Feynman–Kac formula,

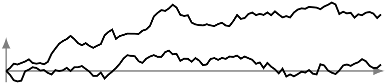
$$U(t,x) = \mathbb{E}\Big[\varphi(u(t))\, e^{-K\int_0^t \|Au(s)\|^2\, ds}\Big]$$

[da prato–debussche,debussche–odasso,odasso]

Non–uniqueness: finite dimensional case

Peano: Additive noise restores uniqueness in most cases,

$$dx_t = \sqrt{|x_t|} + dW_t.$$



This is true under rather general conditions ($b \in L^p$, $\sigma = 1$).

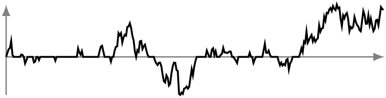
Tanaka: The equation

$$dx_t = \operatorname{sgn}(x_t)\, dW_t$$

has unique solution in law but no path–wise uniqueness.

Girsanov: Non–uniqueness in law happens again if the effect of noise is weakened.

$$dx_t = |x_t|^\alpha\, dW_t, \quad \alpha < \tfrac{1}{2}.$$



[krylov–röckner, engelbert–schmidt]

Some stochastic PDE examples

The finite dimensional theory clearly suggests that one can cook up examples such as

$$\partial_t u = \Delta u + 2\sqrt{|u|}, \qquad \frac{\partial u}{\partial n} = 0.$$

which clearly have the two solutions $u \equiv 0$ and $u \equiv t^2$. Again, the noise restores uniqueness,

$$\dot{u} = \Delta u + 2\sqrt{|u|} + \dot{W}.$$

Similarly, one can argue that

$$\dot{u} = \Delta u + |u|^{\alpha}\,\dot{W}$$

may have different distributions.

What does create non–uniqueness or blow–up?

[gyöngy–krylov, burdzy–mytnik–mueller–perkins]

A simple toy model for blow–up

We consider the formulation in Fourier variables of the surface growth problem,

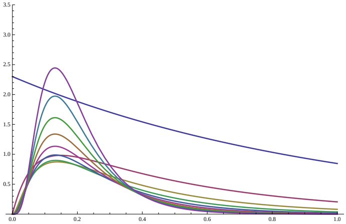
$$\dot{h}_k + k^4 h_k + k^2 \sum_{m=1}^{k-1} m(k-m) h_m h_{k-m} = 0, \qquad k \geqslant 1.$$

on the (invariant) subspace

$$\{h_k = 0 \text{ for } k \leqslant 0 \text{ and } h_k \geqslant 0 \text{ for } k \geqslant 1\}$$

We have

- global solutions for "small" initial data,
- blow–up if the initial data is large in a finite patch.



[mr–blömker]

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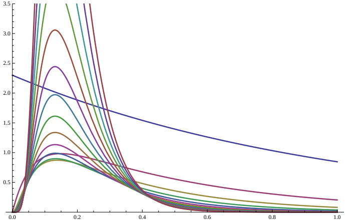

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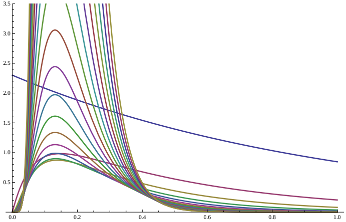


[mr–blömker]

The main characteristics

What does create non–uniqueness or blow–up?

We are essentially interested in the general problem

$$\dot{u} + \nu A u + B(u, u) = \text{forcing},$$

- viscous linear part,
- quadratic nonlinearity
- purely rotational nonlinearity (balance of energy),
- global weak solutions,
- local unique smooth solutions for regular initial conditions,
- existence of an invariant state.

The dyadic model

The system of differential equations

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^\beta x_{n-1}^2 - \lambda_n^\beta x_n x_{n+1}$$

where $x_0 \equiv 0$ and $\lambda_n = 2^n$ has the following characteristics:

- formal balance of energy (whatever it is!),
- existence of weak solutions,
- smoothness for short times.

In fact,

$$\frac{d}{dt}x_n^2 + 2\nu\lambda_n^2 x_n^2 = \lambda_{n-1}x_{n-1}^2 x_n - \lambda_n^\beta x_n^2 x_{n+1}$$

and

$$\frac{d}{dt}\Big(\frac{1}{2}\sum_{n=1}^N x_n^2\Big) + \nu\sum_{n=1}^N \lambda_n^2 x_n^2 = -\lambda_N^\beta x_N^2 x_{N+1}$$

[cheskidov–friedlander–katz–pavlovic]

The dyadic model: known facts

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1}$$

It is also known that:

- positive initial conditions give positive solutions.
- if $\beta \leqslant 2$, there is well–posedness (2DNS–regime)
- if $\beta > 3$ there is blow–up (for large enough positive initial conditions).

By similarity (scaling properties), the three dimensional case corresponds to $\beta \approx \frac{5}{2}$.

[cheskidov]

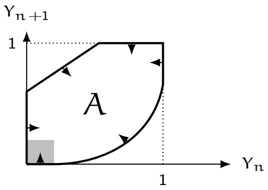
Smoothness and non–uniqueness

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1}$$

The range between 2 and 3 is the difficult one. From the scaling point of view neither the linear nor the nonlinear term are dominant in magnitude.

Theorem

Well–posedness for positive solutions if $\beta \in (2, \frac{5}{2}]$



Moreover there exists a (negative) solution, which is stationary,

$$\lambda_{n-1}^{\beta}\gamma_{n-1}^2 - \lambda_n^{\beta}\gamma_n\gamma_{n+1} = \nu\lambda_n^2\gamma_n, \qquad n \geqslant 1,$$

and non smooth: $\gamma_n \approx \lambda_n^{\beta-2}$.

[barbato–morandin–mr]

Playing with noise

Let $\sigma_n \in \mathbf{R}$,

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1} + \sigma_n \dot{w}_n$$

It is known that

- if $\beta \leqslant 2$ trivial well-posedness,
- if $\beta > 3$ and $\{\sigma_n \neq 0\}$ is finite, then blow–up,
- if $\beta \leqslant 2$ and $\nu \equiv 0$, well posedness with a **special** multiplicative noise.

Assume that $\sigma_n \neq 0$ for all $n \geqslant 1$. We will show that

- path–wise uniqueness for all initial conditions if $\beta \in (2, \frac{5}{2})$,
- blow–up with positive probability starting from each initial condition if $\beta > 3$.

[flandoli–barbato–morandin, mr]

An example of blow–up

Consider

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1} + \sigma_n \dot{w}_n$$

and assume

1. $\beta > 3$,
2. $\sigma_n \neq 0$ for all $n \geqslant 1$,

In analogy with Sobolev spaces, we define

$$V_\alpha = \left\{ x \in \ell^2(\mathbf{R}) : \|x\|_\alpha^2 := \sum_{n=1}^{\infty} (\lambda_n^\alpha x_n)^2 < \infty \right\}$$

Without noise there is blow–up if $\|x(0)\|_\alpha \geqslant M$ for some $\alpha > 0$. With noise the underlying idea is that the deterministic drift dominates and the stocasticity is only a perturbation.

Problem: the set of positive states is **thin**.

An example of blow–up

Theorem

For every x "smooth" and every martingale solution starting at x,

$$\mathbb{P}_x[\tau_\infty < \infty] > 0.$$

Three ideas:

- Solutions with positive initial condition are **almost** positive with positive probability on a time interval.
- Positivity kicks in the deterministic dynamics:

$$\sum_n \lambda_n^{2p} x_n^2 \approx \sum_n \lambda_n^{\beta+2p} x_n^2 x_{n+1} - \nu \sum_n \lambda_n^{2+2p} x_n^2 \approx \|x\|_{p+1}^3 - \nu \|x\|_{p+1}^2$$

- the system is irreducible, hence blowing up initial conditions are reachable.

The trick is to switch between ℓ^2–like and ℓ^∞–like topologies.

[de bouard–debussche, mr]

Almost positivity

Let $z = (z_n)_{n \geqslant 1}$ be the solution to

$$\begin{cases} dz_n + \nu\lambda_n^2 z_n = \sigma_n \, dw_n, \\ z_n(0) = 0 \end{cases}$$

Lemma

If

- $\tau_\infty > T$,
- $\sup_{t \in [0,T]} \lambda_n^{\beta-2} |z_n(t)| \leqslant \frac{\nu}{6}$ *for all* $n \geqslant 1$,

then

$$x_n(t) \geqslant z_n(t) - \frac{1}{2}\nu\lambda_n^{2-\beta}$$

for all $n \geqslant 1$ *and* $t \in [0, T]$.

Everything fine?

There are a few reason that make the counterexample not completely satisfactory:

1. It is a counterexample to **smoothness**, not to **uniqueness**!
2. Does the example cover all the required properties?
   - **[OK]** viscous linear part,
   - **[OK]** quadratic nonlinearity,
   - **[OK]** purely rotational nonlinearity (balance of energy),
   - **[OK]** global weak solutions,
   - **[OK]** local unique smooth solutions for regular initial conditions,
   - [NO!] existence of an invariant state.

The crucial assumption $\beta > 3$ makes the linear part too weak. Smooth solutions live in the "critical" space with decay (at least)

$$\lambda_n^{\beta-2}|x_n| \approx O(1)$$

and $\beta - 2 > 1$.

Path–wise uniqueness

Consider again

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1} + \sigma_n \dot{w}_n$$

and assume this time that

- $\beta \in (2, \frac{5}{2}]$,
- $\sigma_n \neq 0$ for all $n \geqslant 1$,

The idea to prove path–wise uniqueness is

1. prove a **qualitative** regularity criterion,
2. show an **almost** positivity result for positive initial conditions,
3. use a trapping area argument,
4. conclude using non–degeneracy of the noise.

A minimal regularity criterion

Let

- $x \in V_\alpha$, with $\alpha > \beta - 2$,
- $\mathbb{P}_x$ a solution starting at x.

Then under $\mathbb{P}_x$,

$$\{\tau_\infty > T\} = \left\{ \lim_{n\to\infty} \left(\sup_{t\in[0,T]} \lambda_n^{\beta-2} |x_n(t)| \right) = 0 \right\}$$

Almost positivity

Let $x \in V_\alpha$, with $\alpha > \beta - 2$, and $\mathbb{P}_x$ a solution starting at x. Given $\delta_n \downarrow 0$ and $\epsilon_n \downarrow 0$ with

$$\delta_n \leqslant \epsilon_n \leqslant \frac{\nu}{4},$$

there exists an integer $N_0 = N_0(\omega)$ such that

- $N_0 < \infty$, $\mathbb{P}_x$–a. s.,
- $\lambda_n^{\beta-2}|z_n| \leqslant \delta_n$ for $n \geqslant N_0$,
- $\lambda_n^{\beta-2} x_n \geqslant \lambda_n^{\beta-2} z_n - \epsilon_n$ for $n \geqslant N_0$

The trapping area

Scale the solutions

$$U_n(t) = \delta^2 u_n(\delta t)$$

where

$$u_n(t) = \frac{1}{\sqrt{\epsilon_n}}\left(\lambda_n^{\beta-2} x_n - \lambda_n^{\beta-2} z_n + \epsilon_n\right) \geqslant 0$$

for $n \geqslant N_0$.
The quantity δ depends on

1. the initial condition,
2. u_{N_0-1} and u_{N_0}.