Convergence of a self-stabilizing process

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Summary

Plan

Mean-field system

$\left(W_t^i\right)_t \perp\!\!\!\perp \left(W_t^j\right)_t \quad \forall i \neq j$. We consider this dynamical system:

$$dX_t^i = \sqrt{\epsilon}\, dW_t^i - V'(X_t^i)\, dt - \frac{1}{N}\sum_{j=1}^{N} F'(X_t^i - X_t^j)\, dt$$

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Propagation of chaos

$\exists \left(\overline{X_t}\right)_t$ such that $d\overline{X_t} = \sqrt{\epsilon} dB_t - \left(V' + F' * \mathcal{L}\left(\overline{X_t}\right)\right)\left(\overline{X_t}\right) dt$ and C_T which verifies:

$$\sup_{t\in[0;T]} \mathbb{E}\left\{\left|X_t^1 - \overline{X_t}\right|^2\right\} \leq \frac{C_T}{N}.$$

Non-linear equation

$$\begin{cases} X_t = X_0 + \sqrt{\epsilon} B_t - \int_0^t \left(V' + F' * u_s\right)(X_s)\, ds \\ \mathcal{L}(X_s) = du_s(x) \end{cases} \tag{1}$$

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with $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \frac{\alpha}{2} x^2$, $\alpha > 0$.

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The equation (1) can be rewritten in this way:

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What is the exit time of this process?

Well-known results

Existence+uniqueness of the process BRTV, HIP.

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Non-uniqueness of the stationary measures

Herrmann and Tugaut. Non-uniqueness of stationary measures for self-stabilizing processes. *Stochastic Processes and their Applications*, (2010).

Non-linear PDE

Set $(X_t)_{t\in\mathbb{R}_+}$ the strong solution of the SDE. Then:

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The Parabolic equation

$d\mathbb{P}\left[X_t = x\right] = u_t(x)dx$ for all $t > 0$. Moreover:

$$\frac{\partial}{\partial t}u_t = \frac{\partial}{\partial x}\left\{\frac{\epsilon}{2}\frac{\partial}{\partial x}u_t + u_t\Big(V' + F' * u_t\Big)\right\}$$

for all $t > 0$ and $u_0(dx) = \mathbb{P}\left(X_0 \in dx\right)$.

Free energy

$$\Upsilon_\epsilon(\mu) := \int_{\mathbb{R}} \left\{ \frac{\epsilon}{2} \ln(\mu(x)) + V(x) + \frac{1}{2} F * \mu(x) \right\} \mu(x) dx$$

and $\mathcal{D}_\epsilon(\mu)(x) := \frac{\epsilon}{2}\mu'(x) + \left[V'(x) + F' * \mu(x) \right] \mu(x)$.

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The energy is decreasing, BCCP (1998)

Under simple conditions, we have:

$$\frac{d}{dt}\Upsilon_\epsilon\left(u_t^\epsilon\right) = -\int_{\mathbb{R}} \left(\mathcal{D}_\epsilon(u_t^\epsilon)(x)\right)^2 \left(u_t^\epsilon(x)\right)^{-1} dx \leq 0 \,.$$

Stationary measures

Integrated form

The eventual stationary measures can be written in this way :

$$u_\epsilon(x) = Z_\epsilon^{-1}\, e^{-\frac{2}{\epsilon}(V(x)+F*u_\epsilon(x))}.$$

With our two potentials V and F:

$$u_\epsilon^m(x) = \frac{\exp\left[-\frac{2}{\epsilon}\left(\frac{x^4}{4}-\frac{x^2}{2}+\frac{\alpha}{2}(x-m)^2\right)\right]}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(\frac{y^4}{4}-\frac{y^2}{2}+\frac{\alpha}{2}(y-m)^2\right)\right]dy}. \qquad (2)$$

Plan

Implicit solution

Let us introduce the two following functions:

$$\Psi_\epsilon(m) := \int_{\mathbb{R}} (x-m)\, e^{-\frac{2}{\epsilon}\left(\frac{x^4}{4}-\frac{x^2}{2}+\frac{\alpha}{2}(x-m)^2\right)} dx$$

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For each m such that $\Psi_\epsilon(m) = 0$, there exists a unique stationary measure u_ϵ^m whose the first moment is m.

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We can remark: $\frac{d}{dm}Z_\epsilon(m) = \frac{2\alpha}{\epsilon}\Psi_\epsilon(m)$.

Link with the free-energy

A computation provides

$$\Upsilon_\epsilon\left(u_\epsilon^m\right) = -\frac{\epsilon}{2}\log\left[Z_\epsilon(m)\right] - \frac{\alpha}{2}\left(\frac{\Psi_\epsilon(m)}{Z_\epsilon(m)}\right)^2 .$$

Consequently:

Link with the derivative of the free-energy

$$\frac{d}{dm}\Upsilon_\epsilon\left(u_\epsilon^m\right) = -\alpha\frac{\mathrm{Var}\left(u_\epsilon^m\right)}{Z_\epsilon(m)}\Psi_\epsilon(m) .$$

Existence and uniqueness

By taking (2) with $m = 0$, we have immediately:

$$u_\epsilon^0(x) = \frac{\exp\left[-\frac{2}{\epsilon}\left(\frac{1}{4}x^4 + \frac{\alpha-1}{2}x^2\right)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(\frac{1}{4}y^4 + \frac{\alpha-1}{2}y^2\right)\right] dy}\,.$$

Consequently, we have the existence and the uniqueness of the symmetrical stationary measure. We call it u_ϵ^0.

Behavior for small ϵ

An asymptotic computation provides the following weak convergence:

$$\lim_{\epsilon\longrightarrow 0} u_\epsilon^0 = \begin{cases} \frac{1}{2}\delta_{\sqrt{1-\alpha}} + \frac{1}{2}\delta_{-\sqrt{1-\alpha}} & \text{if} \quad \alpha \leq 1 \\ \delta_0 & \text{if} \quad \alpha \geq 1 \end{cases} .$$

Moreover:

$$\lim_{\epsilon\longrightarrow 0} \Upsilon_\epsilon\left(u_\epsilon^0\right) = \Upsilon_0^0 := \begin{cases} \frac{-(1-\alpha)^2}{4} & \text{if} \quad \alpha \leq 1 \\ 0 & \text{if} \quad \alpha \geq 1 \end{cases} .$$

Study of Ψ_ϵ - I

By proceeding a series expansion of $m \mapsto \exp\left[\frac{2\alpha m}{\epsilon}\right]$:

$$e^{\frac{\alpha}{\epsilon}m^2}\Psi_\epsilon(m) = 2\sum_{n=0}^{\infty}\frac{I_\epsilon(2n)}{(2n)!}\left(\frac{2\alpha m}{\epsilon}\right)^{2n+1}\overbrace{\left[\frac{I_\epsilon(2n+2)}{(2n+1)I_\epsilon(2n)} - \frac{\epsilon}{2\alpha}\right]}^{\gamma_\epsilon(n)}$$

with $\quad I_\epsilon(x) := \int_{\mathbb{R}_+} t^x \exp\left[-\frac{2}{\epsilon}\left(\frac{t^4}{4} + \frac{\alpha-1}{2}t^2\right)\right]dt\,.$

Study of Ψ_ϵ - II

- $\forall\epsilon > 0$, an integration by parts provides

$$\gamma_\epsilon(n) = \frac{\epsilon}{2}\left(\frac{I_\epsilon(2n+4)}{I_\epsilon(2n+2)} + (\alpha - 1)\right)^{-1} - \frac{\epsilon}{2\alpha}\,.$$

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- The derivation of the functions $x \mapsto \frac{I_\epsilon(x+2)}{I_\epsilon(x)}$ and $x \mapsto \frac{I'_\epsilon(x)}{I_\epsilon(x)}$ and finally the Cauchy-Schwarz's inequality tell us that the sequence $(\gamma_\epsilon(n))_{n\in\mathbb{N}}$ is decreasing.

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- However, $\Psi_\epsilon(1) < 0$. We deduce the existence of $n_\epsilon \geq 0$ such that $\Psi_\epsilon^{(2k+1)}(0) > 0$ if and only if $k < n_\epsilon$.

Existence of a boundary $\epsilon_c(\alpha)$

$\Longrightarrow \Psi_\epsilon(m) = e^{-\frac{\alpha}{\epsilon}m^2}\left\{\sum_{n=0}^{n_\epsilon-1} C_n m^{2n+1} - \sum_{n=n_\epsilon}^{\infty} C_n m^{2n+1}\right\}$ with $C_n \geq 0$.

By considering $m \mapsto m^{-(2n_\epsilon+1)} e^{\frac{\alpha}{\epsilon}m^2} \Psi_\epsilon(m)$, we deduce Ψ_ϵ vanishes 0 or 1 time on $\mathbb{R}_+$. Moreover, the *sine qua none* condition for having such a solution is $\Psi_\epsilon'(0) > 0$.

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Boundary betweeen Uniqueness and Thirdness

There exists a threshold ϵ_c such that over we have the uniqueness: u_ϵ^0 ; and under we have the thirdness: u_ϵ^0, u_ϵ^+ and u_ϵ^- with $\pm\int_{\mathbb{R}} x u_\epsilon^\pm(x) > 0$. Moreover, ϵ_c is the unique solution of

$$\int_{\mathbb{R}} \left(2\alpha y^2 - 1\right) \exp\left[\left(1-\alpha\right) y^2 - \frac{\epsilon}{2} y^4\right] dy = 0\,.$$

Boundary $\epsilon_c(\alpha)$

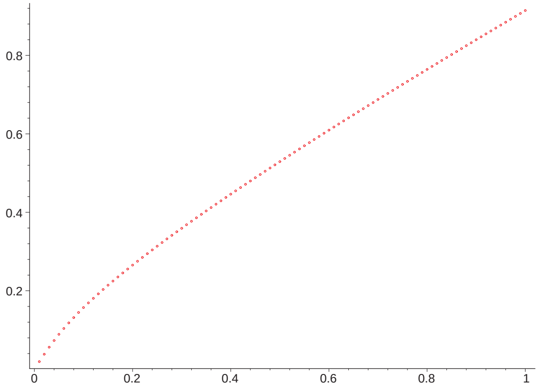

Figure: $\epsilon_c(\alpha)$

Plan

Global convergence over $\epsilon_c(\alpha)$

Global convergence

Let a measure u_0 such that $du_0(x) = u_0(x)dx$ and $\sup\left\{\Upsilon_\epsilon(u_0)\,;\,\int x^{32}u_0(x)dx\right\} < \infty$. Then u_t^ϵ converges weakly towards the unique stationary measure u_ϵ^0.

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Idea of the proof :

- There exists a sequence $(t_k)_k$ s.t. $u_{t_k}^\epsilon$ converges weakly towards a stationary measure that implies towards u_ϵ^0.

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- $\Upsilon_\epsilon(u_t^\epsilon) \longrightarrow \Upsilon_\epsilon(u_\epsilon^0)$ for $t \longrightarrow +\infty$.
- The free energy is decreasing so u_ϵ^0 is its unique minimizer.
- We conclude by using the Prohorov's theorem because the family $\left\{u_t^\epsilon\,;\,t \in \mathbb{R}_+\right\}$ is tight.

Global convergence under the critical value

Global convergence theorem

Let a measure u_0 such that $du_0(x) = u_0(x)dx$ and $\sup\left\{\Upsilon_\epsilon(u_0)\,;\,\int x^{32}u_0(x)dx\right\} < \infty$. Then u_t^ϵ converges weakly towards a stationary measure $u_\infty^\epsilon \in \left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}$.

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- First, we admit that $\int x^n u_0(x)dx < \infty$ for all $n \in \mathbb{N}$.

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Idea of the proof:

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- If $\epsilon < \epsilon_c$, $\Upsilon_\epsilon(u_\epsilon^\pm) < \Upsilon_\epsilon(u)$ for all $u \neq u_\epsilon^\pm$. If u_ϵ^+ (or u_ϵ^-) is an adherence value, it is unique so u_t^ϵ converges weakly towards u_ϵ^+ (or u_ϵ^-).

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- There exists a polynomial function φ such that
  $0 = \int_{\mathbb{R}} \varphi(x) u_\epsilon^0(x) dx < \int_{\mathbb{R}} \varphi(x) v_\infty^\epsilon(x) dx =: 3\rho$.

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- We deduce there exist two sequences $(r_k)_k$ and $(s_k)_k$ which go to ∞ such that for all $r_k \leq t \leq s_k$ and for all $k \in \mathbb{N}$: $\rho = \int_{\mathbb{R}} \varphi u_{r_k}^\epsilon \leq \int_{\mathbb{R}} \varphi u_t^\epsilon \leq \int_{\mathbb{R}} \varphi u_{s_k}^\epsilon = 2\rho$.

Global convergence III

- By using the Cauchy-Schwarz's inequality, we prove $s_k - r_k \longrightarrow \infty$ so we obtain a sequence $(q_k)_{k\in\mathbb{N}}$ such that $u_{q_k}^{\epsilon}$ converges weakly towards a stationary measure $\widetilde{u_\infty^\epsilon}$ which verifies $\int_{\mathbb{R}} \varphi(x)\widetilde{u_\infty^\epsilon}(x)dx \in [\rho; 2\rho]$.

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- $\widetilde{u_\infty^\epsilon} \neq u_\epsilon^\pm$ because $u_\epsilon^\pm$ are not adherence values. $\widetilde{u_\infty^\epsilon} \neq u_\epsilon^0$ because $\int_{\mathbb{R}} \varphi u_\epsilon^0 \notin [\rho; 2\rho]$.

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- This is impossible.

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- This is impossible.
- For all $t > 0$ and all $n \in \mathbb{N}$, $\int_{\mathbb{R}} x^n u_t(x)dx < \infty$.

Weak convergence of the process if $\epsilon < \epsilon_c(\alpha)$

Symmetrical case

Symmetrical local convergence

Let a **symmetrical** measure u_0 such that $du_0(x) = u_0(x)dx$ and $\sup\left\{\Upsilon_\epsilon(u_0)\,;\,\int x^{32}u_0(x)dx\right\} < \infty$. Then u_t^ϵ converges weakly towards the unique **symmetrical** stationary measure u_ϵ^0.

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We apply directly the global convergence theorem.

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We apply directly the global convergence theorem.

Here, the equation is just $dX_t^\epsilon = \sqrt{\epsilon}dB_t - \left(X_t^3 + (\alpha - 1)X_t\right)dt$.
So, there is not self-stabilizing term.

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Let an **asymmetrical** measure u_0 such that $du_0(x) = u_0(x)dx$ and $\sup\left\{\Upsilon_\epsilon(u_0)\,;\,\int x^{32}u_0(x)dx\right\} < \infty$. Moreover, we assume

$$\frac{1}{4}\mathbb{E}\left[X_0^4\right] - \frac{1}{2}\mathbb{E}\left[X_0^2\right] + \frac{\alpha}{2}\mathrm{Var}(X_0) < \Upsilon_0^0 \quad \text{and} \quad \mathbb{E}\left[X_0\right] > 0\,.$$

Then, for ϵ small enough, u_t^ϵ converges weakly towards u_ϵ^+.

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Then, for ϵ small enough, u_t^ϵ converges weakly towards u_ϵ^+.

The key-idea is the existence of $\epsilon_0 > 0$ such that

$$\Upsilon_\epsilon(u_0) < \min_{\mathbb{E}[\mu]=0} \Upsilon_\epsilon(\mu) \quad \text{for all} \quad \epsilon < \epsilon_0\,.$$

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