Rough Burgers-like equations with multiplicative noise

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Case $n = 1$, $g(u) = u$ Burgers equation.

Linear case - Regularity

Linear case: $g = 0$ and $\theta = 1$: Stochastic heat equation

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Difficulty: How to interpret the nonlinear term $g(u)\partial_x u$?

Classical approach

Gradient case: Assume $DG = g$.

u is a weak solution if for φ smooth, periodic

$$\langle u(t), \varphi \rangle - \langle u_0, \varphi \rangle = \int_0^t \Big[\langle u(s), \partial_{xx}\varphi \rangle + \langle g(u)\partial_x u, \varphi \rangle \Big] \mathrm{d}s + \int_0^t \langle \varphi\, \theta(u(s)), \mathrm{d}W_s \rangle.$$

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$\longrightarrow$ Existence and Uniqueness OK (e.g. Gyöngy ’98).

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$\longrightarrow$ If $n = 1$ primitive G always exists, if $n \geq 2$ not.

Unstable Approximations

Observation:(Hairer, Maas '10; Hairer, Voss '10) n=1

$$\mathrm{d}u_\varepsilon = \left[\partial_{xx} u_\varepsilon + g(u_\varepsilon) D_\varepsilon u_\varepsilon\right]\mathrm{d}t + \mathrm{d}W$$

$D_\varepsilon =$ approximation of derivative, e.g.

$$D_\varepsilon u(x) = \frac{1}{\varepsilon}\big(u(x+\varepsilon) - u(x)\big).$$

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u_ε converges to solution of wrong equation:

$$\mathrm{d}\tilde{u} = \big[\partial_{xx}\tilde{u} + g(\tilde{u})\,\partial_x \tilde{u} + c\, g'(\tilde{u})\big]\mathrm{d}t + \mathrm{d}W$$

Constant c depends on the approximation.

Second look at nonlinearity

$$\langle g(u)\partial_x u, \varphi\rangle = \int_0^1 g(u(x))\,\varphi(x)\,\mathrm{d}_x u(x)$$

$u(x)$ same regularity as Brownian motion! Looks like a stochastic integral!

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$\longrightarrow$ Extra term in the unstable approximation looks like Itô -Stratonovich correction: $g'(u)\,\mathrm{d}[u]$ with "quadratic variation" $[u]$.

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Observations:

→ Extra term in the unstable approximation looks like Itô -Stratonovich correction: $g'(u)\,\mathrm{d}[u]$ with "quadratic variation" $[u]$.

→ In gradient case Itô integral:

$$\int_0^1 DG(B_t)\,\mathrm{d}B_t \;=\; G(B_1) - G(B_0) - \int_0^1 \Delta G(B_t)\,\mathrm{d}t$$

can be defined pathwise.

Rough integrals

Use a stochastic integration theory to make sense of

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Problem: Itô theory: Convergence of $\sum_i g(u_i)\varphi_i(u_{i+1} - u_i)$ requires that u_i is adapted to a filtration.

In space integral no natural time direction.

Solution: "Pathwise" approach using Lyons' rough path theory.

Outline

- Brief account of rough path theory à la Gubinelli
- Concept of solution, main result
- Construction of solutions

Young integration

Aim: Define $\int_0^1 Y_s \,\mathrm{d}X_s$ for $X, Y \in C^\alpha$, $\alpha > \frac{1}{2}$:

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\begin{aligned}
I_0 &= Y_0(X_1 - X_0) \\
I_1 &= Y_0(X_{\frac{1}{2}} - X_0) + Y_{\frac{1}{2}}(X_1 - X_{\frac{1}{2}}) \\
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I_n &= \sum_{i=0}^{2^n-1} Y_{t_i^n}(X_{t_{i+1}^n} - X_{t_i^n}) \qquad\qquad (t_i^n = i \cdot 2^{-n})
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Convergence

$$
\begin{aligned}
|I_n - I_{n-1}| &= \Big| \sum_{\substack{i=0 \\ i \text{ even}}}^{2^n-1} (Y_{t_{i+1}^n} - Y_{t_i^n})(X_{t_{i+2}^n} - X_{t_{i+1}^n}) \Big| \\
&\le \frac{1}{2} 2^n \, |X|_\alpha 2^{-n\alpha} \, |Y|_\alpha \, 2^{-n\alpha}
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Theorem (Young '36)

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\Big| \int_0^1 (Y_t - Y_0) \, \mathrm{d}X_t \Big| \le \frac{1}{2^{2\alpha} - 2} \, |X|_\alpha \, |Y|_\alpha .
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Rough Paths

Idea: If $\alpha < \frac{1}{2}$ higher order approximation is necessary!

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Think of iterated integral:

$$\mathbf{X}_{s,t} = \int_s^t (X_u - X_s)\, \mathrm{d}X_u$$

Controlled rough paths

$(X, \mathbf{X}) \in \mathcal{D}^\alpha = \{$rough paths$\}$ fixed. $Y \in C^\alpha$ is controlled by X if

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Y' is uniquely determined if X is sufficiently rough.

Example: $Y_t = g(X_t)$ for $g \in C^2_b$.

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$$\leq \frac{1}{2} 2^n \left(|Y'|_\alpha \, 2^{-n\alpha} |\mathbf{X}|_{2\alpha} 2^{-2n\alpha} + |R^Y|_{2\alpha} 2^{-2n\alpha} |X|_\alpha 2^{-n\alpha} \right)$$

Rough integrals 2

Theorem (Gubinelli '05)

$(X, \mathbf{X}) \in \mathcal{D}^\alpha$, $Y \in \mathcal{C}_X^\alpha$ *for* $\frac{1}{3} < \alpha < \frac{1}{2}$. *Then*

$$\Big| \int_0^1 (Y_t - Y_0)\, \mathrm{d}X_t - Y_0' \mathbf{X}_{0,1} \Big| \leq \frac{1}{2^{3\alpha} - 2} \Big(|Y|_\alpha |\mathbf{X}|_{2\alpha} + |R^Y|_{2\alpha} |X|_\alpha \Big).$$

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Also possible to construct $\int Y\,\mathrm{d}Z$ for $Y, Z \in \mathcal{C}_X^\alpha$ with the same strategy.

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Agenda: Construction of $\int_0^1 g(u(x))\mathrm{d}u(x)$ separated into two parts:

- $\rightarrow$ Construct for every t reference rough path $(X(t), \mathbf{X}(t))$ and show that $u(t)$ is controlled by $X(t)$.

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→ Construct for every t reference rough path $(X(t), \mathbf{X}(t))$ and show that $u(t)$ is controlled by $X(t)$.

→ Apply Gubinelli's result to $Y = g(u)$ and $Z = u$.

Reference rough path

Solution of linear stochastic heat equation

$$X(t,x) = \int_0^t S(t-s)\,\mathrm{d}W_s(x).$$

$S(t) =$ heat semigroup on $[0,1]$.

Existence results for Gaussian rough paths can be applied.

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Theorem (Friz, Victoir '05, Hairer '10)

For fixed t there is a canonical definition for

$$\mathbf{X}(t,x,y) \;=\; \int_x^y \big(X(t,z) - X(t,x)\big)\mathrm{d}_z X(t,z).$$

Furthermore $t \mapsto \mathbf{X}(t,\cdot)$ is a.s. continuous w.r.t $|\cdot|_{2\alpha}$.

Weak Solutions

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A weak solution to (1) is an adapted process taking values in $C([0,T],C)\cap L^1([0,T],\mathcal{C}_X^\alpha)$ such that for every smooth, periodic test function φ

$$\langle \varphi u(t)\rangle = \langle \varphi, u_0\rangle + \int_0^t \langle \varphi\,\theta(u(s)), \mathrm{d}W(s)\rangle + \int_0^t \langle \partial_{xx}\varphi, u(s)\rangle \mathrm{d}s$$
$$+ \int_0^t \Big(\int_0^1 \varphi(x) g(u(s,x))\, \mathrm{d}_x u(s,x)\Big)\,\mathrm{d}s.$$

Non-linear term rough integral!

Mild solutions

A mild solution to (1) is an adapted process u taking values in $C([0,T],C) \cap L^1([0,T],\mathcal{C}^\alpha_X)$ such that

$$
\begin{aligned}
u(x,t) &= S(t)u_0(x) + \int_0^t S(t-s)\,\theta(u(s))\,\mathrm{d}W(s)(x) \\
&\quad + \int_0^t \Big(\int_0^1 \hat{p}_{t-s}(x-y)\, g\big(u(s,x)\big)\, \mathrm{d}_y u(s,y) \Big)\, \mathrm{d}s.
\end{aligned}
$$

$\hat{p}_t =$ heat kernel on $[0,1]$.

Every weak solution is a mild solution an vice versa.

Main result

Existence/Uniqueness:

$\rightarrow$ Initial data $u_0 \in C^\beta$ for $\frac{1}{3} < \alpha < \beta < \frac{1}{2}$.

$\rightarrow$ $g \in C^3, \theta \in C^2$ bounded derivatives.

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Theorem

For every time interval $[0, T]$ there exists a unique weak/mild solution to (1).

If u_ε is a solution to (1) with smoothened noise, then u_ε converges to u in probability.

Extends the construction of Hairer '10 to the multiplicative noise case.

Dependence on $\mathbf{X}$

Different choices for $\mathbf{X}$ e.g. Itô -Stratonovich correction

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$$+ \varphi(x_i)\, g'(u(x_i)) \theta(u(x_i))^2 \Big(\mathbf{X}(x_i, x_{i+1}) + c(x_{i+1} - x_i)\Big)$$

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Extra term $c\, g'(u)\, \theta(u)^2$ appears.

We have the right solution

The rough path $(X, \mathbf{X})$ is geometric i.e.

$$\mathrm{Sym}\,(\mathbf{X}(x,y)) = \frac{1}{2}\Big(\mathbf{X}(x,y) + \mathbf{X}(x,y)^T\Big) = \frac{1}{2}\delta X(x,y) \otimes \delta X(x,y).$$

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In the non-gradient case even different geometric rough paths give rise to different solutions, e.g. for $n = 2$:

$$\widetilde{\mathbf{X}}(t,x,y) = \mathbf{X}(t,x,y) + (y - x)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and}$$

$$g(x,y) = \begin{pmatrix} -\sin(y) & \cos(y) \\ \cos(x) & \sin(x) \end{pmatrix}.$$

Stochastic convolution 1

θ adapted $L^2[0,1]$-valued process. Stochastic convolution:

$$\Psi^\theta(t) = \int_0^t S(t-s)\,\theta(s)\,\mathrm{d}W(s).$$

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Ψ^θ is controlled by X with derivative process θ

$$\Psi^\theta(t,y) - \Psi^\theta(t,x) = \theta(t,x)\Big(X(t,y) - X(t,x)\Big) + R^\theta(t,x,y).$$

Stochastic convolution 2

Additional regularity for θ

$$\|\theta\|_{p,\alpha} = \mathbb{E}\Big[\sup_{x\neq y, s\neq t} \frac{|\theta(t,x)-\theta(s,y)|^p}{\left(|t-s|^{\alpha/2}+|x-y|^{\alpha}\right)^p} + \sup_{x,t}|\theta(t,x)|^p\Big]^{1/p}.$$

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By definition of R^θ:

$$R^\theta(t,x,y) = \int_0^t\int_0^1 \Big(\hat{p}_{t-s}(z-y) - \hat{p}_{t-s}(z-x)\Big)$$
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Then one has for p large and ϑ small:

$$\mathbb{E}\Big[\|R^\theta\|^p_{C^\vartheta\left([0,\tau_K^{\|X\|_\alpha}];\Omega C^{2\alpha}\right)}\Big] \leq C(1+K^p)\|\theta\|^p_{p,\alpha}.$$

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Observation: v is more regular than u.

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Fixed point v^Ψ depends continuously on Ψ.

Outer fixed point

For T small enough mapping

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is a contraction w.r.t.

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Construction continuous in reference rough path $(X, \mathbf{X})$ +
Stability of Friz-Victoir construction $\Rightarrow$ Stability!

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Conclusion

- → Spatial rough integrals give a way to define solutions to equations that are not well-posed classically .
- → Our construction gives an interpretation for extra terms that appear even in well-posed equations.
- → It should also give a different method to prove such approximation results.
- → Rough path machinery very convenient, as it allows to separate the construction into a stochastic part and a deterministic part.