Pathwise regularity of solutions to some SPDEs

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Part I:
The deterministic case

Two examples of partial differential equations

Linear transport equation:

$$\partial_t u = \sum_{i=1}^{n} b_i \mathrm{D}_i u + b_0 u \qquad \text{in } D \times (0,T)$$

This is a first order PDE and models the evolution of the particle density u, driven by the vector field $b\colon D \times [0,T] \to \mathbb{R}^n$ and with initial condition $u(0,x) = u_0(x)$.

Two examples of partial differential equations

Heat equation:

$$\partial_t u = \Delta u := \operatorname{div} \mathrm{D}u := \sum_{i=1}^{n} \mathrm{D}_i \mathrm{D}_i u \qquad \text{in } D \times (0,T)$$

This equation describes the distribution of temperature in the given domain $D \subset \mathbb{R}^n$ over time, starting from the initial heat distribution $u(0,x) = u_0(x)$. It is the prototype of the second order parabolic PDE

$$\partial_t u = \operatorname{div}\big(A(x,t)\mathrm{D}u\big)$$

with bounded, positive definite coefficients $A\colon D \times [0,T] \to \mathbb{R}^{n^2}$.

Weak solutions

A **classical solution** usually refers to a function u for which all derivatives occurring in the PDE exist in C^0. This notion is in general too strong to guarantee existence.

Definition

A map $u \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(D)) \cap \mathrm{L}^2(0,T;\mathrm{W}^{1,2}(D))$ is called **weak solution** to the parabolic system with initial values $u_0 \in \mathrm{L}^2(D)$ if

$$\langle\, u(t) - u_0, \varphi \,\rangle_{\mathrm{L}^2(D)} = -\int_0^t \langle\, A(x,t)\mathrm{D}u, \mathrm{D}\varphi \,\rangle_{\mathrm{L}^2(D)} \mathrm{d}s$$

for all $\varphi \in \mathrm{C}_0^\infty(D,\mathbb{R}^N)$ and all $t \in (0,T)$.

(Similarly, again via integration by parts formula, one can define also weak solutions for the transport equation).

Weak solutions

- Existence of weak solutions can be established easily by compactness and monotonicity methods
- Weak solutions by definition are not a priori regular, but rather have a certain degree of integrability and weak differentiability.

Note:
$w(x) = |x|^\alpha$ for $1 - n/p < \alpha < 1$ (with $p \geq 1$) belongs to $\mathrm{W}^{1,p}(B_1(0))$, but is not differentiable in the classical sense!

Regularity theory

Aim in regularity theory:

Study regularity of weak solutions $u\colon D \times [0,T] \to \mathbb{R}^N$ to general parabolic systems

$$\partial_t u = \operatorname{div}\big(A(x,t)\mathrm{D}u\big) \tag{P}$$

with measurable, elliptic, bounded coefficients A, in the sense that

$$A(x,t)\xi \cdot \xi \geq \lambda_0 |\xi|^2 \quad \text{and} \quad |A(x,t)| \leq \lambda_1$$

for some $0 < \lambda_0 \leq \lambda_1$ and all $\xi \in \mathbb{R}^{Nn}$.

Classical regularity results

- **The scalar case:**
  all *scalar-valued* weak solutions are of class C^0_{loc}, without any regularity of the coefficients $A(x,t)$
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- **The vectorial case:**
  Different regimes: there exists constants $c_1(n) \geq c_2(n) > 1$ such that
  - whenever $\lambda_1/\lambda_0 < c_1(n)$ and $x \mapsto A(x,t)$ is Lipschitz or whenever $\lambda_1/\lambda_0 < c_2(n)$, then all *vector-valued* weak solutions are of class $\mathrm{C}^0_{\mathrm{loc}}$
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    ("Cordes-type condition", Koshelev 1993, Kalita 1994);
  - there exist coefficients $A(x,t)$, with ellipticity constant λ_0 and upper bound λ_1 with $\lambda_1/\lambda_0 > c_1(n)$, such that (P) admits a discontinuous solution, starting from smooth initial data
    (Stará-John 1995).

Part II:
The stochastic case

Possible effects of random perturbations

Aim: Study the pathwise behavior of weak solutions with PDE techniques (applicable in more general cases than semi-group approaches). Possible scenarios:

- conservation of regularity?
- regularization by noise (in cases where the underlying deterministic system admits irregular solutions)?
- roughening effects of noise?

Possible effects of random perturbations

Aim: Study the pathwise behavior of weak solutions with PDE techniques (applicable in more general cases than semi-group approaches). Possible scenarios:

- conservation of regularity?
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- roughening effects of noise?

The answers will depend on the type of SPDE and of noise, but the interplay between two facts might be crucial:

– the noise is irregular and might favor singularities;

+ development of coherent structures is prevented (in known counterexamples coefficients and solution interact in a very particular way!).

Setting with stochastic perturbations

Setting:

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$, and let $(B_t)_{t\geq 0}$ be a standard Brownian motion. We now study SPDEs of the form

$$\mathrm{d}u = \operatorname{div}\big(A(x,t)\mathrm{D}u\big)\mathrm{d}t + H(x,t,\mathrm{D}u)\mathrm{d}B_t \quad \text{in } D \times (0,T)\,, \qquad \text{(N)}$$

with H regular, $u\colon D \times (0,T) \times \Omega \to \mathbb{R}^N$ a random function, and the stochastic integral understood in the Itô sense.

Weak solutions

Definition

An $\mathcal{F}_t$-progressively measurable process u on $[0,T]\times\Omega$ is called a *weak solution* to (N) with initial values $u_0 \in L^2(D,\mathbb{R}^N)$ if

(i) P-a. e. path satisfies $u(\cdot,\omega) \in \mathrm{L}^\infty(0,T;\mathrm{L}^2) \cap \mathrm{L}^2(0,T;\mathrm{W}^{1,2})$;

(ii) for all $t \in [0,T]$, we have P-a. s. the identity

$$\langle u(t) - u_0, \varphi \rangle_{L^2(D)} = -\int_0^t \langle A(\cdot,s)\mathrm{D}u, \mathrm{D}\varphi \rangle_{L^2} \mathrm{d}s + \int_0^t \langle \varphi, H(\cdot,s,\mathrm{D}u)\mathrm{d}B_s \rangle_{L^2}$$

for all $\varphi \in \mathrm{C}_0^\infty(D,\mathbb{R}^N)$.

Conservation of regularity

Theorem (B.-Flandoli 2011)

Let u_0 *be regular, let* $H(x,t,z)$ *be Lipschitz continuous in* z *with small Lipschitz constant, and let the coefficients* A *satisfy the Cordes-type condition (i. e. the underlying deterministic system has only regular solutions).*

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Note:

- Full extension of Kalita's result to the stochastic case;
- Includes additive and multiplicative noise;
- Holds for more general system equation with principle part which is "close" to the Laplace system.

“Toy”-tool

The proof of this theorem is based on a combination of the PDE-techniques used by Kalita and stochastic methods, which finally leads to the pathwise regularity result (but the implementation is quite technical).

“Toy”-tool

Toy case of a pathwise regularity criterion:

Kolmogorov’s criterion:

A process u has a Hölder continuous version if

$$E\Big[|u(x_1) - u(x_2)|^q\Big] \leq c|x_1 - x_2|^{n+\alpha q} \quad \text{for all } x_1, x_2 \in \mathbb{R}^n\,.$$

Campanato’s criterion:

A function u is Hölder continuous if

$$\int_{B_r(x_0)} |u - (u)_{B_r(x_0)}|^2 \,\mathrm{d}x \leq cr^{n+2\alpha} \quad \text{for all } x_0 \in \mathbb{R}^n, r \in (0,1)\,.$$

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A combination of the proofs shows that pathwise Hölder continuity of a process u is guaranteed if

$$E\Big[\Big(r^{-n}\int_{B_r(x_0)} |u - (u)_{B_r(x_0)}|^2 \,\mathrm{d}x\Big)^q\Big] \leq C r^{n+2q\alpha}\,.$$

Regularization by noise

With F. Flandoli we have (partial) results in two directions.

Linear PDE with Stratonovich multiplicative noise:

In the setting of the Stara'-John counterexample, we can show that the *average* solves a system with better parabolicity and is hence regular.

Open question: Pathwise regularity?

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Open question: Pathwise regularity?

Linear transport equation with Stratonovich multiplicative noise:

Under the Ladyzhenskaya-Prodi-Serrin condition on the diffusion coefficients we can show no blow-up of derivatives (not true in the deterministic case).

Thanks for your attention!!!