Random and Deterministic perturbations of dynamical systems

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- M. Freidlin, L. Koralov "Metastability for Nonlinear Random Perturbations of Dynamical Systems", Stochastic Processes and Applications 120 (2010), no. 7, 11941214.

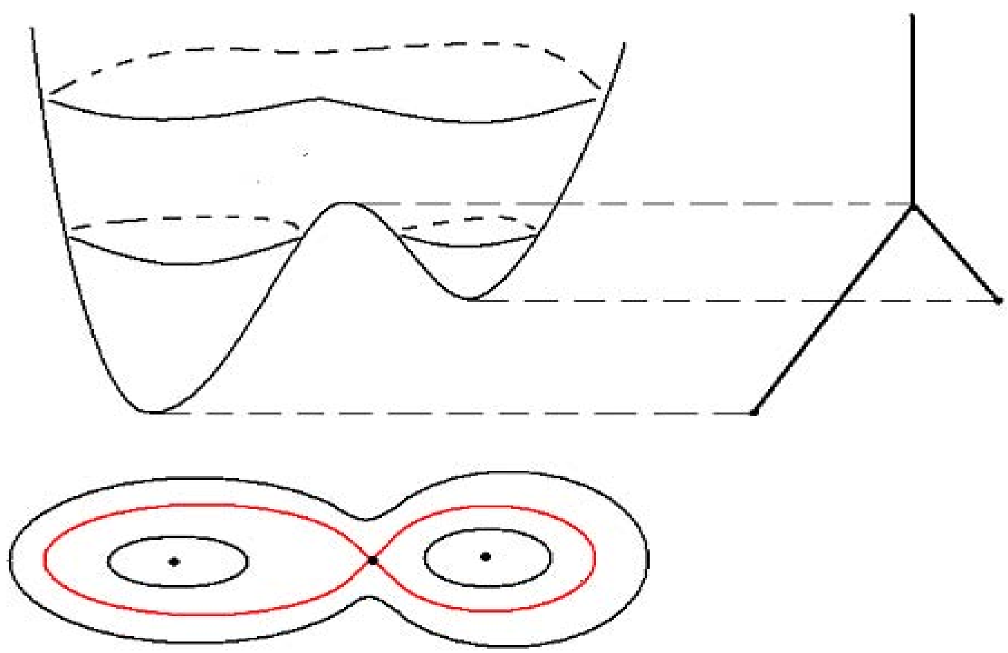
- D. Dolgopyat, L. Koralov "Averaging of incompressible flows on 2-d surfaces", submitted to Journal of American Math. Society.

- D. Dolgopyat, M. Freidlin, L. Koralov "Deterministic and stochastic perturbations of area-preserving flows on a 2-d torus" (to appear in Ergodic Theory and Dynamical Systems).

Part 1. Incompressible flow:

$$\dot{x}(t) = v(x(t)), \quad x(0) = x_0 \in \mathbb{R}^2 \quad \text{or} \quad x_0 \in M.$$

(a) Hamiltonian flows.



Perturbation:

$$dX_t^\varepsilon = \frac{1}{\varepsilon} v(X_t^\varepsilon)dt + \sigma(X_t^{\varkappa,\varepsilon})dW_t \qquad \text{(random)},$$

$$dX_t^\varepsilon = \frac{1}{\varepsilon} v(X_t^\varepsilon)dt + b(X_t^{\varkappa,\varepsilon})dt \qquad \text{(deterministic)}.$$

The dynamics consists of the fast motion (with speed of order $1/\varepsilon$) along the unperturbed trajectories together with the slow motion (with speed of order 1) in the direction transversal to the unperturbed trajectories.

Averaging - consider $h : \mathbb{R}^2 \to \mathbb{G}$. Then

$$h(X_t^\varepsilon) \to Y_t \quad \text{as} \quad \varepsilon \downarrow 0.$$

Locally (away from the vertices of the graph):

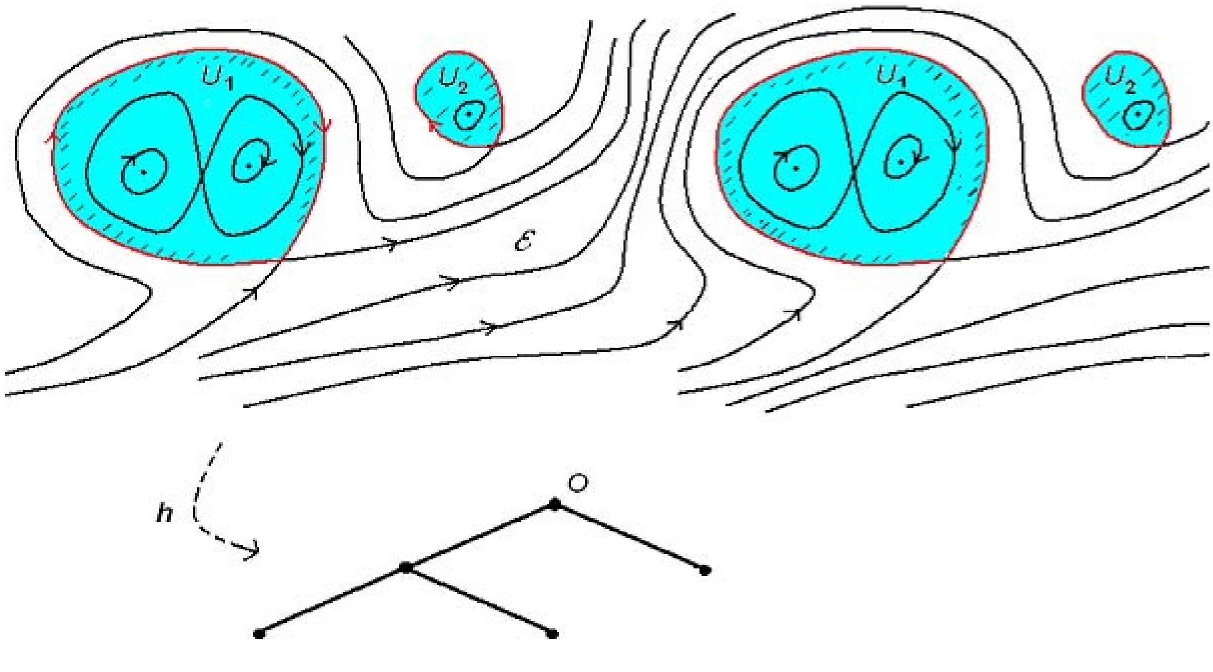
$$\frac{dY_t}{dt} = \frac{\widetilde{b}(Y_t)}{T(Y_t)}, \quad \text{(deterministic)}, \quad \text{where}$$

$$T(h) = \int_{\gamma(h)} \frac{dl}{|\nabla H|}, \quad \widetilde{b}(h) = \int_{\gamma(h)} \frac{\langle b, \nabla H \rangle}{|\nabla H|} dl \quad ,$$

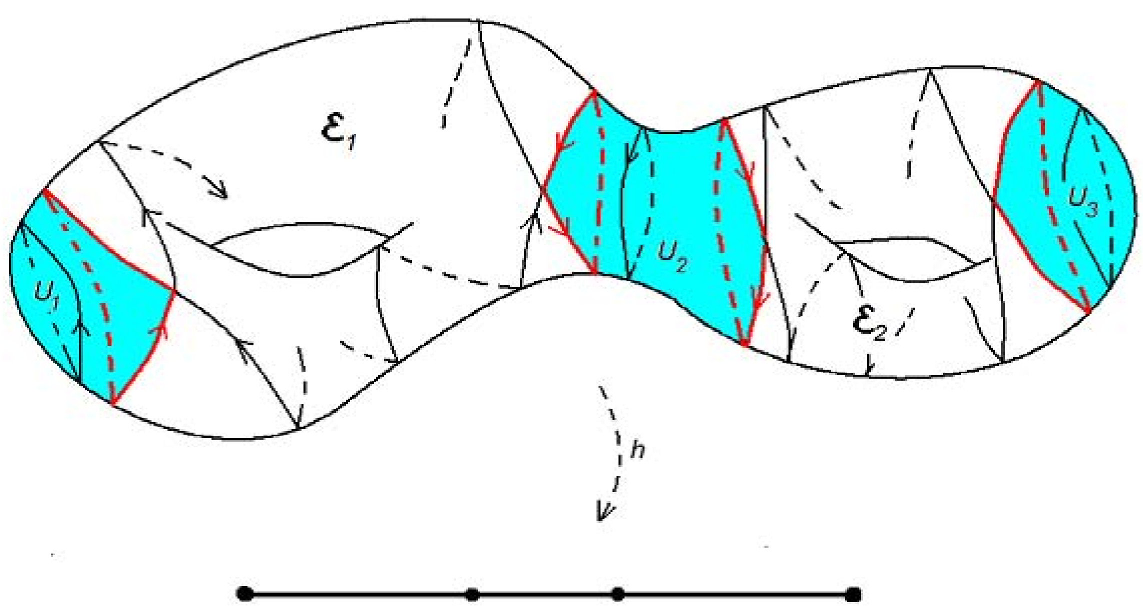
$$dY_t = \overline{\sigma}(Y_t) dW_t + \overline{b}(Y_t) dt \quad \text{(random perturbations)}.$$

Behavior at the vertices. Random perturbations - Freidlin and Wentzell. Deterministic perturbations - regularization required. (Brin and Freidlin).

(b) Locally Hamiltonian flows (there are regions where the unperturbed dynamimcs is ergodic). Example: $H = H_0(x_1, x_2) + \alpha x_1 + \beta x_2$, α/β - irrational.



M - manifold with an area form,
v - incompressible vector field,
X_t^ε - process with generator $L^\varepsilon = \frac{1}{\varepsilon} L_v + L_D$.



Unperturbed dynamics:

$U_1, ..., U_m$ - periodic sets

$\mathcal{E}_1, ..., \mathcal{E}_n$ - 'ergodic components'

Flow on $\mathcal{E}_i$ is isomorphic to a special flow over an interval exchange transformation.

Graph:

- Each edge corresponds to one of U_k
- Three types of vertices:

(a) Those corresponding to $\mathcal{E}_i$,

(b) Those corresponding to saddle points,

(c) Those corresponding to equilibriums (but not saddles).

The flow is Hamiltonian on U_k with a Hamiltonian H. Denote: h_k - coordinate on I_k.

Theorem 1 *The measure on on* $C([0,\infty),\mathbb{G})$ *induced by the process* $Y_t^\varepsilon = h(X_t^\varepsilon)$ *converges weakly to the measure induced by the process with the generator* $\mathcal{L}$ *with the initial distribution* $h(X_0^\varepsilon)$*.*

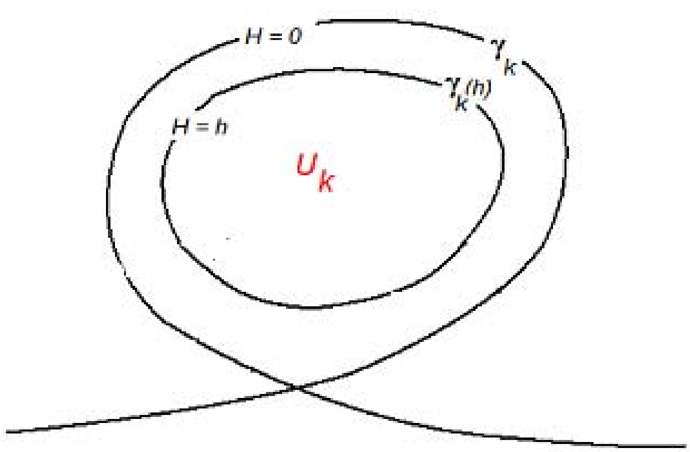
The limiting process is described via its generator $\mathcal{L}$, which is defined as follows.
Let $$L_k f(h_k) = a_k(h_k) f'' + b_k(h_k) f'$$
be the differential operator on the interior of the edge I_k (coefficients are defined below).

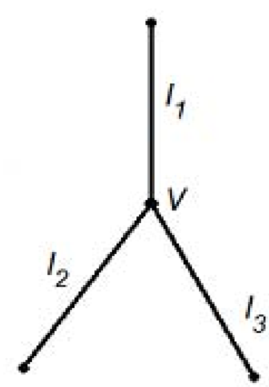
For $f \in D(\mathcal{L})$, we define $\mathcal{L}f = L_k f$ in the interior of each edge, and as the limit of $L_k f$ at the endpoints of I_k.

$D(\mathcal{L})$ consists of $f \in C(\mathbb{G}) \cap C^2(I_k)$ such that (a) $\lim_{h_k \to 0} L_k f(h_k) = q^V$ exist and are the same for all edges entering the same vertex V.
(b) At vertices corresponding to $\mathcal{E}_i$:

$$\sum_{k=1}^{n} p_k^V \lim_{h_k \to 0} f'(h_k) = \lim_{h_k \to 0} L_k f(h_k).$$

(the same with 0 instead of q^V for vertices corresponding to saddles).





Coefficients:

In local coordinates in U_k ($\boldsymbol{\omega} = dxdy$):

$$dX_t^\varepsilon = \frac{1}{\varepsilon} v(X_t^\varepsilon)dt + u(X_t^\varepsilon)dt + \sigma(X_t^\varepsilon)dW_t.$$

Then,

$$a_k(h_k) = \frac{1}{2} T^{-1}(h_k) \int_{\gamma_k(h_k)} \frac{\langle \alpha \nabla H, \nabla H \rangle}{|\nabla H|} dl \quad \text{and}$$

$$b_k(h_k) = \frac{1}{2} T^{-1}(h_k) \int_{\gamma_k(h_k)} \frac{2\langle u, \nabla H \rangle + \alpha \cdot H''}{|\nabla H|} dl,$$

where $\alpha = \sigma\sigma^*$.

$$p_k^V = \pm \frac{1}{2} \int_{\gamma_k} \frac{\langle \alpha \nabla H, \nabla H \rangle}{|\nabla H|} dl.$$

Ingredients of the proof.

(1) Assume (temporarily) that the area measure λ is invariant for the process for each ε.

For the limit Y_t of $Y_t^\varepsilon = h(X_t^\varepsilon)$, we should have

$$\mathbb{E}[f(Y_T) - f(Y_0) - \int_0^T \mathcal{L}f(Y_s)ds] = 0.$$

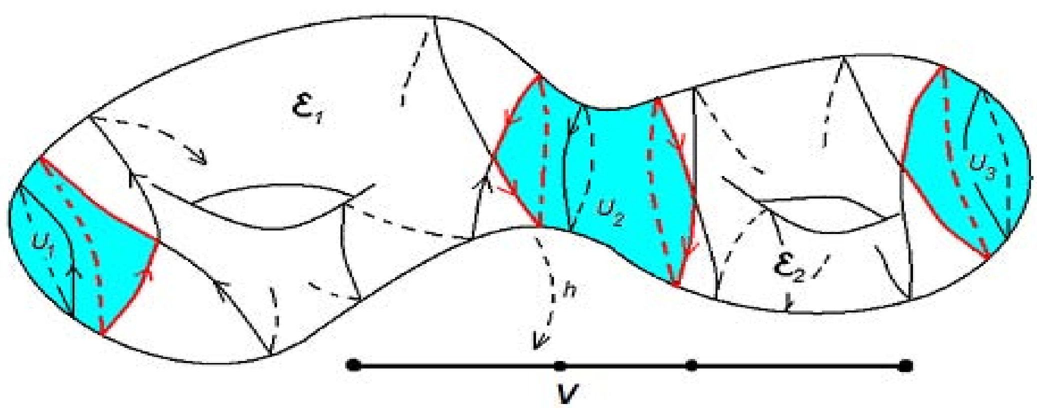
Need to prove the following lemma.

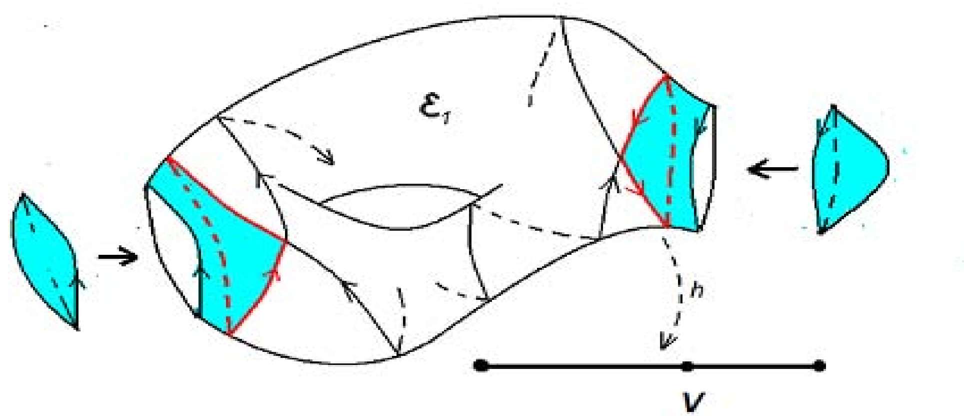
Lemma1 *For each function $f \in D(\mathcal{L})$ and each $T > 0$ we have*

$$\mathbb{E}_x[f(h(X_T^\varepsilon)) - f(h(X_0^\varepsilon)) - \int_0^T \mathcal{L}f(h(X_s^\varepsilon))ds] \to 0$$

uniformly in $x \in \mathbb{T}^2$ as $\varepsilon \to 0$.

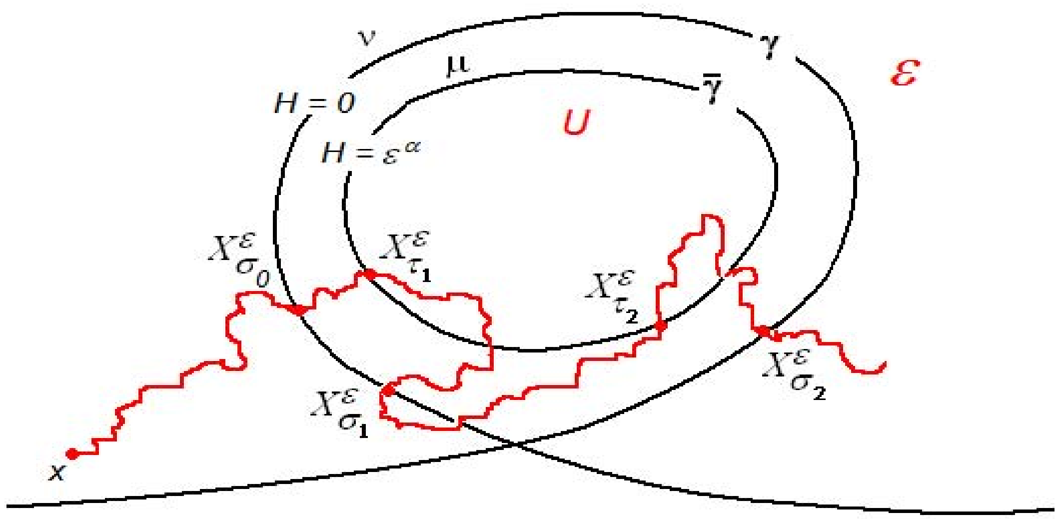
(2) Localization (can deal with a star-shaped graph with one accessible vertex)





(3) Need:

$$\mathbb{E}_x[f(h(X_T^\varepsilon)) - f(h(X_0^\varepsilon)) - \int_0^T \mathcal{L}f(h(X_s^\varepsilon))ds] \to 0$$



Split $[0, T]$ into intervals:
$[0, \sigma_0]$, $[\sigma_0, \tau_1]$, $[\tau_1, \sigma_1]$, $[\sigma_1, \tau_2]$, ...

On intervals $[\tau_n, \sigma_n]$ (inside periodic component) - averaging (Freidlin-Wentzell) with small modifications.

On intervals $[\sigma_n, \tau_{n+1}]$ (getting from the ergodic component into the periodic component):

$$\mathbb{E}_x[f(h(X^\varepsilon_{\tau_{n+1}})) - f(h(X^\varepsilon_{\sigma_n})) - \int_{\sigma_n}^{\tau_{n+1}} \mathcal{L}f(h(X^\varepsilon_s))ds] \approx$$

$$\mathbb{E}_\nu[f(h(X^\varepsilon_\tau)) - f(h(X^\varepsilon_0)) - \int_0^\tau \mathcal{L}f(h(X^\varepsilon_s))ds] \approx$$

$$f'(0)\varepsilon^\alpha - \mathbb{E}_\nu\tau \cdot \mathcal{L}f(0).$$

- How can we calculate $\mathbb{E}_\nu\tau$?
- Why can we assume that we start with the invariant measure ν?

If λ is invariant: $\frac{\mathbb{E}_\nu \tau}{\lambda(\mathcal{E})} \approx \frac{\mathbb{E}_\mu \sigma}{\lambda(U)}$, so

$$\mathbb{E}_\nu \tau \approx \frac{\lambda(\mathcal{E})}{\lambda(U)} \cdot \mathbb{E}_\mu \sigma \approx \text{const} \cdot \varepsilon^\alpha.$$

If λ is not invariant: consider

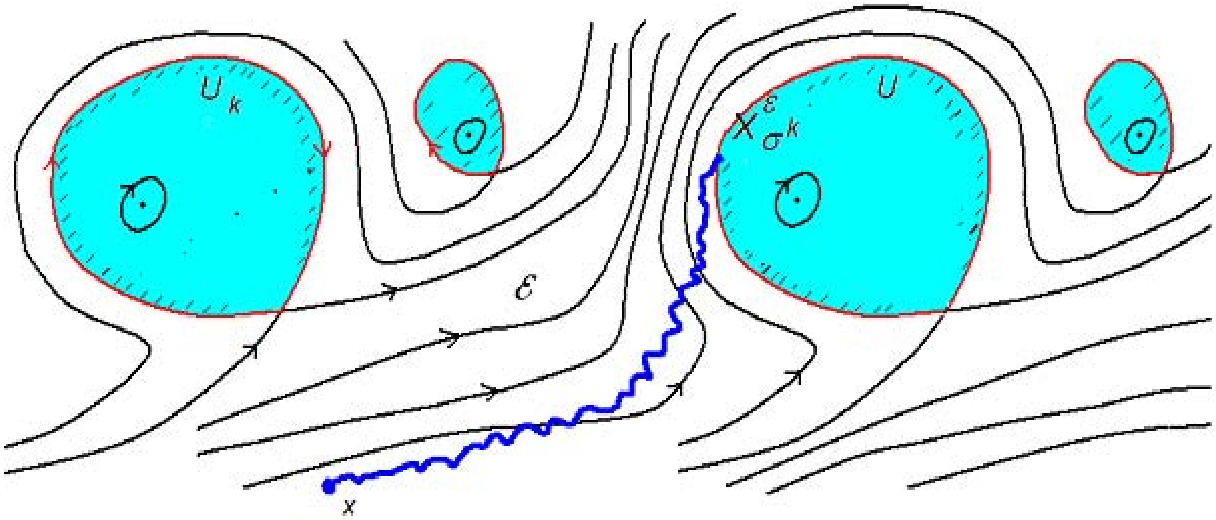
$$d\widetilde{X}_t^\varepsilon = \frac{1}{\varepsilon} v(\widetilde{X}_t^\varepsilon)dt + \widetilde{u}(\widetilde{X}_t^\varepsilon)dt + \sigma(\widetilde{X}_t^\varepsilon)dW_t,$$

(replace u by some $\widetilde{u}$ so that λ is invariant for the new process).

By the Girsanov Theorem:

$$\widetilde{\nu} \approx \nu, \quad \mathbb{E}_{\widetilde{\nu}} \widetilde{\tau} \approx \mathbb{E}_\nu \tau.$$

So, $\mathbb{E}_\nu \tau \approx \frac{\lambda(\mathcal{E})}{\lambda(U)} \cdot \mathbb{E}_{\widetilde{\mu}} \widetilde{\sigma}$ (the gluing conditions are the same as for the measure-preserving process).



(4) Why does $\mathbb{E}_x \sigma^k \to 0$ as $\varepsilon \downarrow 0$? (time to reach U^k)

Let $u^\varepsilon(t, y)$, $y \in M \backslash U_k$, be the probability that the process starting at y does not reach U_k before time t.

$$\frac{\partial u^\varepsilon(t, y)}{\partial t} = \left(L_D + \frac{1}{\varepsilon} L_v \right) u^\varepsilon$$

$$u^\varepsilon(0, y) = 1, \; y \in M \setminus U_k, \quad u^\varepsilon(t, y) = 0, \quad t > 0.$$

(a) **Lemma** (Zlatos): All $H_0^1(M\backslash U_k)$-eigenvalues for $v\nabla$ are zero on $\mathcal{E}$ implies that the $L^2(\mathcal{E})$-norm (and so $L^1(\mathcal{E})$-norm) of $u^\varepsilon(t,\cdot)$ tends to zero as $\varepsilon \downarrow 0$ for each $t > 0$.

(b) A uniform bound on fundamental solution doesn't get affected by adding an incompressible drift term.

(a) and (b) imply that $\mathbb{E}_x\sigma \to 0$. With some effort possible to show that $\mathbb{E}_x\sigma^k \to 0$.

Part 2: Averaging of deterministic perturbations

Recall

$$dX_t^{\varkappa,\varepsilon} = \frac{1}{\varepsilon} v(X_t^{\varkappa,\varepsilon})dt + b(X_t^{\varkappa,\varepsilon})dt +$$

$$\varkappa u(X_t^{\varkappa,\varepsilon})dt + \sqrt{\varkappa}\sigma(X_t^{\varkappa,\varepsilon})dW_t.$$

Let $Y_t^{\varkappa,\varepsilon} = h(X_t^{\varkappa,\varepsilon})$ be the corresponding process on the graph $\mathbb{G}$. We demonstrated that the distribution of $Y_t^{\varkappa,\varepsilon}$ converges, as $\varepsilon \downarrow 0$, to the distribution of a limiting process, which will be denoted by $Z_t^{\varkappa}$. $Z_t^{\varkappa}$, in turn, converges to the distribution of a limiting Markov process on $\mathbb{G}$ when $\varkappa \downarrow 0$.

The limiting process Z_t can be described as follows. It is a Markov process with continuous trajectories which moves deterministically along an edge I_k of the graph with the speed

$$\bar{b}_k(h_k) = \frac{1}{2}(T_k(h_k))^{-1} \int_{\gamma_k(h_k)} \frac{2\langle b, \nabla H \rangle}{|\nabla H|} dl.$$

If the process reaches V corresponding to an ergodic component, then it either remains at V forever or spends exponential time in V and then continues with deterministic motion away from V along a randomly selected edge (with probabilities which can be specified). The same if V corresponds to a saddle point, but no exponential delay.

Theorem 2 *The measure on on* $C([0,\infty),\mathbb{G})$ *induced by the process* $Z_t^{\varkappa}$ *converges weakly to the measure induced by the process* Z_t *with the initial distribution* $h(X_0^{\varepsilon})$.

The process Z_t is defined by the deterministic system. The stochastic perturbations are used just for regularization purposes.

Part 3.

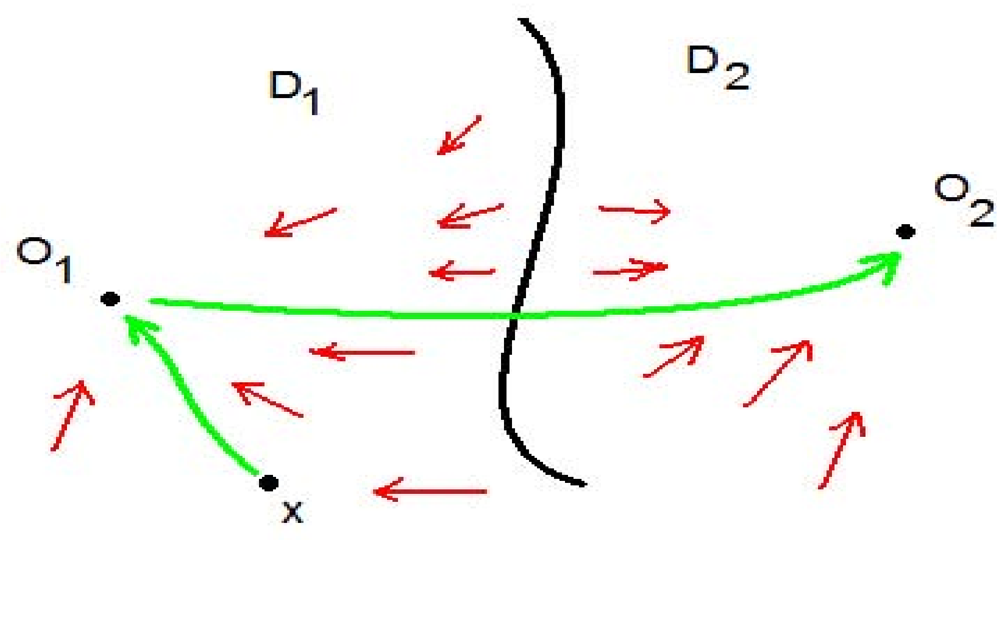
$$\dot{X}_t^x = b(X_t^x), \quad X_0^x = x \in \mathbb{R}^d;$$

$$dX_t^{x,\varepsilon} = b(X_t^{x,\varepsilon})dt + \varepsilon\sigma(X_t^{x,\varepsilon})dW_t, \quad X_0^{x,\varepsilon} = x.$$

in terms of PDEs:

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon^2}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + b(x)\cdot\nabla_x u^\varepsilon,$$

$$u^\varepsilon(0,x) = g(x), \ x \in \mathbb{R}^d.$$



$$u^\varepsilon(t,x) = \mathsf{E}g(X_t^{x,\varepsilon}).$$

Action functional:

$$S_{0,T}(\varphi) = \frac{1}{2}\int_0^T \sum_{i,j=1}^{d} a^{ij}(\varphi_t)(\dot{\varphi}_t^i - b_i(\varphi_t))(\dot{\varphi}_t^j - b_j(\varphi_t))dt,$$

if φ - absolutely continuous,

$$S_{0,T}(\varphi) = +\infty, \qquad \text{otherwise.}$$

$a^{ij} = (a^{-1})_{ij}$.

Quasi-potential:

$$V_{mn} = V(O_m, O_n) =$$

$$\inf_T \{S_{0,T}(\varphi) : \varphi(0) = O_m, \varphi(T) = O_n\}.$$

τ^ε_{mn} - the time it takes the process to go from O_m to a small neighborhood of O_n.

$$\tau^\varepsilon_{mn} \sim \exp(V_{mn}/\varepsilon^2),$$

Consider the process (and solution of PDE) at times $t(\varepsilon)$ with $\ln(t(\varepsilon)) \sim \lambda/\varepsilon^2$. Suppose, for example, that $x \in D_1$ and $V_{12} < V_{21}$.

If $\lambda < V_{12}$, then $u^\varepsilon(t(\varepsilon), x) \to g(O_1)$.
If $\lambda > V_{12}$, then $u^\varepsilon(t(\varepsilon), x) \to g(O_2)$.

Nonlinear problem:

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon^2}{2} \sum_{i,j=1}^{d} a_{ij}(x, u^\varepsilon) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + b(x) \cdot \nabla_x u^\varepsilon,$$

$$u^\varepsilon(0, x) = g.$$

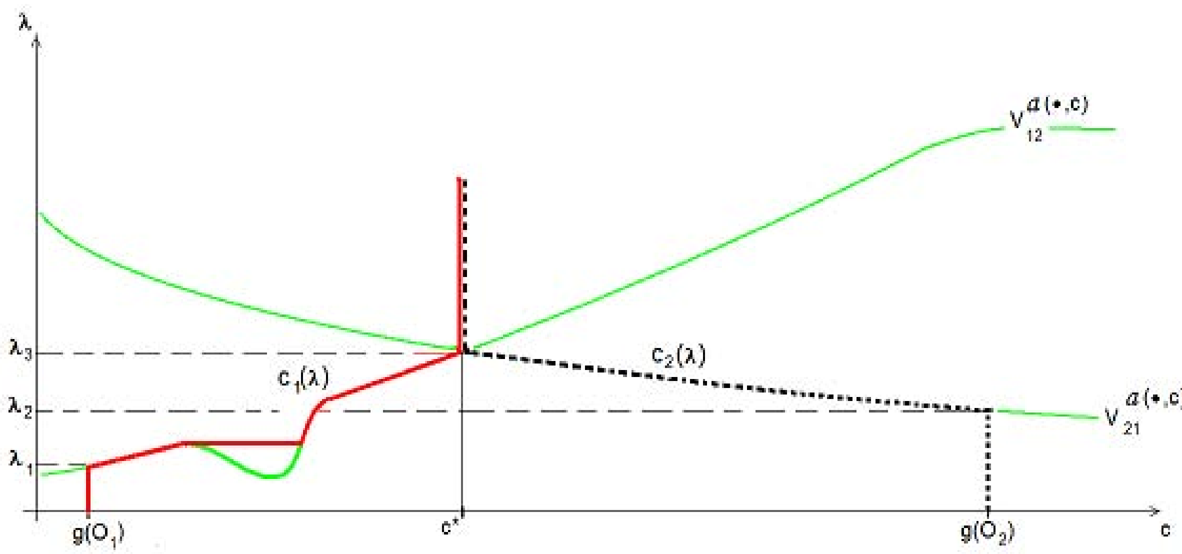
equivalent to the system

$$dX_s^{t,x,\varepsilon} = b(X_s^{t,x,\varepsilon})dt + \varepsilon\sigma(X_s^{t,x,\varepsilon}, u^\varepsilon(t-s, X_s^{t,x,\varepsilon}))dW_s,$$

$$u^\varepsilon(t, x) = \mathsf{E}g(X_t^{t,x,\varepsilon}).$$

Construct $\widetilde{V}_{12}$ using $a(t, g(x_0))$.
If $\lambda < \widetilde{V}_{12}$, then still $u^\varepsilon(t(\varepsilon), x) \to g(O_1)$.
If $\lambda > \widetilde{V}_{12}$, then new effects appear for u^ε and the processes.

Result in the non-linear case (2 equilibriums):

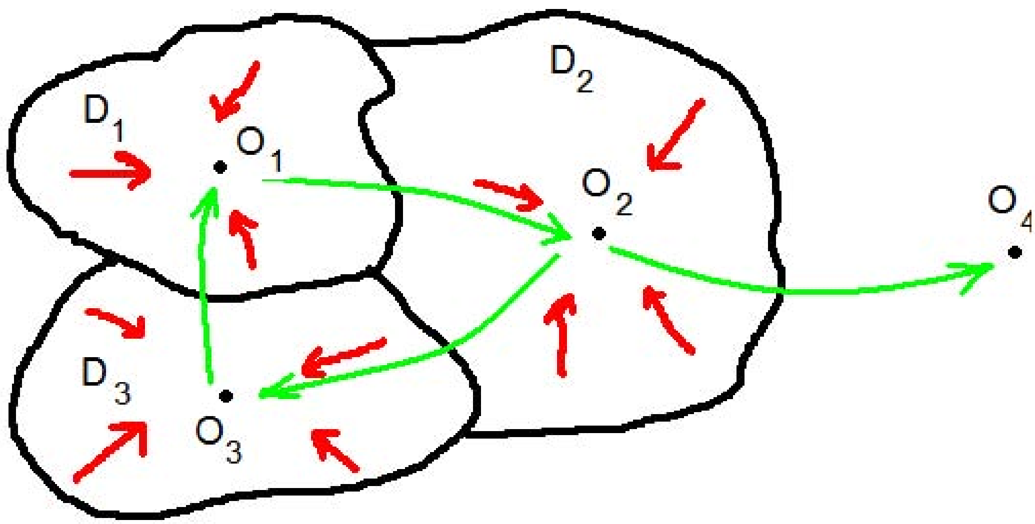


Theorem:

$$\lim_{\varepsilon \downarrow 0} u^{\varepsilon}(\exp(\lambda/\varepsilon^2), x) = c_n(\lambda), \quad x \in D_n.$$

Corollary: If $x \in D_1$, then the distribution of $X_{\exp(\lambda/\varepsilon^2)}^{\exp(\lambda/\varepsilon^2),x,\varepsilon}$ converges to the measure $\mu_1^{\lambda} = a_1 \delta_{O_1} + a_2 \delta_{O_2}$, where the coefficients a_1 and a_2 can be found from the equations $c_1(\lambda) = a_1 g(O_1) + a_2 g(O_2)$, $a_1 + a_2 = 1$.

Multiple equilibriums.



- Need to look at $V_{mn}^{a(x,c)} = V^{a(x,c)}(O_m, O_n)$, which determine the hierarchy of cycles;

- The hierarchy of cycles may evolve in time (i.e., depends on λ).