Small noise asymptotics of integrated Ornstein–Uhlenbeck processes driven by α-stable Lévy processes

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1. Source of randomness: Lévy process L

L is a **Lévy process** if $L_0 = 0$, is stochastically continuous and has

independent stationary increments

(and right continuous paths with left limits).

$$L = \underbrace{\text{Brownian motion} + \text{drift}} + \underbrace{\text{jumps}}$$

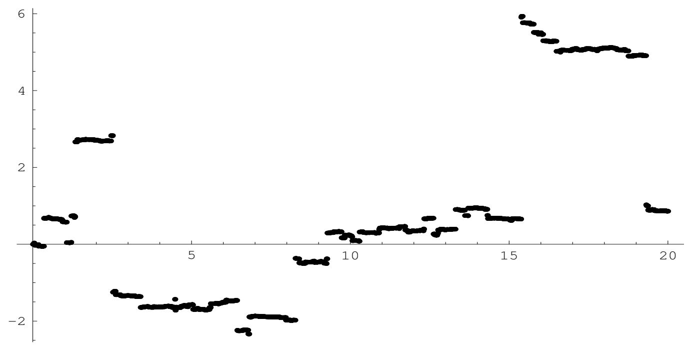
Lévy–Khintchine formula for $L \in \mathbb{R}^m$: $\langle x, y \rangle = \sum_{i=1}^m x_i y_i$

$$\mathbf{E}e^{i\langle L_t,\lambda\rangle} = \exp\Big[\underbrace{-\frac{t}{2}\langle A\lambda, \lambda\rangle}_{\text{Brownian motion}} + \underbrace{it\langle \lambda, \mu\rangle}_{\text{drift}} + \underbrace{t\int \Big(e^{i\langle\lambda,y\rangle} - 1 - \frac{i\langle \lambda, y\rangle}{1+\|y\|^2}\Big)\,\nu(dy)}_{\text{jumps}} \Big]$$

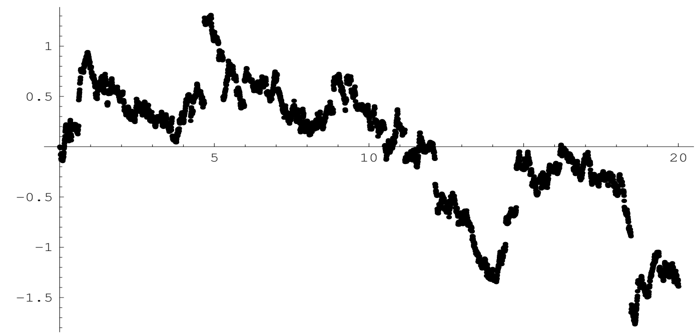
2. α-stable Lévy–Processes (Lévy Flights)

$L = (L_t)_{t \geqslant 0}$ is a one-dimensional α-stable Lévy process (symmetric: $\beta = 0$)

$$\mathbf{E} e^{\mathrm{i}uL_t} = \exp\left\{ -tc|u|^\alpha \left(1 - \mathrm{i}\beta \,\mathrm{sgn}(u) \tan\frac{\pi\alpha}{2}\right)\right\}, \quad \alpha \in (0,1) \cup (1,2)$$



$\alpha = 0.75$



$\alpha = 1.75$

Pure jump process with enumerable many (small) jumps on any time interval, jump times are dense.

$\alpha = 1$	Cauchy–process	$\frac{1}{\pi}\frac{1}{1+x^2}$
$\alpha = 2$	Brownian motion	$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

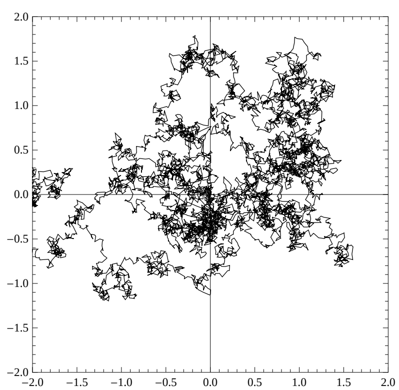
3. α-stable Lévy process (Lévy flights)

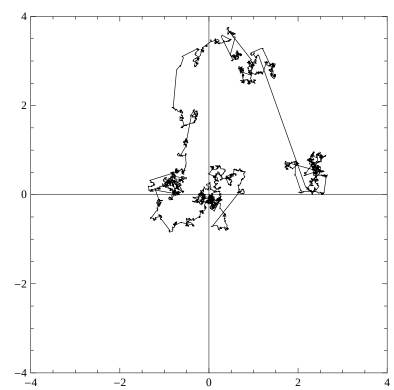
Isometric α-stable LP in $\mathbb{R}^m$:

$$\mathbf{E}e^{i\langle L_t,\lambda\rangle} = \exp\Big[-tc_{m,\alpha}\|\lambda\|^\alpha\Big], \quad \alpha\in(0,2), \quad c_{m,\alpha} = \frac{\pi^{m/2}}{2^\alpha}\frac{\Gamma(-\frac{\alpha}{2})}{\Gamma(\frac{m+\alpha}{2})}$$

Jump measure: $\nu(dy) = \dfrac{dy}{\|y\|^{\alpha+m}}$, $\alpha\in(0,2)$

Cauchy process: $\alpha=1$, probability density $p(x)\sim \dfrac{1}{1+\|x\|^2}$





Brownian motion

1.50-stable Lévy process

4. Motivation and Setting

Chechkin, Gonchar, Szydłowski, Physics of Plasmas 2002.

$l=(l_t)_{t\geqslant 0}$ is an isometric α-stable Lévy process in $\mathbb{R}^3$,

$$\mathbf{E}\mathrm{e}^{\mathrm{i}\langle u,l_t\rangle}=\mathrm{e}^{-t\|u\|^\alpha},\quad u\in\mathbb{R}^3,\quad \alpha\in(0,2).$$

Langevin equation for a particle in a external magnetic field $\mathbf{B}$ and Lévy electric field $\dot{l}$:

$$\ddot{x}=[\dot{x}\times\mathbf{B}]-\nu\dot{x}+\varepsilon\dot{l}$$

or

$$\begin{cases}\dot{x}^\varepsilon=v^\varepsilon,\\ \dot{v}^\varepsilon=\underbrace{[v^\varepsilon\times\mathbf{B}]-\nu v^\varepsilon}_{=:-Av^\varepsilon}+\varepsilon\dot{l}\end{cases},\quad A=\begin{pmatrix}\nu & -B_3 & B_2\\ B_3 & \nu & -B_1\\ -B_2 & B_1 & \nu\end{pmatrix}$$

In other words, x^ε is an integrated OU process:

$$x_t^\varepsilon=x_0+\int_0^t v_s^\varepsilon\,\mathrm{d}s,\quad v_t^\varepsilon=v_0-\int_0^t Av_s^\varepsilon\,\mathrm{d}s+\varepsilon l_t$$

5. ε-dependent timescale

Interesting events should occur on the time intervals of the order $\mathcal{O}(\frac{1}{\varepsilon^\alpha})$, $\varepsilon \to 0$.

Time transformation: $t \mapsto \frac{t}{\varepsilon^\alpha}$.

Self-similarity of an α-stable process: $\mathrm{Law}(\varepsilon l_{\frac{t}{\varepsilon^\alpha}}, t \geqslant 0) = \mathrm{Law}(l) = \mathrm{Law}(L)$

$$V_t := v_{\frac{t}{\varepsilon^\alpha}} = -\int_0^{\frac{t}{\varepsilon^\alpha}} A v_s \, \mathrm{d}s + \varepsilon l_{\frac{t}{\varepsilon^\alpha}} = -\frac{1}{\varepsilon^\alpha}\int_0^t A v_{\frac{s}{\varepsilon^\alpha}} \, \mathrm{d}s + \varepsilon l_{\frac{t}{\varepsilon^\alpha}} \overset{\text{Law}}{=} -\frac{1}{\varepsilon^\alpha}\int_0^t A V_s \, \mathrm{d}s + L_t,$$

$$X_t := x_{\frac{t}{\varepsilon^\alpha}} = \int_0^{\frac{t}{\varepsilon^\alpha}} v_s \, \mathrm{d}s = \frac{1}{\varepsilon^\alpha}\int_0^t v_{\frac{s}{\varepsilon^\alpha}} \, \mathrm{d}s = \frac{1}{\varepsilon^\alpha}\int_0^t V_s \, \mathrm{d}s$$

From now on: on some probability space consider an α-stable Lévy process L and a family of processes $\{V^\varepsilon, X^\varepsilon\}$ (with big friction parameter $\frac{1}{\varepsilon^\alpha} \to \infty$)

$$\begin{cases} V_t^\varepsilon = -\dfrac{1}{\varepsilon^\alpha}\displaystyle\int_0^t A V_s^\varepsilon \, \mathrm{d}s + L_t, \\ X_t^\varepsilon = \dfrac{1}{\varepsilon^\alpha}\displaystyle\int_0^t V_s^\varepsilon \, \mathrm{d}s \end{cases} \qquad \mathrm{Law}(V_t^\varepsilon, X_t^\varepsilon, t \geqslant 0) = \mathrm{Law}(v^\varepsilon_{\frac{t}{\varepsilon^\alpha}}, x^\varepsilon_{\frac{t}{\varepsilon^\alpha}}, t \geqslant 0)$$

6. Explicit solution

Ornstein–Uhlenbeck process:

$$V_t^\varepsilon = -\frac{1}{\varepsilon^\alpha}\int_0^t AV_s^\varepsilon \, \mathrm{d}s + L_t \quad \Rightarrow \quad V_t^\varepsilon = \int_0^t \mathrm{e}^{-\frac{t-s}{\varepsilon^\alpha}A} \, \mathrm{d}L_s$$

Integrated Ornstein–Uhlenbeck process (Fubini):

$$\begin{aligned} AX_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t AV_s^\varepsilon \, \mathrm{d}s &= \frac{1}{\varepsilon^\alpha}\int_0^t \Big[\int_0^s A\mathrm{e}^{-\frac{s-u}{\varepsilon^\alpha}A} \, \mathrm{d}L_u\Big] \, \mathrm{d}s \\ &= \frac{1}{\varepsilon^\alpha}\int_0^t \Big[\int_u^t A\mathrm{e}^{-\frac{s-u}{\varepsilon^\alpha}A} \, \mathrm{d}s\Big] \, \mathrm{d}L_u \\ &= A^{-1}A\int_0^t \Big(1 - \mathrm{e}^{-\frac{t-u}{\varepsilon^\alpha}A}\Big) \, \mathrm{d}L_u \end{aligned}$$

The process X^ε is absolutely continuous, non-Markovian, semimartingale.

7. Convergence of f.d.d.

Theorem 1. For any $n \geqslant 1$, $0 \leqslant t_1 < \cdots < t_n < \infty$

$$(AX^{\varepsilon}_{t_1}, \ldots, AX^{\varepsilon}_{t_n}) \xrightarrow{\mathbf{P}} (L_{t_1}, \ldots, L_{t_n}), \quad \varepsilon \to 0.$$

Assume:

$$\mathbf{E}^{\mathrm{i}\langle u, L_t \rangle} = \mathrm{e}^{-\|u\|^{\alpha}}, \quad \alpha \in (0,1), \quad u \in \mathbb{R}^d.$$

Show:

$$AX^{\varepsilon}_{t} \xrightarrow{\mathbf{P}} L_t, \quad \varepsilon \to 0, \quad t \geqslant 0.$$

8. Proof (convergence of one-dimensional distributions)

$$AX_t^\varepsilon - L_t = -\int_0^t \mathrm{e}^{-\frac{t-s}{\varepsilon^\alpha}A}\,\mathrm{d}L_s$$

$$\begin{aligned}
\mathbf{E}\mathrm{e}^{\mathrm{i}u(AX_t^\varepsilon - L_t)} &= \mathbf{E}\exp\Big\{-\mathrm{i}u\lim_n \sum_{k=1}^n \mathrm{e}^{-\frac{t-s_k}{\varepsilon^\alpha}A}\Delta L_{s_k}\Big\} \\
&= \lim_n \prod_{k=1}^n \mathbf{E}\mathrm{e}^{-\mathrm{i}u\mathrm{e}^{-\frac{t-s_k}{\varepsilon^\alpha}A}\Delta L_{s_k}} \\
&= \lim_n \prod_{k=1}^n \mathrm{e}^{\Delta s_k \left\|-u\mathrm{e}^{-\frac{t-s_k}{\varepsilon^\alpha}A}\right\|^\alpha} \\
&= \exp\Big\{\lim_n \sum_{k=1}^n \Delta s_k \left\|u\mathrm{e}^{-\frac{t-s_k}{\varepsilon^\alpha}A}\right\|^\alpha\Big\} \\
&= \exp\Big\{\|u\|^\alpha \int_0^t \underbrace{\left\|\mathrm{e}^{-\frac{t-s}{\varepsilon^\alpha}A}\right\|^\alpha}_{\to 0,\ s\neq t,\ \varepsilon\to 0}\,\mathrm{d}s\Big\} \to 1,\ \varepsilon \to 0
\end{aligned}$$

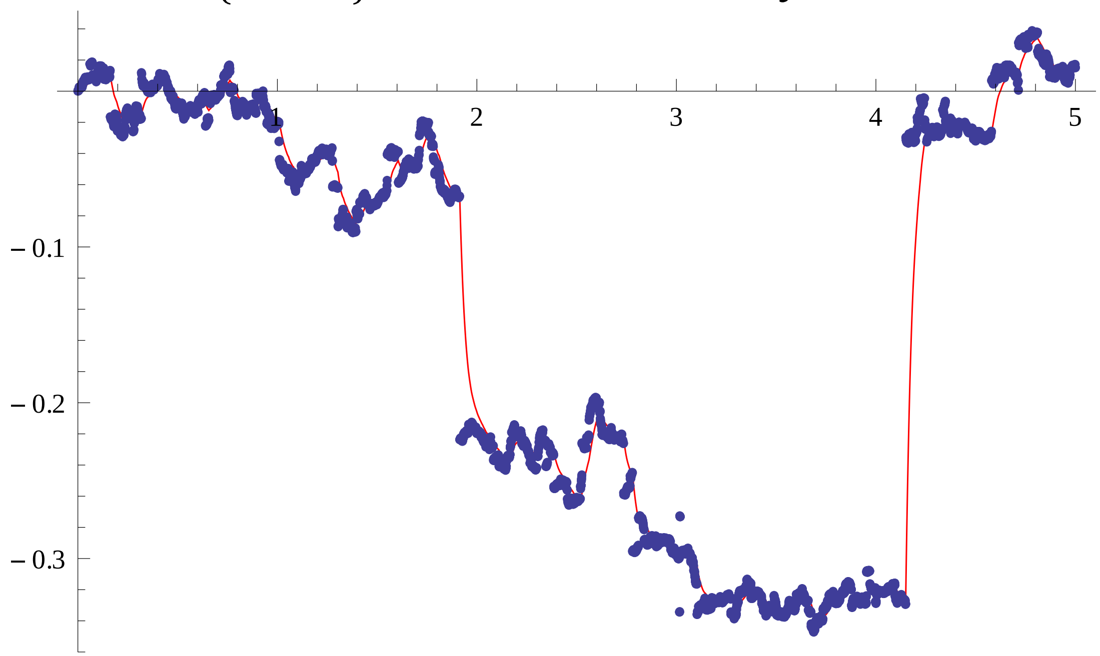
9. Functional limit theorem?

Convergence of f.d.d. does not imply convergence of the first passage times.

$$\mathbf{P}(\tau_a(X^\varepsilon) \leqslant t) = \mathbf{P}(\sup_{s\leqslant t} X^\varepsilon > a)$$

Need convergence in a path space $D([0,\infty),\mathbb{R})$ with an appropriate metric.

Problem: the limit α-stable Lévy process L is (in general) càdlàg
the processes $\{AX^\varepsilon\}_{\varepsilon>0}$ are absolutely continuous.



10. Uniform convergence does not hold

Consider the space $D([0,\infty),\mathbb{R})$ with a (local) uniform topology associated with the metric

$$d_{U,T}(x,x') := \sup_{t\in[0,T]} |x_t - x'_t|, \quad T>0,$$

$$d_U(x,x') := \int_0^\infty \mathrm{e}^{-T}(1 \wedge d_{U,T}(x,x'))\,\mathrm{d}T$$

No U-convergence unless L is continuous (Brownian motion with drift):

$$d_{U,T}(AX^\varepsilon, L) := \sup_{t\in[0,T]} |AX^\varepsilon_t - L_t| \overset{\mathbf{P}}{\nrightarrow} 0, \quad \varepsilon \to 0.$$

11. Skorohod J_1-convergence does not hold

Skorohod (1956): J_1-topology (as well as J_2, M_1, M_2 topologies)

Consider continuous time changes

$$\Lambda = \Big\{\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+,\ \text{strictly increasing and continuous}, \lambda(0) = 0,\ \lambda(+\infty) = +\infty)\Big\}$$

$$x^n \to x \quad \Leftrightarrow \quad \text{there exists a sequence } \{\lambda^n\} \subset \Lambda \text{ such that}$$

$$\sup_{t\geqslant 0} |\lambda^n(t) - t| \to 0,$$

$$\sup_{t\in[0,T]} |x^n(\lambda^n(t)) - x(t)| \to 0 \text{ for all } T > 0.$$

This topology in metrizable and the space D is Polish.

No J_1-convergence unless L is continuous (Brownian motion with drift):

$$d_{J_1,T}(AX^\varepsilon, L) \overset{\mathbf{P}}{\not\to} 0, \quad \varepsilon \to 0.$$

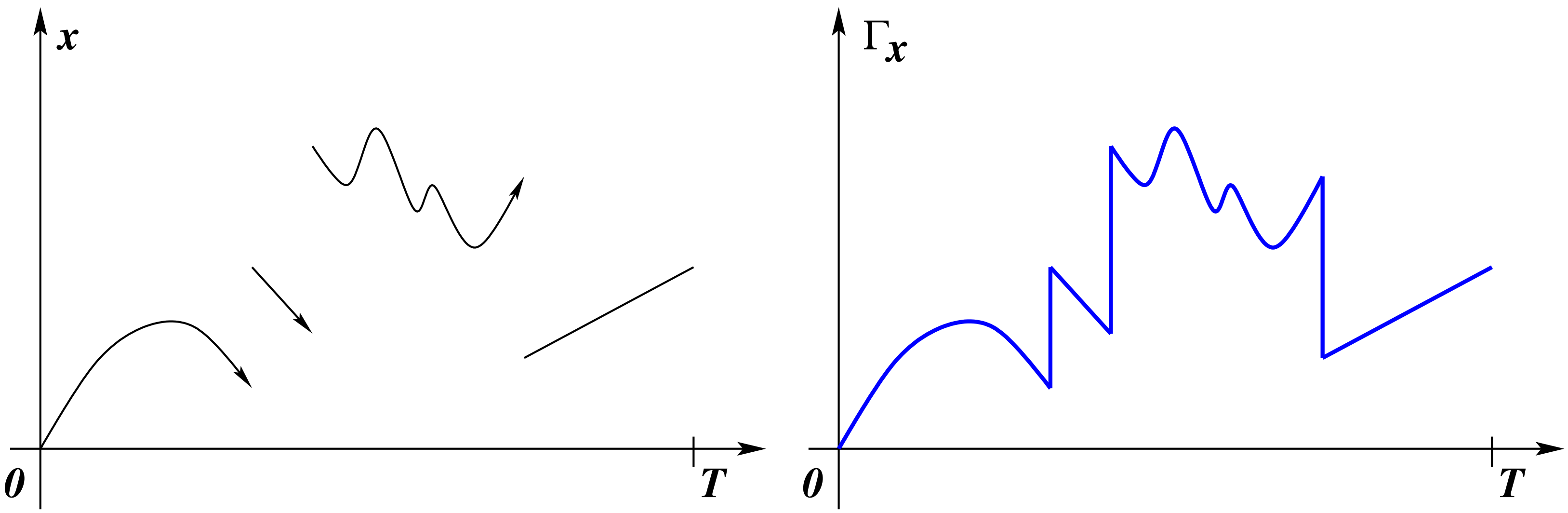
We need a weaker metric, such that the $\sup$-functional is still continuous.

12. Skorohod M_1-covergence I

For $x \in D([0,T],\mathbb{R})$ define a *completed graph* Γ_x:

$$\Gamma_x := \{(x_0, 0)\} \cup \{(z,t) \in \mathbb{R} \times (0,T] \colon z = cx_{t-} + (1-c)x_t \text{ for some } c,\ c \in [0,1]\},$$

$\Gamma_x \subset \mathbb{R}^2$.



Natural order on Γ_x:

$$(z,t) \leqslant (z',t') \quad \text{if} \quad t < t' \text{ or } t = t' \text{ and } |x_{t-} - z| \leqslant |x_{t-} - z'|.$$

13. Skorohod M_1-convergence II

Parametric representation of Γ_x: continuous nondecreasing w.r.t. order mapping

$$(z_u, t_u)\colon [0,1] \to \Gamma_x.$$

Denote Π_x the set of all parametric representations of Γ_x.

Skorohod M_1-convergence on $D([0,T],\mathbb{R})$:

$$x^n \to x \quad \Leftrightarrow \quad \text{for any } (z,t) \in \Pi_x \text{ there is } (z^n, t^n) \subset \Pi_{x^n} \text{ such that}$$

$$\max\Big\{ \sup_{u\in[0,1]} |z_u^n - z_u|, \sup_{u\in[0,1]} |t_u^n - t_u| \Big\} \to 0, \quad n\to\infty.$$

This topology in metrizable and the space $D(\mathbb{R}_+,\mathbb{R};M_1)$ is Polish (see Whitt, Chapter 12.8).

The $\sup$-functional is continuous.

Goal: Prove convergence $AX^\varepsilon \to L$ in $D([0,\infty),\mathbb{R};M_1)$ in probability

i.e. **convergence of f.d.d.** (done) and **tightness**.

14. M_1-oscillation function

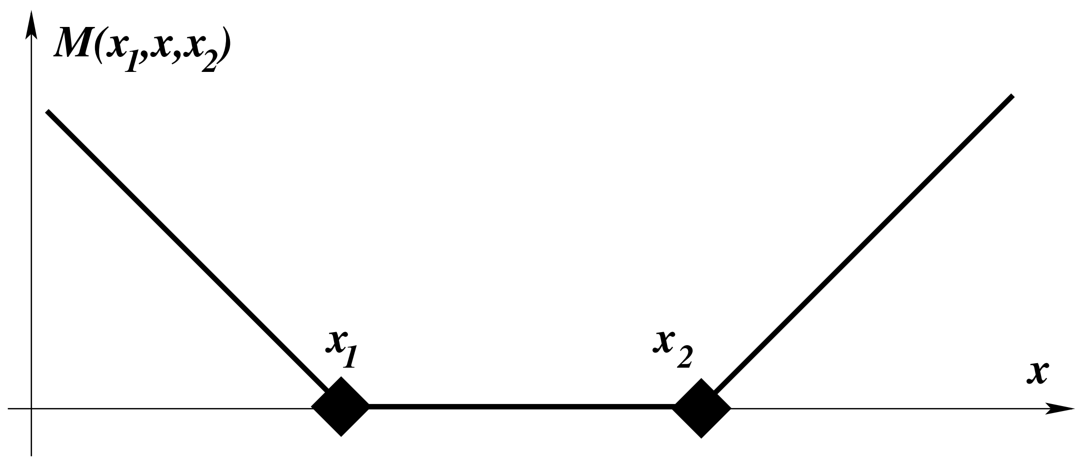
For $x, y \in \mathbb{R}$ denote the segment

$$[\![x, y]\!] := \{z \in \mathbb{R}\colon\ z = x + c(y - x),\ c \in [0, 1]\}.$$

M_1-oscillation function $M\colon \mathbb{R}^3 \to [0, \infty)$,

$$M(x_1, x, x_2) := \begin{cases} \min\{|x - x_1|, |x_2 - x|\}, & \text{if } x \notin [\![x_1, x_2]\!], \\ 0, \quad x \in [\![x_1, x_2]\!]. \end{cases}$$

$M(x_1, x, x_2) =$ euclidean distance between the point x and the segment $[\![x_1, x_2]\!]$.



15. M_1-tighness criterium

Tightness of $\{AX^\varepsilon\}_{\varepsilon>0}$ in $D([0,\infty),\mathbb{R};M_1)$:

1. Boundedness: For every $T>0$ and $K>0$

$$\lim_{K\to\infty}\sup_{\varepsilon>0}\mathbf{P}\Big(\sup_{t\in[0,T]}|AX_t^\varepsilon|>K\Big)=0$$

2. M_1-oscillations: For every $T>0$ and $\Delta>0$

$$\lim_{\delta\downarrow 0}\limsup_{\varepsilon\to 0}\mathbf{P}\Big(\sup_{\substack{0\leqslant t_1<t<t_2\leqslant T,\\ t_2-t_1\leqslant\delta}} M(AX_{t_1}^\varepsilon,AX_t^\varepsilon,AX_{t_2}^\varepsilon)>\Delta\Big)=0,$$

16. Idea of the proof I

1. Boundedness: straightforward.

2. M_1-oscillations: decompose

$$L_t = \xi_t + Z_t,$$

ξ_t : zero-mean martingale with small jumps and $\mathbf{P}\Big(\sup_{t\in[0,T]} |\xi_t| > \dfrac{\Delta}{4}\Big) \leqslant \theta$

Z_t : compound Poisson process with drift

Linearity of equations:

$$\begin{aligned} AX_t^\varepsilon &= AX_t^{\varepsilon,\xi} + AX_t^{\varepsilon,Z} \\ &:= \int_0^t \big(1 - \mathrm{e}^{-\frac{t-s}{\varepsilon^\alpha}A}\big)\,\mathrm{d}\xi_s + \int_0^t \big(1 - \mathrm{e}^{-\frac{t-s}{\varepsilon^\alpha}A}\big)\,\mathrm{d}Z_s \end{aligned}$$

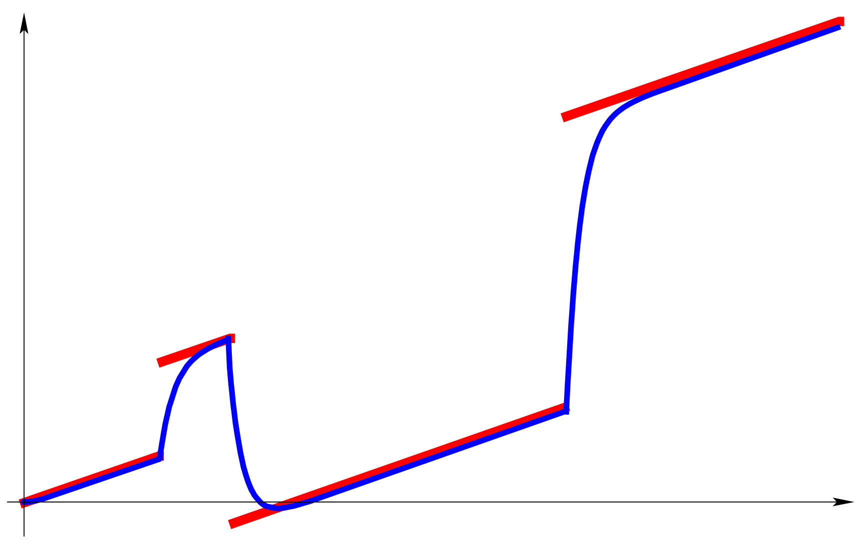
17. Idea of the proof II

Gaussian part: converges in the local uniform metric.

$AX^{\gamma,\xi}$ is small in the local uniform metric.

Control M_1-oscillations of $AX^{\varepsilon,Z}$

$$\sup_{\substack{0\leqslant t_1<t<t_2\leqslant T,\\ t_2-t_1\leqslant\delta}} M(AX^{\varepsilon,Z}_{t_1}, AX^{\varepsilon,Z}_{t}, AX^{\varepsilon,Z}_{t_2})$$



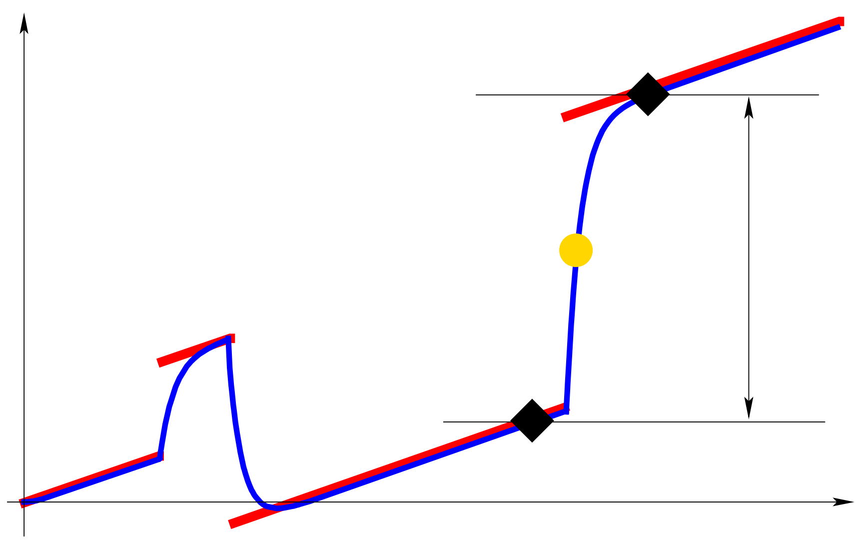
18. Idea of the proof II

Gaussian part: converges in the local uniform metric.

$AX^{\gamma,\xi}$ is small in the local uniform metric.

Control M_1-oscillations of $AX^{\varepsilon,Z}$

$$M(AX^{\varepsilon,Z}_{t_1}, AX^{\varepsilon,Z}_{t}, AX^{\varepsilon,Z}_{t_2}) = 0$$



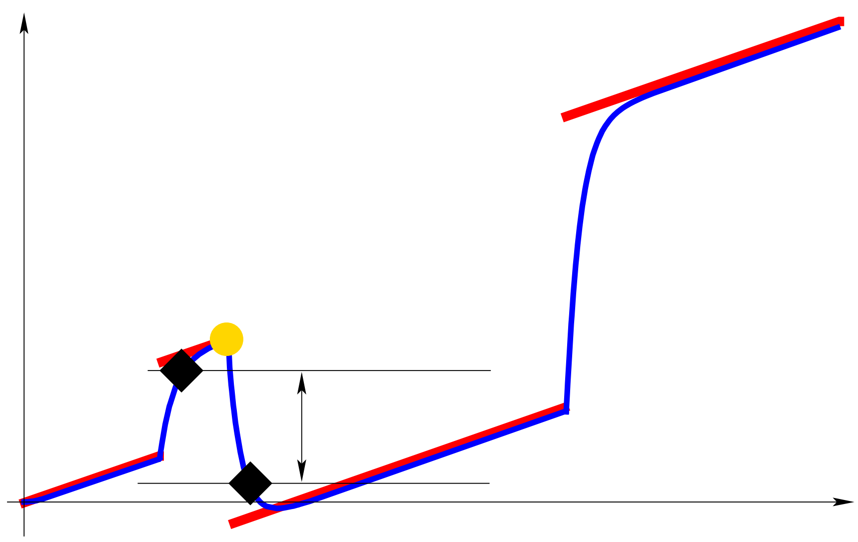
19. Idea of the proof II

Gaussian part: converges in the local uniform metric.

$AX^{\gamma,\xi}$ is small in the local uniform metric.

Control M_1-oscillations of $AX^{\varepsilon,Z}$

$$M(AX_{t_1}^{\varepsilon,Z}, AX_t^{\varepsilon,Z}, AX_{t_2}^{\varepsilon,Z}) \leqslant |AX_t^{\varepsilon,Z} - AX_{t-\delta}^{\varepsilon,Z}|$$



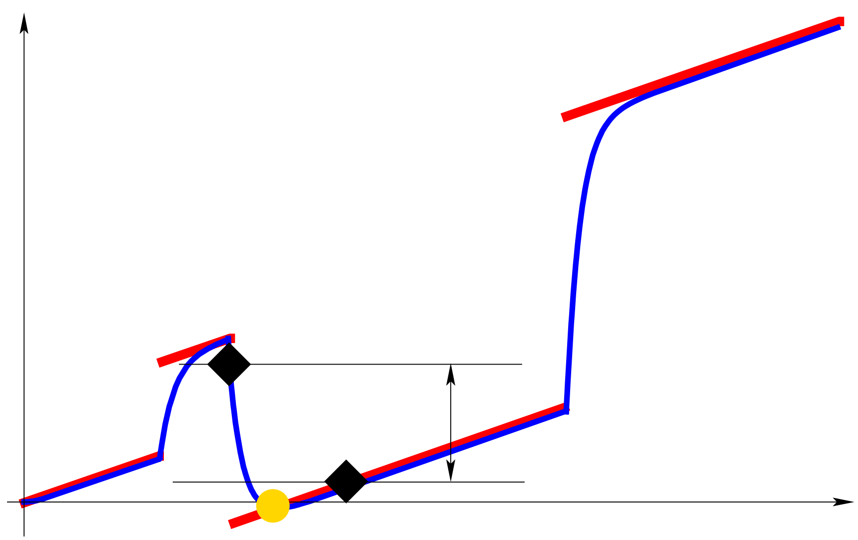
20. Idea of the proof II

Gaussian part: converges in the local uniform metric.

$AX^{\gamma,\xi}$ is small in the local uniform metric.

Control M_1-oscillations of $AX^{\varepsilon,Z}$

$$M(AX^{\varepsilon,Z}_{t_1}, AX^{\varepsilon,Z}_{t}, AX^{\varepsilon,Z}_{t_2}) \leqslant |AX^{\varepsilon,Z}_{t} - AX^{\varepsilon,Z}_{t+\delta}|$$



21. M_1-convergence in $\mathbb{R}^1$

Theorem 2. Let L be a one-dimensional α-stable Lévy process, $\alpha \in (0,2)$, and let X^ε be an integrated OU-process with zero initial conditions. Then

$$AX^\varepsilon \xrightarrow{\mathbf{P}} L \text{ in } D([0,\infty),\mathbb{R};M_1) \text{ as } \varepsilon \to 0.$$

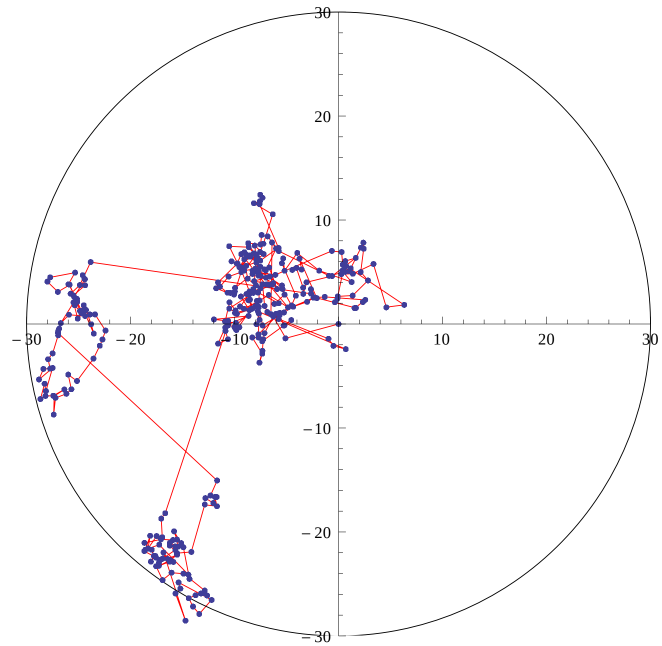
Theorem 3. Let $l^{(\alpha)} = (l_t^{(\alpha)})_{t\geqslant 0}$ be a one-dimensional α-stable Lévy process, $\alpha \in (0,2)$, and let x^ε be the integrated OU process with zero initial conditions. Then

$$\Big(Ax^\varepsilon_{\frac{t}{\varepsilon^\alpha}}\Big)_{t\geqslant 0} \Rightarrow (l_t^{(\alpha)})_{t\geqslant 0} \quad \text{in } D([0,\infty),\mathbb{R};M_1) \text{ as } \varepsilon \to 0.$$

Corollary. Let $l^{(\alpha)}$ be a one-dimensional α-stable process with $\limsup_{t\to\infty} l_t^{(\alpha)} = +\infty$ a.s. Then for any $a > 0$

$$\varepsilon^\alpha \tau_a(x^\varepsilon) \xrightarrow{d} \tau_{\frac{a}{A}}(l^{(\alpha)}) \text{ as } \varepsilon \to 0.$$

22. SM_1-convergence in $\mathbb{R}^2$

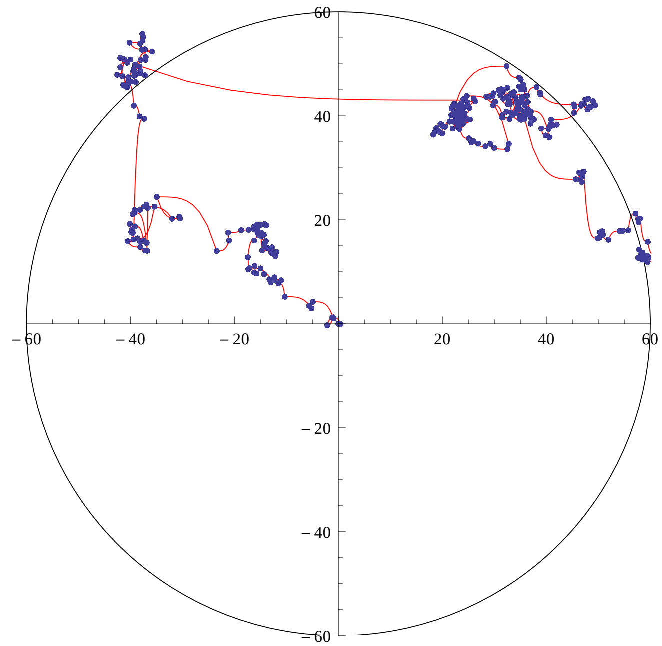


$$X_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t V_s^\varepsilon\, ds$$

$$V_t^\varepsilon = -\frac{1}{\varepsilon^\alpha}\int_0^t AV_s^\varepsilon\, ds + L_t, \qquad A = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}$$

$$AX_t^\varepsilon \overset{SM_1}{\Rightarrow} L$$

23. WM_1-convergence in $\mathbb{R}^2$

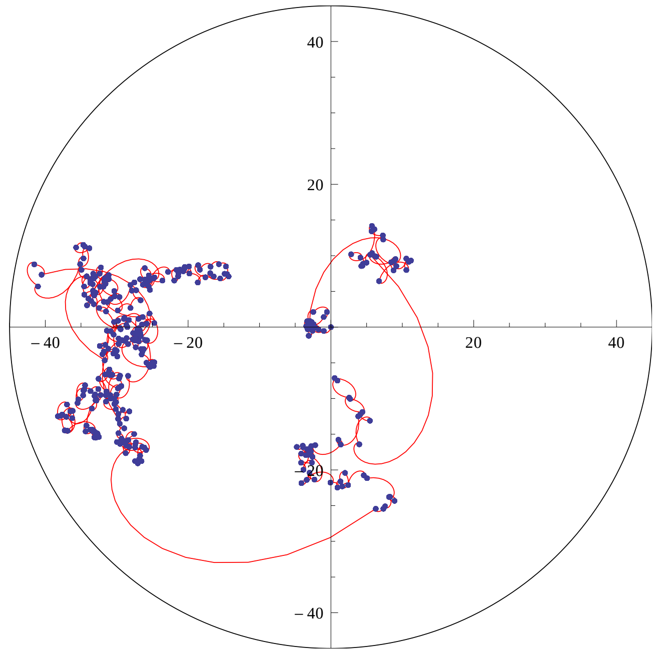


$$X_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t V_s^\varepsilon \, ds$$

$$V_t^\varepsilon = -\frac{1}{\varepsilon^\alpha}\int_0^t AV_s^\varepsilon \, ds + L_t, \qquad A = \begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix}, \quad \nu, \mu > 0, \quad \nu \neq \mu$$

$$AX_t^\varepsilon \overset{WM_1}{\Rightarrow} L$$

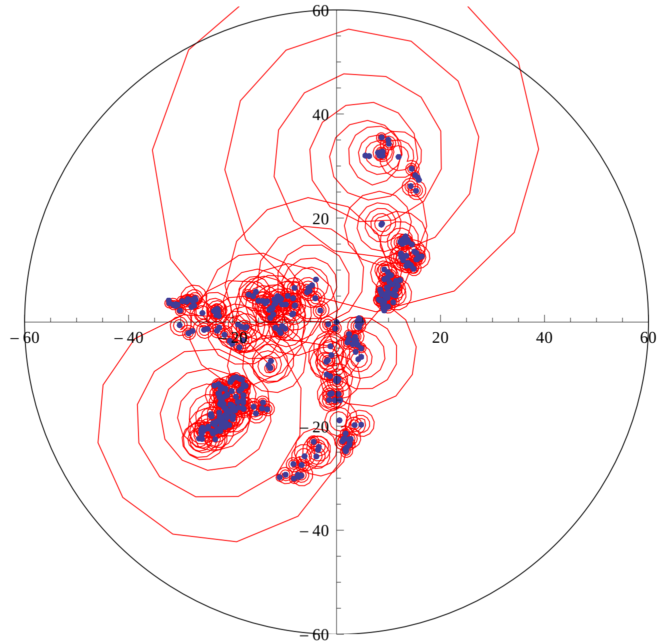
24. No good convergence in $\mathbb{R}^2$



$$X_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t V_s^\varepsilon \, ds$$

$$V_t^\varepsilon = -\frac{1}{\varepsilon^\alpha}\int_0^t A V_s^\varepsilon \, ds + L_t, \qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \lambda_{1,2} = 1 \pm \mathrm{i}$$

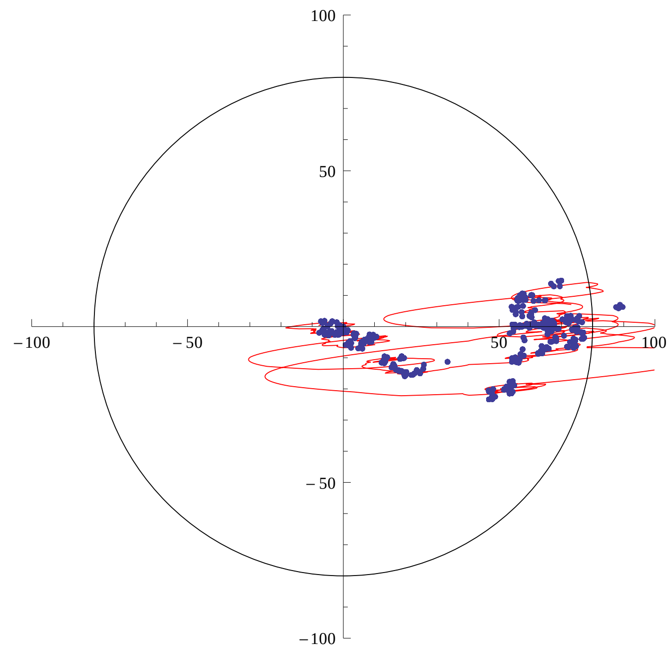
25. Even worse, $\mathbb{R}^2$



$$X_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t V_s^\varepsilon\, ds$$

$$V_t^\varepsilon = -\frac{1}{\varepsilon^\alpha}\int_0^t A V_s^\varepsilon\, ds + L_t, \qquad A = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}, \quad \lambda_{1,2} = 1 \pm 3\mathrm{i}$$

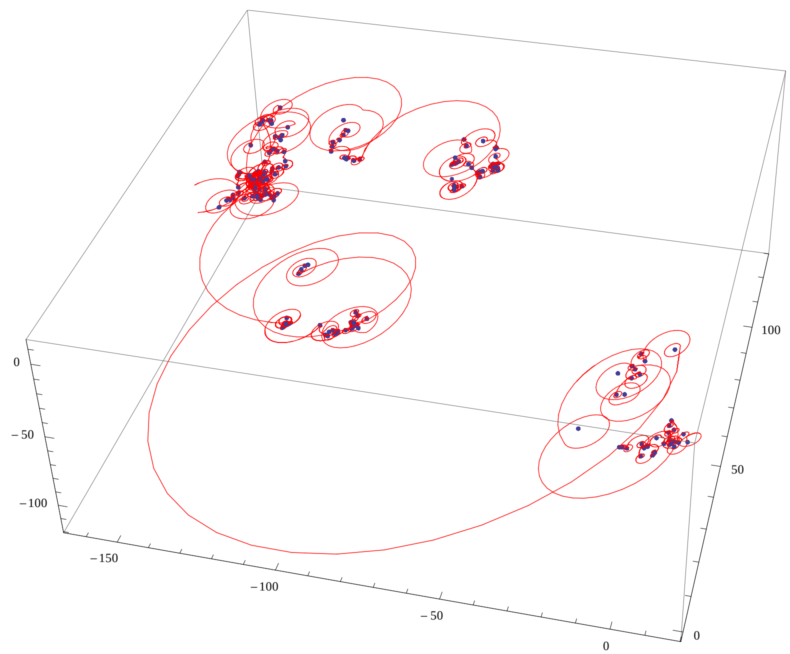
26. Real eigenvalues, $\mathbb{R}^2$



$$X_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t V_s^\varepsilon\, ds$$

$$V_t^\varepsilon = -\frac{1}{\varepsilon^\alpha}\int_0^t AV_s^\varepsilon\, ds + L_t, \qquad A = \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}, \quad \nu > 0.$$

27. External magnetic field, $\mathbb{R}^3$



$$X_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t V_s^\varepsilon\, ds, \qquad -AV = -\nu V + [V \times \mathbf{B}],$$

$$V_t^\varepsilon = -\frac{1}{\varepsilon^\alpha}\int_0^t AV_s^\varepsilon\, ds + L_t, \quad A = \begin{pmatrix} \nu & -B_3 & B_2 \\ B_3 & \nu & -B_1 \\ -B_2 & B_1 & \nu \end{pmatrix}, \ \nu = 1, \mathbf{B} = (2, -3, 1)$$

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