Ergodic Properties of Stochastic Curve Shortening Flows[1]

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Joint works with

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Agenda

- Introduction to (Stochastic) Mean Curvature Flow
- Well-Posedness in 1+1 D ('Stochastic Curve Shortening Flow')
- Long Term Behavior for *Homogeneous Normal Noise*
- Ergodicity & Polynomial Stability for *Additive Vertical Noise*

Mean Curvature Flow

Definition

Let $t \to M_t \subset \mathbb{R}^d$ be a family of (d-1)-dim. Submanifolds, then (M_t) evolves according to MCF if

$$\langle \dot{M}_t, \nu_{M_t} \rangle (p) = -\kappa_{M_t}(p) \quad \forall t \in [0, T], p \in M_t$$

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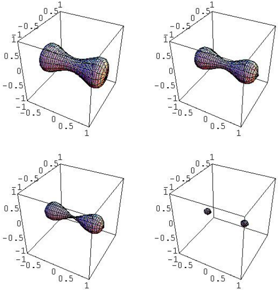

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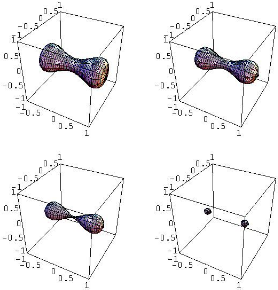

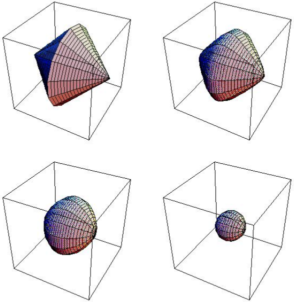

MCF - Modelling Aspects

• Gradient Flow Structure

$$\mathcal{M} = \{\ (d-1)\text{-dim. submanifolds } \Sigma \subset \mathbb{R}^d\ \}$$

Riem. structure

$$T_M\Sigma = \{V : M \to \mathbb{R}^d\}, \quad \|V\|^2_{T_M} = \int_M V^2(x)\,d\sigma_M(x)$$

$$\text{MCF} \iff \dot{\Sigma} = -\nabla\Phi(\Sigma), \quad \Phi(S) = |S|$$

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• Sharp Interface Limit of Allen-Cahn Eqaution

$$d\varphi^\epsilon = \Delta\varphi^\epsilon - \frac{1}{\epsilon^2} F'(\varphi^\epsilon),$$

where

$$F(\varphi) = (1-\varphi^2)^2 \text{ Double-Well Potential.}$$

Then

$$\varphi^\epsilon \stackrel{\epsilon \to 0}{\longrightarrow} \chi_\Omega, \qquad M_t := \partial\Omega_t \text{ solves MCF}$$

MCF - Deterministic Theory

- Level Set PDE for $U = U(x,t)$ s.th. $M_t = \{U(.,.t) = 0\}$
  - General case

$$dU = |\nabla U| \text{ div} \Big(\frac{\nabla U}{|\nabla U|}\Big)$$

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• Properties

- Qausi-linear, degenerate parabolic PDE
- No 'abstract' functional analysis theory
- Weak maximum principle $\rightsquigarrow$ viscosity solutions
- No uniqueness in general, 'fattening phenomenon'

Basic Model of Stochastic MCF

MCF perturbed by random flow

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where $W(x,t)$ Gaussian time-dependent random field.

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Basic Models in 1+1 D

1) Vertical Noise Model, Dirichlet BC

$$du = \frac{\partial_x^2 u}{1 + (\partial_x u)^2}\, dt + \sum_{i=1}^{\infty} \phi_i(., u(.))\, db_t^i. \qquad (*)$$

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Noise coefficients: $\phi_i : [0,1] \times \mathbb{R} \to \mathbb{R}$, $\phi(0,z) = \phi(1,z) = 0$.

$$\sum_{i=1}^{\infty} (\phi_i(z_1) - \phi_i(z_2))^2 \leq \Lambda^2 |z_1 - z_2|^2 \qquad (K)$$

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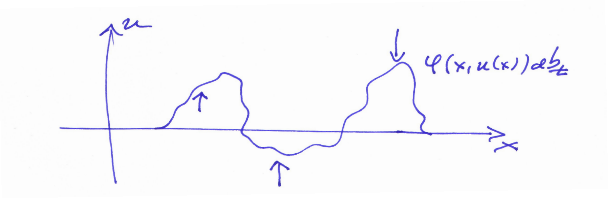

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2) Homogeneous Normal Noise Model, (Periodic BC)

$$du(x) = \frac{\partial_x^2 u}{1+(\partial_x u)^2}(x)\, dt + \epsilon\sqrt{1+(\partial_x u)^2}\varphi(x, u(x)) \circ db_t$$

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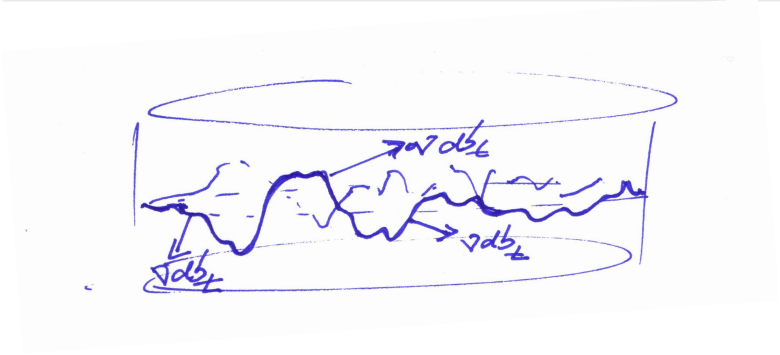

Well-Posedness for Vertical Noise Model, (Dirichlet BC)

Theorem (Strong Solution)

If $u_0 \in H_0^1([0,1])$ and regularity condition (K) holds, then there exists a unique variational strong solution of

$$u(t) = u(0) + \int_0^t A(\bar{u}(s))\, ds + \int_0^t \sigma(\bar{u}(s))\, dW_s, \quad t \in [0, T],$$

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$$A : H_0^1([0,1]) \to H^{-1}([0,1])$$

$${}_{H^{-1}}\langle Au, v\rangle_{H_0^1} = -\int_0^1 \arctan(u')(x) \cdot v'(x) dx = \int_0^1 \frac{u''}{1+(u')^2} \cdot v(x) dx$$

and

$$\sigma(u(s))(.) = \sum_i \phi(., u(s,.)) \in H^{-1}([0,1]).$$

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Theorem (Generalized Markov Solution)

The familiy of solutions $((u_t)_{t \geq 0}, u_0 \in H^1_0([0,1]))$ induces a unique Feller process on $L^2([0,1])$.

Aspects of Proof - No Coercivity

- Variational frame work

$$V \subset H \subset V^* \Leftrightarrow H_0^1([0,1]) \subset L^2([0,1)] \subset H^{-1}$$

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- Degeneracy of MCF-Operator: $A : H^1 \to H^{-1} \rightsquigarrow$ *no coercivity*

$${}_{H^{-1}}\langle Au, u\rangle_{H_0^1} \simeq -\int_{[0,1]} |\partial_x u| dx = -\|u\|_{1,1},$$

$$2\ {}_{V^*}\langle Au, u\rangle_V + \|\sigma(u)\|^2_{L_2(U,H)} \not\lesssim c_2\|u\|_H^2 - c_4\|u\|_V^\alpha, \quad \forall v \in V$$

Aspects of Proof - Lyapunov Function

- *Remedy:*

$$\langle Au, u\rangle_{H_0^1} = -\int_{[0,1]} \frac{\partial_x^2 u}{1+(\partial_x u)^2}\partial_x^2 u(x)dx \leq 0 \quad \forall u \in C_0^\infty(]0,1[).$$

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- A satisfies an alternative *Lyapunov-type condition:* For $n \in \mathbb{N}$, the operator A maps $span\{e_1, \ldots, e_n\} \subset V$ into V and $c_2 \in \mathbb{R}$ such that

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$\rightsquigarrow$ Compactness in L^2 of spectral Galerkin-Approximations.
Conclude by standard monotonicity arguments

Well-Posedness for Vertical Noise Model, Periodic BC

Stratonovich Formulation

$$du(x) = \frac{\partial_x^2 u}{1+(\partial_x u)^2}(x)\,dt + \epsilon\sqrt{1+(\partial_x u)^2} \circ db_t,$$

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Theorem (Strong and Generalized Solutions)

i) For $\epsilon \leq \sqrt{2}$ *and* $u \in \tilde{H}_0^1([0,1])$ *the Ito-model has a unique strong variational solution.*
ii) The familiy of solutions $((u_t)_{t \geq 0}, u_0 \in \tilde{H}_0^1([0,1]))$ *induces a unique Feller process on* $L^2([0,1])$.

Long-Term Behavior of Normal Noise Model

Theorem

For $u_0 \in \tilde{H}^1([0,1])$ *and* $\epsilon \leq \sqrt{2}$ *then*

$$(u^T)_{t\geq 0} := (u_{T+t} - u_T)_{t\geq 0} \overset{T\to\infty}{\Longrightarrow} \epsilon(1\cdot\beta_t)_{t\geq 0} \text{ on } C(\mathbb{R}_{\geq 0}, L^2([0,1])),$$

where β *is standard real Brownian motion.*

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Proof.

$$\tilde{u} = u - [u], \quad [u] = \int_{[0,1]} u dx$$

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Proof.

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$$d\tilde{u}(x) = \Big(\frac{\epsilon^2}{2}\partial_x^2\tilde{u} + (1 - \frac{\epsilon^2}{2})\partial_x \arctan(\partial_x\tilde{u})\Big)(x)dt$$
$$+ \epsilon\left(\sqrt{1+(\partial_x\tilde{u}(x))^2} - [\sqrt{1+(\partial_x\tilde{u}(x))^2}]\right)db_t$$

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$$d[u] = \epsilon\left[\sqrt{1 + (\partial_x\tilde{u}(x))^2}\,\right]db_t$$

□

Ergodicity in the Additive Noise Case

Model

$$du = \frac{\partial_x^2 u}{1+(\partial_x u)^2}dt + QdW_t, \quad u(0) = u_0 \in H_0^{1,2}([0,1]),$$

where $W \triangleq$ cyl. White noise on some U and $Q \in L_2(U, H_0^{1,2}([0,1]))$.
Example: $U = L^2([0,1])$ and $Q = (-\Delta)^{-\beta}$ for $\beta > 3/4$.

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Theorem - Ergodicity

The semigroup Let $(P_t)_{t\geq 0}$ on $L^2([0,1])$ corresponding to $\hat{u}$ is ergodic:
$\exists!\ \nu \in \mathcal{M}_1(L^2([0,1]))$ s.th.

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \langle P_t \varphi, \mu \rangle = \langle \varphi, \nu \rangle$$

for any $\mu \in \mathcal{M}_1(L^2([0,1]))$ and $\varphi : L^2([0,1]) \mapsto \mathbb{R}$ bdd. cont.

Proof of Ergodicity – 'Lower Bound Technique'

- L^2-compactness of time-averages:
  ${}_{H^{-1}}\langle Av, v\rangle_{H^1} = -\int_0^1 \arctan(\partial_x v) \cdot \partial_x v \, dx \leq -c\|v\|_{W^{1,1}(0,1)} + \alpha$

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  - But weakly dissipative/non-expanding (*e-property*).

$$|P_t\varphi(x) - P_t\varphi(y)| \leq \mathrm{Lip}(\varphi)\|x-y\| \quad \forall x, y \in L^2.$$

Proof of Ergodicity – 'Lower Bound Technique'

- L^2-compactness of time-averages:
  $_{H^{-1}}\langle Av, v\rangle_{H^1} = -\int_0^1 \arctan(\partial_x v)\cdot\partial_x v\, dx \leq -c\|v\|_{W^{1,1}(0,1)} + \alpha$
- Uniqueness of the limit:
  - Degeneracy of MCF-Operator: SPDE is not 'strongly dissipative' (cf. da Prato/Zabczyk).
  - But weakly dissipative/non-expanding (*e-property*).

$$|P_t\varphi(x) - P_t\varphi(y)| \leq \mathrm{Lip}(\varphi)\|x-y\| \quad \forall x, y \in L^2.$$

  - $\rightsquigarrow$ suffices to check *lower-bound-condition:*
    (cf. [Komorowski/Peszat/Szarek] AOP 2010):
    For all $\delta > 0$ and every $x \in L^2([0,1])$ and

$$Q^T(x, A) := \frac{1}{T}\int_0^T P_s(x, A)ds$$

    it holds that

$$\boxed{\liminf_{T\to\infty} Q^T(x, B_\delta(0)) > 0.}$$

Polynomial Mixing

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Theorem

- The inv. measure ν is concentrated on

$$\left\{u \in W^{1,1}_{loc}(0,1) \,\middle|\, (\arctan(u_x))_x \in L^2(0,1),\ u_x \in BV(0,1)\right\}$$

and

$$\int |u_x|^{\frac{1}{2}}_{TV}\,\nu(du) + \int \|u\|^{\frac{1}{2}}_{H^1}\,\nu(du) + \int \|(\arctan u_x)_x\|^2_{L^2}\,\nu(du) < \infty.$$

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- For two sol. (u_t), (v_t) with $u_0, v_0 \in H^{1,2}_0(0,1)$

$$\mathbb{E}(\|u_t - v_t\|^{\frac{1}{2}}_{L^2}) \leq C \cdot t^{-\frac{1}{4}} \|u_0 - v_0\|^{\frac{1}{2}}_{L^2} \times \left(1 + \mathbb{E}\left(\frac{1}{t}\int_0^t \|u_s\|^{\frac{1}{2}}_{H^{1,2}_0}\, ds + \frac{1}{t}\int_0^t \|v_s\|^{\frac{1}{2}}_{H^{1,2}_0}\, ds\right)\right).$$

Proof – Three Basic Lemmas

Lemma 1

$$\left(\int_{[0,1]} |u_{xx}(x)|\, dx\right)^{\frac{1}{2}} \leq \frac{3}{2} + \frac{1}{2}\int_{[0,1]} \frac{(u_{xx}(x))^2}{1+(u_x(x))^2}\, dx + \frac{1}{2}\int_{[0,1]} |u_x|\, dx$$

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$${}_{H^{-1}}\langle V(u) - V(v), u - v\rangle_{H^1} \leq -\frac{1}{\left(1 + \|u\|_{H^1}^2 + \|v\|_{H^1}^2\right)} \|u - v\|_{L^2}^2 .$$

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Lemma 3

Let $f, g : \mathbb{R}_{\ge 0} \mapsto \mathbb{R}_{\ge 0}$ such that $f'(t) \le -\frac{f}{g}$. Then for $\alpha \in (0,1]$

$$f(t) \le c_\alpha \left(\frac{1}{t}\right)^\alpha \left(\frac{1}{t}\int_0^t g^\alpha(s)ds\right) f(0).$$

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