A new non-Markovian approach to weak convergence for SPDEs

Adam Andersson
Joint work with
Raphael Kruse and Stig Larsson

Mathematical Sciences
Chalmers University of Technology
Göteborg, Sweden

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Outline

- Stochastic integration in Hilbert space,

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- Strong convergence in a dual Watanabe-Sobolev norm.

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$W \colon [0, T] \times U_0 \to L_2(\Omega)$ cylindrical Q-Wiener process:

$$W(t)u := I(\chi_{[0,t]} \otimes u) = \sum_{i=1}^{\infty} \langle u, u_i\rangle_0 \beta_i(t),$$

where $(u_i)_{i\in\mathbf{N}} \subset U_0$ is an ON-basis and $(\beta_i)_{i\in\mathbf{N}}$ are independent standard Brownian motions.

The H-valued Wiener integral

Wiener integral for simple integrands:

$$\int_0^T \chi_{[s,t]} \otimes (h \otimes u) \, \mathrm{d}W = [(W(t) - W(s))u] \otimes h \in L_2(\Omega) \otimes H = L_2(\Omega, H)$$

Extends directly to linear combinations.

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By density the integral extends to all of $L_2([0, T], \mathcal{L}_2^0)$. For stochastic equations driven by additive noise this definition of the integral suffices.

Malliavin calculus

Let $C_{\mathrm{p}}^{\infty}(\mathbf{R}^n)$ denote the space of all C^{∞}-functions over $\mathbf{R}^n$ with polynomial growth. Define

$$\mathcal{S} = \{X = f(I(\phi_1), \dots, I(\phi_n))\colon f \in C_{\mathrm{p}}^{\infty}(\mathbf{R}^n), \\ \phi_1, \dots, \phi_n \in L_2([0, T], U_0),\ n \geq 1\}$$

and

$$\mathcal{S}(H) = \Big\{F = \sum_{k=1}^{n} X_k \otimes h_k \colon X_1, \dots, X_n \in \mathcal{S},\ h_1, \dots, h_n \in H,\ n \geq 1\Big\}.$$

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We define the Malliavin derivative of $F \in \mathcal{S}(H)$ as the process

$$D_t F = \sum_{k=1}^{m} \sum_{i=1}^{n} \partial_i f_k(I(\phi_1), \ldots, I(\phi_n)) \otimes (h_k \otimes \phi_i(t))$$

and let, for $v \in U_0$,

$$D_t^v F = D_t F v = \sum_{k=1}^{m} \sum_{i=1}^{n} \partial_i f_k(I(\phi_1), \ldots, I(\phi_n)) \otimes \langle \phi_i(t), v \rangle_0 \otimes h_k$$

Malliavin calculus: integration by parts

For all $F \in \mathcal{S}(H)$ and $\Phi \in L_2([0,T], \mathcal{L}_2^0)$,

$$\langle DF, \Phi \rangle_{L_2([0,T]\times\Omega, \mathcal{L}_2^0)} = \left\langle F, \int_0^T \Phi(t)\,\mathrm{d}W(t) \right\rangle_{L_2(\Omega,H)}.$$

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Let $\mathbf{D}^{1,p}(H)$ be the closure of $\mathcal{S}(H)$ with respect to the norm

$$\|F\|_{\mathbf{D}^{1,p}(H)} = \Big(\mathbf{E}[\|F\|_H^p] + \mathbf{E}\Big[\int_0^T \|D_t F\|_{\mathcal{L}_2^0}^p \,\mathrm{d}t \Big] \Big)^{\frac{1}{p}}.$$

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Let $(\delta, \mathcal{D}(\delta))$ be the adjoint of $D\colon L_2(\Omega, H) \to L_2([0,T] \times \Omega, \mathcal{L}_2^0)$.

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$\mathcal{D}(\delta) \subset L_2([0,T] \times \Omega, \mathcal{L}_2^0)$ is large and contains in particular all predictable $\mathcal{L}_2^0$-valued processes. In this case $\delta(\Phi) = \int_0^T \Phi(t)\,\mathrm{d}W(t)$.

The stochastic equation

An easy equation for a difficult problem:

$$\begin{aligned} &\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(X(t))\,\mathrm{d}t + \mathrm{d}W(t), \quad t \in (0, T], \\ &X(0) = X_0. \end{aligned}$$

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- $F \in \mathcal{C}_{\mathrm{b}}^2(H, H)$.
- $X_0 \in H$.

The stochastic equation

There exist for every $p \geq 2$ a unique solution $X \in \mathcal{C}([0,T], L_p(\Omega, H))$ satisfying the integral equation

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))\,\mathrm{d}s + \int_0^t S(t-s)\,\mathrm{d}W(s), \quad t \in [0,T].$$

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Spatial regularity [Kruse, Larsson]:

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Regularity in the Malliavin sense [Fuhrman, Tessitore]:

$X(t) \in \mathbf{D}^{1,p}(H)$ for almost all $t \in [0,T]$ and $p < \frac{2}{1-\beta}$.

Approximation by the finite element method

A discretized equation:

$$\begin{cases} \mathrm{d}X_h(t) + [A_h X_h(t) - P_h F(X_h(t))]\,\mathrm{d}t = P_h\,\mathrm{d}W(t), & t \in (0, T] \\ X_h(0) = P_h X_0. \end{cases}$$

Finite element spaces $(V_h)_{h\in(0,1]}$ of continuous piecewise linear functions corresponding to a quasi-uniform family of triangulations of D.

A_h is the discrete Laplacian satisfying

$$\langle A_h \psi, \chi \rangle_H = \langle \nabla \psi, \nabla \chi \rangle_H, \quad \forall \psi, \chi \in V_h.$$

$P_h \colon H \to V_h$ orthogonal projection w.r.t. $\langle \cdot, \cdot \rangle_H$.

Mild solution of spatially discretized equation

Let $(S_h(t))_{t\geq 0}$ be the analytic semigroup generated by $-A_h$.

For every $h \in (0,1]$ $\exists!$ solution $X_h \in C([0,T], L_2(\Omega, S_h))$ to the mild equation

$$\begin{aligned} X_h(t) = S_h(t)P_h X_0 + \int_0^t S_h(t-s)P_h F(X_h(s))\,\mathrm{d}s \\ + \int_0^t S_h(t-s)P_h\,\mathrm{d}W(s), \quad t \in (0,T]. \end{aligned}$$

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Error estimate for $E_h(t) = S(t) - S_h(t)P_h$:

$$\|E_h(t)A^{\frac{\varrho}{2}}\|_{\mathcal{L}} \leq Ct^{-\frac{\varrho+\theta}{2}}h^{\theta}, \quad 0 \leq \theta \leq 2,\ 0 \leq \varrho \leq 1,\ \varrho+\theta \leq 2.$$

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Strong convergence:

$$\|X(T) - X_h(T)\|_{L_p(\Omega,H)} \leq Ch^{\beta-\epsilon}, \quad n \in \mathbf{N}.$$

Weak convergence: Results

Theorem

For every $\gamma \in [0, \beta)$ the following weak convergence holds:

$$\left|\mathbf{E}\left[\varphi(X(T)) - \varphi(X_n(T))\right]\right| \leq Ch^{2\gamma}, \quad h \in (0,1).$$

under either of the following assumptions:

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- *Additive noise,* $\beta \in [0,1]$ *and* $\varphi \in \mathcal{C}^2_{\mathrm{b}}(H, \mathbf{R})$ *(FEM) [A., Larsson, 2012], on ArXiv.*

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- *Linear multiplicative noise, $\beta = 1$ and $\varphi \in \mathcal{C}_{\mathrm{b}}^2(H, \mathbf{R})$ (Spectral) [A., Jentzen, Larsson, Schwab, 2014], writing in progress.*

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- *Additive noise,* $\beta \in [0,1]$ *and* $\varphi \in \mathcal{C}_{\mathrm{b}}^2(H, \mathbf{R})$ *(FEM) [A., Larsson, 2012], on ArXiv.*
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Weak convergence: Results

Theorem

For every $\gamma \in [0, \beta)$ the following weak convergence holds:

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Open question: Is the rate of weak convergence the same for all $G \in \mathcal{C}_{\mathrm{b}}^2(H, \mathcal{L}_2^0)$?

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Here I present the third method!

Proof: Important spaces

Let $p \geq 2$. We define the space

$$\mathbf{M}^{1,p}(H) = \mathbf{D}^{1,p}(H) \cap L_{2p}(\Omega, H),$$

with norm

$$\|X\|_{\mathbf{M}^{1,p}(H)} = \max(\|X\|_{\mathbf{D}^{1,p}(H)}, \|X\|_{L_{2p}(\Omega,H)}).$$

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The dual space $\mathbf{M}^{1,p}(H)^*$ is equipped with the norm

$$\|X\|_{\mathbf{M}^{1,p}(H)^*} = \sup_{\Upsilon \in B} \langle \Upsilon, X \rangle_{L_2(\Omega,H)},$$

where B denote the unit ball in $\mathbf{M}^{1,p}(H)$.

Proof: Bound of the weak error

Linearization: By a first order Taylor expansion

$$\mathbf{E}\big[\varphi(X(T)) - \varphi(X_n(T))\big] = \mathbf{E}\langle \varphi'(X(T)), X_n(T) - X(T)\rangle$$
$$+ \int_0^1 (1-\varrho)\varphi''(X(T) + \lambda(X_n(T) - X(T))) \cdot (X(T) - X_n(T))^2 \,\mathrm{d}\varrho.$$

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For $p < \frac{2}{1-\beta}$: $R = \|\varphi'(X(T))\|_{\mathbf{M}^{1,p}(H)} < \infty$.

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Therefore

$$\big|\mathbf{E}\big[\varphi(X(T)) - \varphi(X_n(T))\big]\big| \leq R\big|\mathbf{E}\langle R^{-1}\varphi'(X(T)), X_n(T) - X(T)\rangle\big|$$
$$+ \|\varphi''(X(T))\|_{L_2(\Omega,\mathcal{L}^{[2]}(H,\mathbf{R}))}\|X(T) - X_n(T)\|^2_{L_4(\Omega,H)}$$
$$\leq R \sup_{\Upsilon\in B} \mathbf{E}\langle\Upsilon, X_n(T) - X(T)\rangle\big| + C\|X(T) - X_n(T)\|^2_{L_4(\Omega,H)}.$$

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Therefore

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Thus,

$$\big|\mathbf{E}\big[\varphi(X(T)) - \varphi(X_n(T))\big] \lesssim \|X_n(T) - X(T)\|_{\mathbf{M}^{1,p}(H)^*} + \|X(T) - X_n(T)\|^2_{L_4(\Omega,H)}.$$

Proof: Key Lemma

Lemma

Let $p, p' \in (1,\infty)$ *satisfy* $\frac{1}{p} + \frac{1}{p'} = 1$.

(i) *For random variables* $Z\colon \Omega \to H$, *we have*

$$\|Z\|_{\mathbf{M}^{1,p}(H)^*} \leq \|Z\|_{L_2(\Omega,H)}.$$

(ii) *If for* $\Phi\colon [0,T]\times\Omega \to \mathcal{L}$ *the map* $\Upsilon \mapsto \Phi(t)^*\Upsilon$ *is bounded in* $\mathbf{M}^{1,p}(H)$ *uniformly in* t, *then under mild assumptions on* ψ

$$\Big\| \int_0^T \Phi(t,\phi(t))\psi(t)\,\mathrm{d}t \Big\|_{\mathbf{M}^{1,p}(H)^*} \leq R \int_0^T \|\psi(t)\|_{\mathbf{M}^{1,p}(H)^*}\,\mathrm{d}t.$$

(iii) *If* $\Phi \in L_2([0,T]\times\Omega, \mathcal{L}_2^0)$ *is predictable, then*

$$\Big\| \int_0^T \Phi(t)\,\mathrm{d}W(t) \Big\|_{\mathbf{M}^{1,p}(H)^*} \leq C\|\Phi\|_{L_{p'}([0,T]\times\Omega,\mathcal{L}_2^0)}.$$

Proof: Strong convergence in the $\mathbf{M}^{1,p}(H)^*$-norm

After a first order Taylor expansion the difference satisfy the equation:

$$
\begin{aligned}
&X(T) - X_h(T) \\
&\quad = E_h(t)X_0 + \int_0^T E_h(T-s)F(X(t))\,\mathrm{d}t \\
&\qquad + \int_0^T S_h(T-t)P_h F'(X(t))\big(X(t)-X_h(t)\big)\,\mathrm{d}t \\
&\qquad + \int_0^T S_h(T-t)P_h \\
&\qquad\quad \times \int_0^1 (1-\varrho)F''(X(t)+\varrho(X_h(t)-X(t)))\cdot(X(t)-X_h(t))^2\,\mathrm{d}\varrho\,\mathrm{d}t \\
&\qquad + \int_0^T E_h(T-t)\,\mathrm{d}W(t).
\end{aligned}
$$

Proof: Strong convergence in the $\mathbf{M}^{1,p}(H)^*$-norm

By the Key Lemma (i) and (iii)

$$
\begin{aligned}
&\|X(T) - X_h(T)\|_{\mathbf{M}^{1,p}(H)^*} \\
&\quad \le \|E_h(t)X_0\| + \int_0^T \big\|E_h(T-s)F(X(t))\big\|_{L_2(\Omega,H)} \,\mathrm{d}t \\
&\qquad + \Big\| \int_0^T S_h(T-t)P_hF'(X(t))(X(t)-X_h(t)) \,\mathrm{d}t \Big\|_{\mathbf{M}^{1,p}(H)^*} \\
&\qquad + \int_0^T \|X(t)-X_h(t)\|^2_{L_2(\Omega,H)} \,\mathrm{d}t \\
&\qquad + \Big(\int_0^T \|E_h(T-t)\|^{p'}_{\mathcal{L}_2^0} \,\mathrm{d}t \Big)^{\frac{1}{p'}}.
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To apply Key Lemma (ii) we need to check that

$$
\Upsilon \mapsto F'(X(t))^* S_h(T-t)P_h\Upsilon, \quad \text{bounded in } \mathbf{M}^{1,p}(H).
$$

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Clearly $\big(F'(X(t))\big)^* S_h(T-t)\Upsilon \in L_{2p}(\Omega, H)$.

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Remains to prove

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\begin{aligned}
\big\|\big(F'(X(t))\big)^* S_h(T-t)\Upsilon\big\|^p_{\mathbf{D}^{1,p}(H)} &\lesssim \big\|\big(F'(X(t))\big)^* S_h(T-t)\Upsilon\big\|^p_{L_p(\Omega,H)} \\
&\quad + \int_0^T \big\|\big(F'(X(t))\big)^* S_h(T-t) D_s\Upsilon\big\|^p_{L_p(\Omega,\mathcal{L}_2^0)} \,\mathrm{d}s \\
&\quad + \int_0^T \mathbf{E}\Big[\Big(\sum_{k\in\mathbf{N}} \big\|\big(F''(X(t)) D_s^{u_k} X(t)\big)^* S_h(T-t)\Upsilon\big\|^2\Big)^{\frac{p}{2}}\Big] \,\mathrm{d}s \\
&\le |F|^p_{\mathcal{C}^1_{\mathrm{b}}(H,H)} \Big(\|\Upsilon\|^p_{L_p(\Omega,H)} + \int_0^T \|D_s\Upsilon\|^p_{L_p(\Omega,H)} \,\mathrm{d}s\Big) \\
&\quad + |F|^p_{\mathcal{C}^2_{\mathrm{b}}(H,H)} \int_0^T \mathbf{E}\Big(\sum_{k\in\mathbf{N}} \|D_s^{u_k} X(t)\|^2_H\Big)^{\frac{p}{2}} \|\Upsilon\|^p \,\mathrm{d}s \\
&\lesssim \|\Upsilon\|^p_{\mathbf{D}^{1,p}(H)} + \|\Upsilon\|^2_{L_{2p}(\Omega,H)} \sup_{t\in[0,T]} \|X(t)\|^2_{\mathbf{D}^{1,2p}(H)} < \infty.
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Proof: Strong convergence in the $\mathbf{M}^{1,p}(H)^*$-norm

Using Key Lemma (ii) we get

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If we fix $\gamma \in [0, \beta)$ and let $p = 2/(1-\gamma)$, then one can show that

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\|X(T) - X_h(T)\|_{\mathbf{M}^{1,p}(H)^*} \leq{}& (t^{-\gamma} + 1)h^{2\gamma} \\
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Gronwall's Lemma applies and we are done!

Path dependent test functions

Let μ be a Borel measure on $[0, T]$ satisfying

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Then, for $\varphi \in \mathcal{C}_{\mathrm{b}}^2(H, \mathbf{R})$ we compute

$$\begin{aligned}
&\Big|\mathbf{E}\Big[\varphi\Big(\int_0^T X(t)\,\mathrm{d}\mu(t)\Big) - \varphi\Big(\int_0^T X_h(t)\,\mathrm{d}\mu(t)\Big)\Big]\Big| \\
&\quad\le \Big|\mathbf{E}\Big\langle \varphi'\Big(\int_0^T X(s)\,\mathrm{d}\mu(s)\Big), \int_0^T X(t) - X_h(t)\,\mathrm{d}\mu(t)\Big\rangle\Big| + \mathrm{remainder} \\
&\quad\le \int_0^T \mathbf{E}\Big|\Big\langle \varphi'\Big(\int_0^T X(s)\,\mathrm{d}\mu(s)\Big), X(t) - X_h(t)\Big\rangle\Big|\,\mathrm{d}\mu(t) + h^{2\gamma} \\
&\quad\lesssim \int_0^T \sup_{\Upsilon \in B} \mathbf{E}\big\langle \Upsilon, X(t) - X_h(t)\big\rangle\,\mathrm{d}\mu(t) + h^{2\gamma} \\
&\quad\lesssim h^{2\gamma} \int_0^T t^{-\gamma}\,\mathrm{d}\mu(t) + h^{2\gamma} \lesssim h^{2\gamma}.
\end{aligned}$$

Future work:

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Thank you for your attention!