Limit theorems for random intermittent maps

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Outline

- Random maps and skew product
- Warmup: Random piecewise expanding maps
- Intermittent maps – non-random limit theorems
- Random intermittent maps; annealed limit theorems

Skew product, deterministic representation

We consider random maps of the form

$$\{T_1, T_2, p_1, p_2\}$$

where the maps T_i are chosen iid with probability p_i. Classical setting: constant probabilities and skew product representation

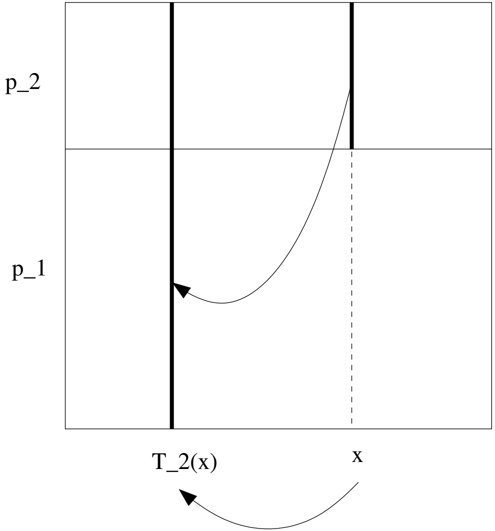
$$T(x,\omega) = (T_{\omega_0}(x), \sigma(\omega)).$$

We will consider (will need!) $p_i = p_i(x)$ spatially dependent probabilities where the associated Markov process is

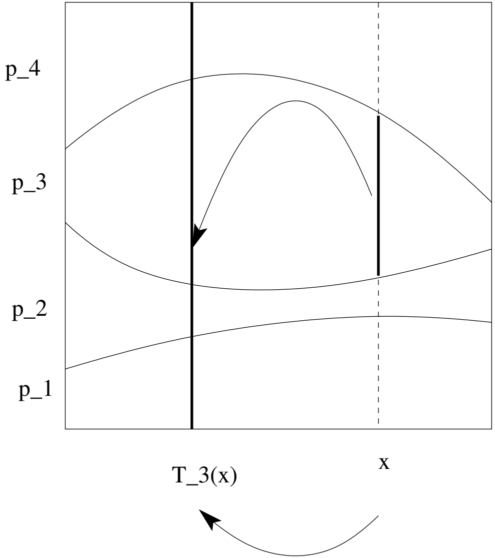
$$\mathbb{P}(x, A) = p_1(x)\mathbf{1}_A(T_1(x)) + p_2(x)\mathbf{1}_A(T_2(x)).$$

To realize this as a 'skew product' we use the following geometric idea

Constant probabilities: $X : (x, \omega) \in [0,1] \times [0,1]$.



Spatially dependent probabilities

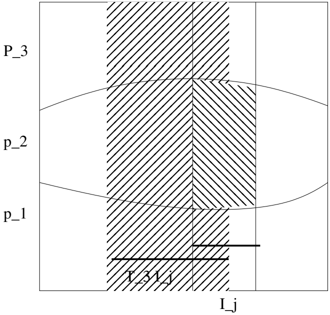


Limit theorems for random expanding

Assume $T_i \in$ expanding Lasota-Yorke maps.

- $[0,1] = \cup[a_j, a_{j+1}] = \cup I_j^{(i)}$
- $T_i : I_j^{(i)} \to [0,1],\ C^2$ and expanding
- $|T_i'| \geq \lambda_i > 1$

Assuming $\inf p_i(x) > 0$ the representation leads to a piecewise expanding, 2D-map of the unit square into itself.



Works best if the p_i are also locally smooth with respect to I_i;

Correlation decay and Central Limit Theorem

With S as above, BV = 2D bounded variation functions, the transfer operator $\mathcal{P}_S$ is quasicompact. Then

1. There is an ACIPM for S, $d\nu = h\,d(m \times m)$ (m= Lebesgue).
2. There is a $\rho < 1$ such that $f \in BV,\ g \in L^\infty$ and $\int f\,dx = 0$ then
$$\left| \int f \cdot g \circ S^n d\nu \right| \le C \|f\|_{BV} \|g\|_\infty \rho^n$$
3. Assume S weakly mixing and $f \in BV$ with $\int f d\nu = A$. There exists $\sigma^2 \ge 0$ such that
$$\frac{S_n f - nA}{\sqrt{n}} \to \mathbb{N}(0, \sigma)$$

Convergence is in distribution and $\sigma^2 > 0$ iff f is not a coboundary for S.

A few remarks on CLT

- Quasicompactness and correlation decay:
  See Liverani (2011): Multidimensional . . . pedestrian approach.
- Spectral approach to CLT (and other limit theorems):
  See Gouëzel (2015?, expository)
- Why is $f \in BV$ natural: Consider the *perturbed* transfer operator

$$\mathcal{P}_t(h) = \mathcal{P}_S(e^{itf} h), t \in \mathbb{R}$$

  and study spectral stability as $t \to 0$.

Embedding the $\Sigma_n f = \sum_{k=0}^{n-1} f \circ S^k$

$$\begin{aligned}
\int \mathcal{P}_t^2 \varphi \cdot \psi \, dm &= \int \mathcal{P}_S e^{itf} \mathcal{P}_S e^{itf} \varphi \cdot \psi \, dm \\
&= \int e^{itf} \mathcal{P}_S e^{itf} \varphi \cdot \psi \circ S \, dm \\
&= \int e^{itf \circ S} e^{itf} \varphi \cdot \psi \circ S^2 \, dm \\
&= \int e^{it\Sigma_2 f} \varphi \cdot \psi \circ S^2 \, dm
\end{aligned}$$

get

$$\int \mathcal{P}_t^n \varphi \cdot \Psi \, dm = \int e^{it\Sigma_n f} \varphi \cdot \psi \circ S^n \, dm$$

Setting $\varphi = \psi = \mathbf{1}$ leads to characteristic function

$$E(e^{it\Sigma_n f}) = \int \mathcal{P}_t^n \varphi \cdot \Psi \, dm = \int \mathcal{P}_t^n \mathbf{1} \, dm$$

Variance and correlation decay

In theorem above, we identify:

$$\sigma^2 = \int \tilde{f}^2 dm + 2 \sum_k \int \tilde{f} \cdot \tilde{f} \circ S^k dm,$$

where $\tilde{f} = f - A$. Key condition to obtain CLT via spectral argument is the summability of correlations:

$$\sum_k \int \tilde{f} \cdot \tilde{f} \circ S^k dm < \infty$$

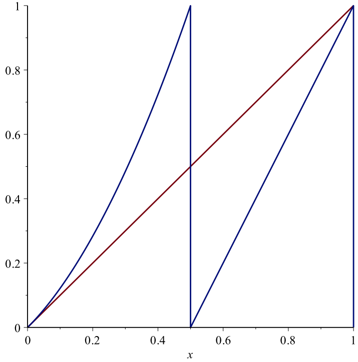
as expected.

Other decay rates like stretched exponential or even polynomial are known for maps with indifferent fixed points. These are the so-called **intermittent** maps.

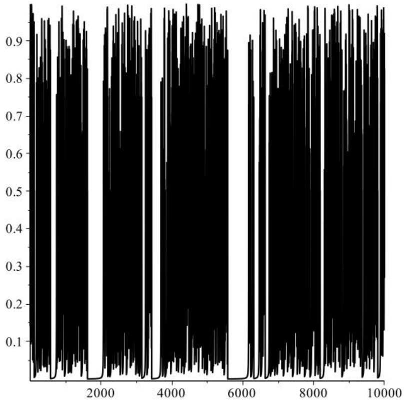
An example.

Fix $0 < \alpha < \infty$. Set

$$T_\alpha(x) := \begin{cases} x + 2^\alpha x^{1+\alpha} & x \in [0, 1/2) \\ 2x - 1 & x \in [1/2, 1) \end{cases}$$

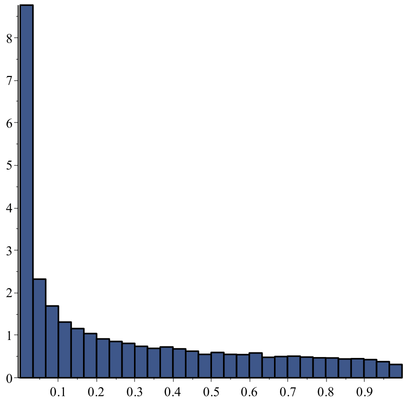


Orbits are mostly spread chaotically throughout $[0, 1)$ interspersed with short periods getting 'stuck' near the neutral fixed point at $x = 0$.



The periods of getting stuck are the *intermittencies*.

An orbit histogram gives a picture of an *invariant density* for the map T_α:



It is known that the density has an order $O(x^{-\alpha})$ singularity near $x = 0$.

History for single map 1

- Liverani, Saussol, Vaienti (ETDS 1999) established regularity properties of the invariant density for T_α and proved sub-exponential decay of correlation in the case of regular fixed point (i.e. $0 < \alpha < 1$) and finite ACIM:

$$Cor_n(g, f) := \int (g \circ T^n) f \, d\mu - \int g \, d\mu \int f \, d\mu$$

$$|Cor_n(g, f)| \leq C(f) ||g||_\infty (\log n)^{\frac{1}{\alpha}} n^{1-\frac{1}{\alpha}}$$

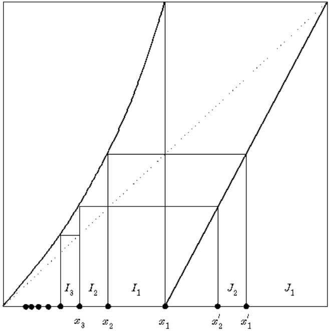
for $f \in \mathcal{C}^1$ and $g \in L^\infty$. μ is the ACIM

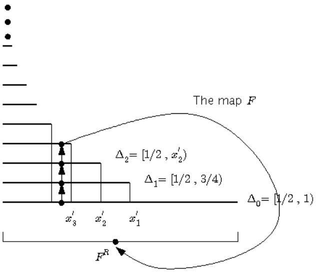
- The maps T_α above are known as LSV-maps. Related: Pomeau-Manneville maps.

History for single map 2

- LS Young (Israel J. Math 1999) induced away from the fixed point and studied return time asymptotices on $\Delta = [1/2, 1]$.
- Led to a systematic approach for many non-uniformly hyperbolic systems known as Young Towers or Markov extensions.
- Links invariant measures, mixing and correlation decay rates to a single intuitive estimate.

Young Towers





If $R(x) = n + 1$ then $F(x) := T_\alpha^{n+1}(x) \in [1/2, 1]$

- ACIM $\nu \sim m$ (m= Lebesgue) for T depends on

$$\sum_k m(\Delta_k) < \infty$$

- For $f \in \mathcal{C}_\beta, \ g \in L^\infty$

$$|Cor_n(g,f)| \leq C(f)||g||_\infty \sum_{k>n} m(\Delta_k)$$

- For LSV, careful calculus estimate gives

$$m(\Delta_k) = \frac{1}{2} x_k = O\left(n^{-\frac{1}{\alpha}}\right)$$

- Distortion control required:

$$\left| \frac{DF(x)}{DF(y)} - 1 \right| \leq C\theta^{d(F(x),F(y))}$$

Summary for LSV maps

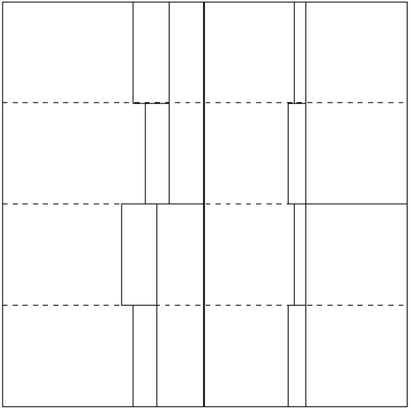
For $T = T_\alpha$, invariant $\nu = hdm$, and $Cor_n(f, g) = O(n^{1-\frac{1}{\alpha}})$

- H. Hu, 0. Sarig and S. Gouëzel (2002-2004) showed the correlation rate is sharp when $0 < \alpha < 1$.
- Central limit theorems hold when $\nu = \frac{1}{\alpha} - 1 > 1 \Leftrightarrow 0 < \alpha < \frac{1}{2}$
- When $\alpha \geq 1$ the ACIM is $\sigma-$ finite. Melbourne &Terhesiu (Invent. 2012) established mixing and correlation decay estimates for suitably normalized correlation.

Random intermittent maps

- Let $0 < \alpha < \beta < \infty$ and T_α, T_β two intermittent LSV maps and consider
$$T := (T_\alpha, T_\beta, p_1, p_2)$$
the associated random dynamical system.
- We can represent T as a deterministic skew product on $[0,1] \times [0,1)$ by
$$S(x,y) = (T_{\alpha(\omega)}(x), \sigma(\omega))$$
Here
$$\sigma(\omega) = \begin{cases} \frac{\omega}{p_1} \text{ if } \omega \in [0, p_1) \\ \frac{\omega - p_1}{p_2} \text{ if } \omega \in [p_1, 1) \end{cases} ; \alpha(\omega) = \begin{cases} \alpha \text{ if } \omega \in [0, p_1) \\ \beta \text{ if } \omega \in [p_1, 1) \end{cases}$$
This is just the independent $(p_1, p_2)-$ shift

In order to apply Young's construction, we need analogues of the intervals I_n and J_n from the single map case. Since the position $x_n = x_n(\omega)$ (similarly $x'_n(\omega)$), instead of intervals we see the following picture:



$\Delta_0 = [1/2, 1) \times [0, 1)$ and the return sets I_n and J_n are unions of 2^n rectangles stacked 'vertically'

The key estimates are again

$$\sum_{k>n}\sum_{j\leq 2^k} m \times m(J_j) = \sum_{k>n}\sum_{j} E_\omega(x'_j(\omega) - x'_{j+1}(\omega))$$
$$= \frac{1}{2}\sum_{k>n}\sum_{j} E_\omega(x_j(\sigma\omega) - x_{j+1}(\sigma\omega))$$
$$= \sum_{k>n}\sum_{j} E_\omega(x_j(\omega)) - E_\omega(x_{j+1}(\omega))$$
$$= \sum_{k>n} E_\omega(x_k(\omega))$$

So we need to calculate the expected position of $x_k(\omega)$ over the randomizing variable ω.

The only completely obvious bounds are

$$x_n(\alpha) \leq x_n(\omega) \leq x_n(\beta)$$

where $x_n(\alpha)$ is the non-random point under parameter α and similar for $x_n(\beta)$.
With a little care, we can derive the following exact asymptotics:

Proposition

For $0 < \alpha \leq \beta < \infty$, *for a.e.* ω*:*

$$\lim_n \frac{n^{\frac{1}{\alpha}} x_n(\omega)}{\frac{1}{2}\alpha^{-\frac{1}{\alpha}} p_1^{-\frac{1}{\alpha}}} = 1$$

So $x_n(\omega) \sim 1/2\alpha^{-\frac{1}{\alpha}} p_1^{-\frac{1}{\alpha}} n^{-\frac{1}{\alpha}}$. We can see this is the 'right' result by setting $p_1 = 1$ where we recover the same sharp estimate due to LS Young for a single map at parameter α.

The heuristic:
For large n, most strings ω_0^n see about $p_1 \cdot n$ occurrences of α, pushing the position strongly toward $x_n(\alpha)$. The fast escape process therefore dominates the asymtpotics.
To make this precise, we need a large deviations result.

Theorem (Hoeffding, 1963)

Let X_k be an independent sequence of RV, with

$$0 \leq X_k \leq 1 \quad \forall k$$

Set $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ and $E_n = E(\bar{X}_n)$ Then for each $0 < t < 1 - p_1$

$$\mathbb{P}\{\bar{X}_n - E_n > t\} \leq \exp(-2nt^2)$$

With exponentially decaying deviations, a simple Borel-Cantelli argument suffices to get pointwise convergence, almost surely.

At any rate, it follows that $E_\omega(x_n(\omega)) = O(n^{-\frac{1}{\alpha}})$. Therefore

Theorem

Let $0 < \alpha < \beta \leq 1$. *Then*

- *there is an ACIPM* $d\nu = hdm \times m$ *for the random skew* S. S *is mixing with respect to* ν.
- $|Cor_n(g, f)| \leq C(f)||g||_\infty n^{1-\frac{1}{\alpha}}$ *for* f *Hölder and* $g \in L^\infty$
- *the CLT holds for Hölder observables when* $0 < \alpha < 1/2$.

$\beta \leq 1 \Leftarrow$ bounded distortion of the return map F.

We have not really used the exact asymptotics. These allow the following extended limit theorems. Here we lean heavily on machinery developed by Gouëzel (ETDS 2007 and earlier partial results).

Theorem

$0 < \alpha < \beta \leq 1$ and $c := \int f(0,\omega)d\omega$. The following (extended) limit theorems hold:

1. *If $\frac{1}{2} \leq \alpha < 1$ and $c = 0$, suppose there exists a $\gamma > \frac{\beta}{\alpha}(\alpha - \frac{1}{2})$ such that $|f(x,\omega) - f(0,\omega)| \leq C_f x^\gamma$. Then there exists $\sigma^2 \geq 0$ such that*

$$\frac{1}{\sqrt{n}} S_n f \to \mathcal{N}(0, \sigma^2).$$

2. *If $\alpha = \frac{1}{2}$ and $c \neq 0$ then $S_n f / \sqrt{c^2 A n \ln n} \to \mathcal{N}(0,1)$.*
3. *If $\frac{1}{2} < \alpha < 1$ and $c \neq 0$ then $S_n f / n^\alpha \to Z$ where the random variable Z has characteristic function given by*

$$E(\exp(itZ)) = \exp\{-A|c|^{\frac{1}{\alpha}}\Gamma(1 - \frac{1}{\alpha})\cos(\pi/2\alpha) \cdot |t|^{\frac{1}{\alpha}}(1 - i\,\mathrm{sgn}(ct)\tan(\pi/2\alpha))\}.$$

Results for the Markov process

Most of the above is framed in terms of the deterministic skew S. What can be factored down to the random map (say, as a Markov process on $[0,1]$)?

- **Stationary measure** It turns out that the invariant density h for S must be almost surely independent of ω. The probability density $\hat{h}(x) = E_\omega(h(x,\alpha)$ is the density of a stationary measure for T. This follows from: $\mathcal{P}_S$ preserves $x-$measurable functions on the square and if $g \in L^1$ depends only on the $x-$coordinate, then $x-$ almost surely:

$$E_\omega(\mathcal{P}_S g(x,\omega)) = \mathcal{P}_T g(x)$$

- **Correlation decay** Same observation allows one to factor the correlation decay down to T:

$$\int g \cdot \mathcal{P}_T^n f dm \leq C(f)\|g\|_\infty n^{1-\frac{1}{\alpha}}$$

Since $h \geq \delta > 0$ can replace dm by $d\nu = h\, dm$.

- **CLT** There is no satisfactory interpretation of the CLT factoring down from the skew. Instead, the natural question is the **quenched** CLT: For almost every fixed ω, setting $S_n f = S_n f(\omega)$ as the sequence of RV, look for a central limit theorem. See, eg: Aimino, Nicol and Vaienti 2014 and references for sample results in the **expanding on average** case.

Thanks!