Exponential growth rates and ergodic theory for jump diffusions

Workshop on Random Dynamical Systems, Bielefeld

Jan Gairing
gairing@mathematik.hu-berlin.de

Institut für Mathematik
Humboldt Universität zu Berlin

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Outline:

Understand dynamical properties of a system of type

$$\xi_t = x + \int_0^t X(\xi_{s-}, \mathrm{d}s) \ , \quad x \in \mathbb{R}^d \ ,$$

$$X(x,t) = \beta(x)t + \sigma(x)W_t + \int_0^t \int g(x,z)\tilde{N}(\mathrm{d}z\mathrm{d}s) + \int_0^t \int g(x,z)N(\mathrm{d}z\mathrm{d}s)$$

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Make accessible to **linear stability analysis**.

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Make accessible to **linear stability analysis**.

- Perturbed dynamical system
- Interpretation of the system as *random dynamical system*
- (path wise) Ergodic theory
- derive *Furstenberg-Khasminskii averaging*.

Random perturbations and linear stability

Consider an (autonomous) DS

$$\dot{x} = v(x) \tag{1}$$

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Consider an (autonomous) DS + " noise "

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How robust is x^* wrt. to the initial condition?

Let v be smooth and denote $\varphi : \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}^d$ be the flow of (1). Consider the linearization instead

$$(\dot{\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi}) = \tfrac{\mathrm{d}}{\mathrm{d}x} v\varphi \ + \ \text{``}\ \tfrac{\mathrm{d}}{\mathrm{d}x}\dot{\eta}\ \text{''} \tag{2}$$

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Stability of (1) can be measured by the (exponential) growth rate of (2)

$$\lambda := \lim_{t \to \infty} \frac{1}{t} \ln \left| \tfrac{\mathrm{d}}{\mathrm{d}x} \varphi(t, \cdot) \right| \quad \text{(Lyapunov exponent)}$$

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(1) is considered

- *exponentially stable* if $\lambda < 0$
- *exponentially unstable* if $\lambda > 0$

Random perturbations and stability

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- In general λ depends on x (and is random).
- Unperturbed case: dependence of eigenspaces of B_0
- Under fairly broad assumptions λ is constant a.s. (for a.a. x).
- Given by ergodic average on the eigenspaces.

Consider the linear jump diffusion

$$\mathrm{d}\xi_t = B_0\xi_t\mathrm{d}t + \sum_{j=1}^{m} B_j\xi_{t-}(\diamond)\mathrm{d}Z_t^j \ , \quad \xi_0 = x \in \mathbb{R}^d \tag{3}$$

- Matrices $B_j \in \mathbb{R}^{d\times d}$
- Lévy process $Z \sim (b, A, \nu)$ in $\mathbb{R}^m$, i.e.

$$Z_t = bt + A^{\frac{1}{2}}W_t + \int_0^t \int_{|z|\leqslant 1} z\tilde{N}(\mathrm{d}z\mathrm{d}s) + \int_0^t \int_{|z|>1} zN(\mathrm{d}z\mathrm{d}s)$$

$\Rightarrow$ Strong solution exists. (Lipschitz!!)

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- (linear) Itô type

$$\Delta\xi_t = \xi_t - \xi_{t-} = \Delta Z_t \boldsymbol{B}\xi_{t-} = \sum_{j=1}^{m} \Delta Z_t^j B_j \xi_{t-}$$

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- (geometric) Marcus/Stratonovich

$$\Delta\xi_t = (\phi^{\Delta Z \boldsymbol{B}} - \mathrm{Id})\xi_{t-} \ ,$$

where $\phi^{z\boldsymbol{B}}$ is the solution flow to the deterministic linear ode

$$\dot{\phi} = z\boldsymbol{B}\phi = \sum_{j=1}^{m} z^j B_j \phi \ , \quad \phi(0) = x \ .$$

Random matrix theory (discrete time)

Let $\Phi_1, \Phi_2, \dots \in \mathbb{R}^{d\times d}$ be i.i.d. and $x_0 \in \mathbb{R}^d \setminus 0$,

(4) $$x_n := \Phi_n x_{n-1} \in \mathbb{R}^d \qquad \text{and} \qquad \bar{x}_n := \frac{\Phi_n \bar{x}_{n-1}}{|\Phi_n \bar{x}_{n-1}|} \in S^{d-1} \; .$$

For $\lambda_n := \ln |x_n|$,

(5) $$\lambda_n = \lambda_{n-1} + \ln |\Phi_n \bar{x}_{n-1}| = \lambda_0 + \sum_{i=1}^{n} \ln |\Phi_i \bar{x}_{i-1}| \; .$$

and the limit

(6) $$\lambda = \lambda(x_0) = \lim_{n\to\infty} \frac{\lambda_n}{n} \; .$$

We want to apply some Krylov-Bogolyubov-type averaging procedure and the Birkhoff ergodic theorem to express the limit as an ergodic mean. [Kha11].

- On the product $(\Omega = \Omega_0^{\mathbb{N}}, \mathbb{P} = \mathbb{P}_0^{\otimes\mathbb{N}}, \mathcal{F} = \mathcal{F}^{\otimes\mathbb{N}})$.

$$\Phi_i(\omega) = \Phi_0(\omega_i)\ , \qquad \text{and} \qquad \Omega = \Omega_0^{\mathbb{N}} = \left\{\omega = (\omega_1, \omega_2, \ldots)\ ,\ \omega_i \in \Omega_0\right\}$$

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- Then $\mathbb{P}$ is invariant under the shift

$$\theta\omega = \theta(\omega_1, \omega_2, \dots) = (\omega_2, \omega_3, \dots) \ .$$

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- $(x_n)_{n \in \mathbb{N}}$, $(\bar{x}_n)_{n \in \mathbb{N}}$ form cocycles over θ

$$\bar{x}_{n+m}(\bar{x}_0, \omega) = \bar{x}_n(\bar{x}_m(\bar{x}_0, \omega), \theta^m \omega) \ .$$

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- $(\lambda_n)_{n\in\mathbb{N}}$ has the *additive cocycle property* over $\varphi_n := (\bar{x}_n, \theta^n)$ on $S^{d-1} \times \Omega$,

$$\lambda_{m+n}(\bar{x}_0, \omega) = \lambda_m(\bar{x}_0, \omega) + \lambda_n \circ \varphi_m(\bar{x}_0, \omega)\ .$$

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- There exists *invariant measure* μ on S^{d-1} and $\mu \otimes \mathbb{P}$ is invariant under φ .

Random matrix theory (ergodic theory)

Theorem (Birkhoff ergodic theorem)

$$\lambda(\bar{x}_0, \omega) := \lim_{n\to\infty} \frac{\lambda_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \lambda_1 \circ \varphi_k(\bar{x}_0, \omega)$$

exists $\mu \otimes \mathbb{P}$ almost surely, is invariant with respect to φ and is constant on the ergodic components of $S^{d-1} \times \Omega$ with respect to φ.

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*In particular we a **Furstenberg-Khasminskii averaging** formula*

$$\lambda(\bar{x}_0,\omega) \equiv \mathbb{E}\int_{S^{d-1}} \ln|\Phi_1(\cdot)\bar{x}|\mu(\mathrm{d}\bar{x})\ ,\ \mu\otimes\mathbb{P}\text{-a.s.}$$

provided it is finite.

Oseledec' *Multiplicative ergodic theorem*

- **linear cocycle** $(\xi_t)_{t\in\mathbb{R}}$ on $\mathbb{R}^d$ over an
- ergodic metric DS $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$
- (logarithmic) integrability conditions

there exists a family of random invariant subspaces $(E_i)_{i=1,\dots,r}$ such that $\mathbb{R}^d = E_1 \oplus \cdots \oplus E_r$ and that

$$x \in E_i(\omega) \setminus \{0\} \quad \text{iff.} \quad \lim_{t\to\infty} \frac{1}{t} \log |\xi_t(\omega)x| = \lambda_i$$

These numbers λ_i are called *Lyapunov exponents*.

[Ose68]

Continuous version

If $(\Phi_t)_{t\geqslant 0}$ is the matrix valued solution flow, then for any $\tau > 0$

$$\lambda(\bar{x}_0, \omega) \equiv \mathbb{E} \int_{S^{d-1}} \ln |\Phi_\tau(\cdot)\bar{x}| \mu(\mathrm{d}\bar{x}) \ , \ \mu \otimes \mathbb{P}\text{-a.s.} \ .$$

We observe that

$$\mathbb{E} \ln |\Phi_\tau(\cdot)\bar{x}| = \frac{1}{2} \mathbb{E}^{\bar{x}} \ln |\xi_\tau|^2 = \frac{1}{2} \left(\mathcal{P}_\tau \ln |\cdot|^2\right)(\bar{x}) \ , \tag{7}$$

$(\mathcal{P}_t)_{t\geqslant 0}$ is the semigroup associated to ξ.

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Recall the **extended generator** of $\mathcal{P}$

$$\begin{aligned} \hat{\mathcal{L}} &= \left\{ (f,g) \ : \ \mathcal{P}_t f - f = \int_0^t \mathcal{P}_s g \mathrm{d}s \right\} \\ &= \left\{ (f,g) \ : \ f(X_\cdot) - \int_0^\cdot g(X_s)\mathrm{d}s \text{ is a martingale } \right\} \ . \end{aligned}$$

[EK05]

Furstenberg-Khasminskii type formula

Theorem

Let ξ be the Markov process solving the stochastic differential equation (3) and $\mathcal{L}$ the generator of the associated semigroup with

$$\ln|\cdot|^2 \in \hat{\mathcal{L}} \ .$$

Then for μ a.e. initial condition the Lyapunov exponents are given by the formula

$$\lambda_\mu := \lim_{T\to\infty} \frac{\lambda(T)}{T} = \int_{S^{d-1}} \frac{1}{2}\mathcal{L}\Big(\ln|\cdot|^2\Big)(\bar{x})\mu(\mathrm{d}\bar{x}) \quad \mathbb{P} \text{ a.s.} \ .$$

For the linear Itô equation

$$\mathrm{d}\xi_t = B_0 \xi_t \mathrm{d}t + \sum_{j=1}^{m} B_j \xi_{t-} \mathrm{d}Z_t^j$$

we have

$$\begin{aligned} \mathcal{L} \log(|\cdot|^2)(x) = 2 &\left(\bar{x}^* B_0 \bar{x} + b^j \bar{x}^* B_j \bar{x} \right) \\ &+ |B_j \bar{x}|^2 - 2 \left(\bar{x}^* B_j \bar{x} \right)^2 \\ &+ 2 \int \log(|\bar{x} + z^j B_j \bar{x}|) - \mathbb{1}_{\{|z| \leqslant 1\}} z^j \bar{x}^* B_j \bar{x} \; \nu(\mathrm{d}z) \; . \end{aligned}$$

Thus simple estimates provide a sufficient condition for $\ln |\cdot|^2 \in \hat{\mathcal{L}}$ by

$$\int_{|z| \geqslant 1} \ln |z| \; \nu(\mathrm{d}z) < \infty \; .$$

For $\mathcal{C}^2$ vector fields $v_j : \mathbb{R}^d \to \mathbb{R}^d$ and

$$\mathrm{d}\xi_t = \sum_{j=0}^{m} v_j(\xi_{t-}) \diamond \mathrm{d}Z_t^j$$

the associated generator takes the form

$$\mathcal{L}^\diamond f = \left(v_0 + b^j v_j + \tfrac{1}{2} v_j^2\right) f + \int_{\mathbb{R}^m} \int_0^1 z^j \left(v_j f \circ \phi_r^{z\boldsymbol{v}} - \mathbb{1}_{\{|z| \leqslant 1\}} v_j f\right) \mathrm{d}r \; \nu(\mathrm{d}z) \; .$$

The **linear** Marcus Generator

In the linear case

$$\mathrm{d}\xi_t = B_0\xi_t\mathrm{d}t + \sum_{j=1}^{m} B_j\xi_{t-} \diamond \mathrm{d}Z_t^j$$

the formula reads

$$\begin{aligned}\mathcal{L}^\diamond \log(|\cdot|^2)(x) &= 2q_0(\bar{x}) + 2b^j q_j(\bar{x}) + h^j(q_j)(\bar{x}) \\ &\quad + 2\int\int_0^1 z^j \Big(q_j\big(\frac{\phi_r^{z\boldsymbol{B}}\bar{x}}{|\phi_r^{z\boldsymbol{B}}\bar{x}|}\big) - \mathbb{1}_{\{|z|\leqslant 1\}} q_j(\bar{x})\Big)\, \mathrm{d}r\nu(\mathrm{d}z)\end{aligned}$$

with the notation

(8) $$q_j(\bar{x}) = \langle B_j\bar{x}, \bar{x}\rangle$$

(9) $$h_j(\bar{x}) = B_j\bar{x} + q_j(\bar{x})\bar{x}$$

(10) $$h^j(q_j)(\bar{x}) = h_j^i \frac{\partial q_j}{\partial x_i}(\bar{x}) = \langle (B_j + B_j^*)\bar{x}, B_j\bar{x}\rangle - 2\langle B_j\bar{x}, \bar{x}\rangle^2$$

Again, for $\ln|\cdot|^2 \in \hat{\mathcal{L}}$ it suffices

$$\int_{|z|\geqslant 1} |z|\ \nu(\mathrm{d}z) < \infty\ .$$

$$\lambda = \int_{S^{d-1}} \frac{1}{2}\mathcal{L}^{\diamond}\Big(\ln(|\cdot|^2\Big)(\bar{x})\mu(\mathrm{d}\bar{x})$$

$$= \int_{S^{d-1}} q_0(\bar{x}) + b^j q_j(\bar{x}) + \frac{1}{2}h^j(q_j)(\bar{x})$$

$$+ \int\int_0^1 z^j\Big(q_j(\frac{\phi_r^{z\boldsymbol{B}}\bar{x}}{|\phi_r^{z\boldsymbol{B}}\bar{x}|}) - \mathbb{1}_{\{|z|\leqslant 1\}}q_j(\bar{x})\Big)\ \mathrm{d}r\nu(\mathrm{d}z)\mu(\mathrm{d}\bar{x})$$

Remark that the dynamic of $\bar{\xi}$ on S^{d-1} is invariant under μ. Following [AK00] we deduce that the non local term is

$$\int_{S^{d-1}}\int_{|z|>1} z^j q_j(\bar{x})\ \mathrm{d}r\nu(\mathrm{d}z)\mu(\mathrm{d}\bar{x})$$

A Furstenberg-Khasminskii type formula: **Marcus** equation

$$\lambda = \int_{S^{d-1}} \left(q_0(\bar{x}) + b^j q_j(\bar{x}) + \frac{1}{2} h^j(q_j)(\bar{x}) + q_j(\bar{x}) \int_{|z|>1} z^j \; \nu(\mathrm{d}z) \right) \mu(\mathrm{d}\bar{x})$$

with the notation

$$\begin{aligned} q_j(\bar{x}) &= \langle B_j \bar{x}, \bar{x} \rangle \\ h_j(\bar{x}) &= B_j \bar{x} + q_j(\bar{x}) \bar{x} \\ h^j(q_j)(\bar{x}) &= h^i_j \frac{\partial q_j}{\partial x_i}(\bar{x}) = \langle (B_j + B_j^*)\bar{x}, B_j \bar{x} \rangle - 2 \langle B_j \bar{x}, \bar{x} \rangle^2 \end{aligned}$$

[AOP86]

Uniqueness of the top exponent

Recall that under the integrability condition on the ν.

$$\lambda_\mu = \int_{S^{d-1}} \frac{1}{2}\mathcal{L}\Big(\ln|\cdot|^2\Big)(\bar{x})\mu(\mathrm{d}\bar{x}) \quad \mathbb{P} \text{ a.s. } .$$

The exponent is *well defined* (independent of x) if μ is unique,
i.e. $\bar{\xi}$ is **ergodic** on S^{d-1}.

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Ergodicity of the projection [MT09]:

1. **Irreducibility**:

$$\bar{\mathcal{P}}_t \mathbb{1}_U(x) > 0 \quad \text{for any open} \quad U \subset S^{d-1},\ x \in S^{d-1}\ .$$

2. **Strong Feller property** (smoothing):

$$\bar{\mathcal{P}}_t f(x) \in C_b(S^{d-1}) \quad \text{for any} \quad f \in \mathscr{B}_b(S^{d-1})\ .$$

FK vs Oseledec

Oseldetec:

- Lyapunov exponents $\lambda_r < \cdots < \lambda_1$ are well defined (non-random).
- Random invariant subspaces $E_i(\omega)$ that realize λ_i.
- Hence: Random invariant measures $\mu_\omega(\mathrm{d}x) \times \mathbb{P}(\mathrm{d}\omega)$ that realize λ_i.

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- Birkhoff:

$$\lambda_\mu = \int_{S^{d-1}} \cdots \mu(\mathrm{d}\bar{x}) \ .$$

- Markovian invariant measures μ realizing $\lambda_\mu = \lambda(\bar{x}_0) \overset{!?}{=} \lambda_1$.

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Conclusion:

λ_μ is **unique** ($\bar{\xi}$ is **ergodic**) *iff.* no **non-random** Oseledec-subspace E_i!

The Projection: **Jump Kernels**

$$\bar{\xi}_t = \pi(\xi_t) = \frac{\xi_t}{|\xi_t|} \in S^{d-1} \ .$$

Jump kernel

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(i) Marcus case:

$$\bar{g}(\bar{x}, z) = \phi^{Hz}(\bar{x}) - \bar{x} \quad \text{with} \quad \dot{\phi} = zH(\phi) = \sum_j z^j h_j(\phi) \ .$$

(recall that $h_j(x) = \nabla\pi(x) B_j x$)

The Projection: **Jump Kernels**

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Jump kernel

$$\Delta\bar{\xi}_t = \bar{g}(\bar{\xi}_{t-}, \Delta Z_t)$$

(i) Marcus case:

$$\bar{g}(\bar{x}, z) = \phi^{Hz}(\bar{x}) - \bar{x} \quad \text{with} \quad \dot{\phi} = zH(\phi) = \sum_j z^j h_j(\phi) \ .$$

(recall that $h_j(x) = \nabla\pi(x) B_j x$)

(ii) Itô case:

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In both cases:

$$g(\bar{x}, z) = zH(\bar{x}) + O(|z|^2)$$

Can write:

$$\bar{\xi}_t = \bar{x}_0 + \int_0^t \boldsymbol{H}(\bar{\xi}_{s-}, \mathrm{d}s) \ .$$

The Projection: $\mathrm{d}\bar{\xi}_t = \boldsymbol{H}(\bar{\xi}_{t-}, \mathrm{d}t)$

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$$\boldsymbol{H}(x,t) = t\Big(h_0(x) + H(x)b + \frac{1}{2}\sum_j \mathrm{D}h_j(x)h_j(x)\Big) + H(x)\mathrm{d}W_t$$
$$+ \int_{|z|\leqslant 1} H(x)z\ \tilde{N}(\mathrm{d}t\mathrm{d}z) + \int_{|z|>1} H(x)z\ N(\mathrm{d}t\mathrm{d}z)$$
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$$+ \int_{\mathbb{R}^m} \pi\Big((\mathrm{Id} + z\boldsymbol{B})x\Big) - x - H(x)z\ N(\mathrm{d}t\mathrm{d}z)\ .$$

$$\tilde{h}_0(x) = \tilde{B}_0 x - \langle x, \tilde{B}_0 x\rangle x\ , \quad \tilde{B}_0 = B_0 - \frac{1}{2}\sum_{j=1}^{m} B_j B_j\ .$$

Assumptions: **Lévy measure**

1. Integrability condition: $(\Rightarrow$ Birkhoff & Oseledec)
   (i) Itô case: $\int_{|z|>1} \log(|z|)\nu(\mathrm{d}z) < \infty$
   (ii) Marcus case: $\int_{|z|>1} |z|\nu(\mathrm{d}z) < \infty$
2. Order condition:

$$\liminf_{\varepsilon \searrow 0} \varepsilon^{-\alpha} \int_{|z|\leqslant\varepsilon} |z|^2 \nu(\mathrm{d}z) > 0 , \qquad \alpha \in (0,2) . \tag{11}$$

[Ore68, Pic96, Kun11]

Can define *infinitesimal covariance*:

$$\Sigma_\nu := \liminf_{\varepsilon \searrow 0} \frac{1}{\sigma_\nu(\varepsilon)} \int_{|z|<\varepsilon} z \otimes z \, \nu(\mathrm{d}z) \qquad \in \mathbb{R}^{m\times m} . \tag{12}$$

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α-stable measures

$\nu_\alpha(\mathrm{d}z) \sim |z|^{-\alpha-1}\mathrm{d}z, \alpha \in (0,2)$ rotationally invariant / iid. concentrated on axes

then (11) for $\alpha' = 2 - \alpha$, $\Sigma_\nu = \mathrm{Id} \in \mathbb{R}^{m\times m}$.

Assumptions: **non-degeneracy**

1. The Lévy triplet (b, A, ν):
$$\det(A + \Sigma_\nu) > 0 \ .$$
[Kun11]

2. The vector fields.
$$\dim \operatorname{span} \{h_1(\bar{x}), \ldots, h_m(\bar{x})\} = d - 1 \ , \quad \bar{x} \in S^{d-1} \ .$$
Recall h_j is the projection of B_j (rotational part)
$$h_j(\bar{x}) = B_j \bar{x} - \langle B_j \bar{x}, \bar{x} \rangle \bar{x} \ , \quad j = 0, \ldots, m \ .$$

Denote

$$H(x) = \Big(h_1(x), h_2(x), \cdots, h_m(x) \Big) \in \mathbb{R}^{d \times m} \ , \quad x \in \mathbb{R}^d \ . \tag{13}$$

Unique ergodicity

Theorem

We ask the following rank condition

(14) $$\operatorname{rank}\left(H(\bar{x})\,(A+\Sigma_\nu)\,H(\bar{x})^*\right) = d-1\ , \quad \forall \bar{x} \in S^{d-1}\ .$$

Then the pojection $\bar{\xi}$ is ergodic.

Ergodic theory

1. Irreducibility: **Support theorems** (Stroock-Varadhan)

$$\Big\{\phi : [0,t] \to \mathbb{R}^d,\ \phi(t) \in U\Big\} \cap \operatorname{supp}(\xi) \neq \varnothing\ .$$

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2. Strong Feller: **Gradient estimates** (Bismut-Elworthy-Li)

Assume

$$\mathbb{E}\Big[\ \nabla f(\xi_t^x)\ \Big] \lesssim \|f\|_\infty\ , \quad \forall\ f \in C_b^\infty$$

Approximate $f \in \mathscr{B}_b$ by $f_1, f_2, \ldots \in C_b^\infty$, then $\nabla \mathcal{P}_t f(x)$ exists.

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$\Rightarrow$ Any ergodic measure is unique!

Embedded equation

$$\xi_t^x = x + \int_0^t X(\xi_{s-}^x, \mathrm{d}s) \ , \quad x \in \mathbb{R}^d \ ,$$

$$X(x,t) = \beta(x)t + \sigma(x)A^{\frac{1}{2}}W_t + \int_0^t \int g(x,z)\tilde{N}(\mathrm{d}z\mathrm{d}s) + \int_0^t \int g(x,z)N(\mathrm{d}z\mathrm{d}s)$$

Assume $\beta, \sigma, g(\cdot, z)$ to be $C_0^\infty(\mathbb{R}^d)$, and

$$g(x,\cdot) \in C_b^\infty(\mathbb{R}^m) \ , \quad g'(x,z)\Big|_{z=0} = \sigma(x) \ .$$

Can take smooth $\rho, \rho(1) = 1, \operatorname{supp} \rho \subset (\frac{1}{2}, \frac{3}{2})$

$$X(x,t) = \rho(|x|^2)\boldsymbol{H}(\bar{x}, t) \ .$$

Bismut's approach to Malliavin calculus

- $u : \Omega \times (0,T] \to \mathbb{R}^{m\times d}$ predictable.

$$W_t^{u.\lambda} = W_t + \int_0^t u_s.\lambda \mathrm{d}s$$

- $v : \Omega \times (0,T] \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ predictable.

$$N_t^{v.\lambda} = N_t\left((w, \boldsymbol{u}^{v.\lambda}(\omega))\right) \ ,$$
$$\boldsymbol{u}^{v.\lambda}(\omega) = \{(t_i, z_i + v(\omega, t_i, z_i).\lambda\} \ .$$

Define the *Girsanov-shift* in direction (u, v) by

$$\begin{aligned} \theta^\lambda : \Omega &\longrightarrow \Omega \\ \omega &\mapsto (w^\lambda, \boldsymbol{u}^\lambda) \end{aligned} \tag{15}$$

F-derivative

For $\Phi : \Omega \to \mathbb{R}^d$ let

$$\mathscr{D}_\theta \Phi = \left(\frac{\partial \Phi^i \circ \theta^\lambda}{\partial \lambda^j}\right)^{ij}\Big|_{\lambda=0}$$

Integration-by-parts

"Finite energy condition": u, v bounded,

(16) $$|v(\omega,t,z)| + |v'(\omega,t,z)| \leqslant 2\rho(z)\ , \quad \rho \in L^1(\nu)\ .$$

Integration-by-parts [BGJ87]

For any $f \in C_p^2(\mathbb{R}^d), \Psi$ F-differentiable,

$$\mathbb{E}\Big[\ \Psi \nabla^* f(\xi_t^x) \mathscr{D}_\theta \xi_t^x\ \Big] = \mathbb{E}\Big[\ f(\xi_t^x) \Gamma_\theta(\Psi)\ \Big]$$

$$\Gamma_\theta(\Psi) = \Psi \left(-\int u(s)\mathrm{d}W_s + \iint \mathrm{div}_z\, v\ \tilde{N}(\mathrm{d}z\mathrm{d}s) \right) + \mathscr{D}_\theta \Psi$$

Integration-by-parts, B-E-L and densities

If ξ^x diffeomorphic ("small jumps")

$$\begin{aligned}\mathscr{D}_\theta \xi_t^x &= \nabla \xi_t^x \Big(\int_0^t (\nabla \xi_{s-}^x)^{-1} \sigma(\xi_{s-}^x) u_s \mathrm{d}s \\ &\qquad + \iint_0^t (\nabla \xi_{s-}^x)^{-1} (\mathrm{Id} + \nabla g(\xi_{s-}^x, z))^{-1} g'(\xi_{s-}^x, z) v(s,z)\ N(\mathrm{d}z \mathrm{d}s) \Big) \\ &= \nabla \xi_t^x \mathscr{A}_{0,t}^\theta \ .\end{aligned}$$

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If we can invert $\mathscr{A}_{0,t}^\theta$, set $\theta = (\mathscr{A}_{0,t}^\theta)^{-1} y$

$$\begin{aligned}\nabla_y \mathcal{P}_t f(x) &= \mathbb{E}\Big[\; \nabla f(\xi_t^x) \nabla \xi_t^x y \; \Big] = \mathbb{E}\Big[\; \nabla f(\xi_t^x) \nabla \xi_t^x \mathscr{A}_{0,t}^\theta \; \Big] \\ &= \mathbb{E}\Big[\; \nabla f(\xi_t^x) \mathscr{D}_\theta \xi_t^x \; \Big] = \mathbb{E}\Big[\; f(\xi_t^x) \Gamma_\theta(1) \; \Big]\end{aligned}$$

Integration-by-parts, B-E-L and densities

If ξ^x diffeomorphic ("small jumps")

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Without jumps and $u \in \mathbb{R}^d$

$$\mathscr{A}_{0,t}^\theta = \int_0^t (\nabla \xi_{s-}^x)^{-1} \sigma(\xi_{s-}^x) \mathrm{d}s \cdot u = \mathscr{A}_{0,t} u \ .$$

and set

$$u = (\mathscr{A}_{0,t})^* (\mathscr{M}_{0,t})^{-1} \xi \ , \quad \mathscr{M}_{0,t} := \mathscr{A}_{0,t} (\mathscr{A}_{0,t})^* \ .$$

Find θ such that $\mathscr{A}^\theta$ invertible. With integration-by-parts and $\Psi = (\mathscr{A}^\theta)^{-1}_{ij}$

$$\mathbb{E}\Big[\ (\mathscr{A}^\theta)^{-1}_{ij}(\nabla f(\xi^x_t)\mathscr{D}^\theta \xi^x_t)^i\ \Big] = \mathbb{E}\Big[\ f(\xi^x_t)\Gamma_{\xi^x_t}\Big((\mathscr{A}^\theta)^{-1}_{ij}\Big)^i\ \Big]$$

summing up over i then gives

$$\begin{aligned}
\sum_i \mathbb{E}\Big[\ f(\xi^x_t)\Gamma_{\xi^x_t}\Big((\mathscr{A}^\theta)^{-1}_{ij}\Big)^i\ \Big] &= \sum_i \mathbb{E}\Big[\ (\mathscr{A}^\theta)^{-1}_{ij}(\nabla f(\xi^x_t)\mathscr{D}^\theta \xi^x_t)^i\ \Big] \\
&= \sum_i \mathbb{E}\Big[\ (\mathscr{A}^\theta)^{-1}_{ij}(\nabla f(\xi^x_t)\nabla \xi^x_t \mathscr{A}^\theta)^i\ \Big] \\
&= \sum_i \mathbb{E}\Big[\ (\mathscr{A}^\theta)^{-1}_{ij}\sum_k(\nabla f(\xi^x_t)\nabla \xi^x_t)^k \mathscr{A}^\theta_{ki}\ \Big] \\
&= \sum_k \mathbb{E}\Big[\ (\nabla f(\xi^x_t)\nabla \xi^x_t)^k \delta_{kj}\ \Big] \\
&= \mathbb{E}\Big[\ (\nabla f(\xi^x_t)\nabla \xi^x_t)^j\ \Big]\ .
\end{aligned}$$

It remains to show that the left hand side is bounded by $C\|f\|_\infty$.

B-E-L type on S^{d-1}

Find θ such that $(\mathscr{A}^\theta)^{-1}_{ij}$ exist for all i.

$$\nabla_{e_j}\mathcal{P}_t f(x) = \mathbb{E}\Big[\ (\nabla f(\xi_t^x)\nabla\xi_t^x)^j\ \Big] \leqslant \|f\|_\infty \mathbb{E}\Big[\ \sum_i \Gamma_{\xi_t^x}\Big((\mathscr{A}^\theta)^{-1}_{ij}\Big)^i\ \Big]$$

Note: We need the estimate only for tangent directions.

B-E-L type on S^{d-1}

Usual choice

$$u_t = \left((\nabla\xi_t^x)^{-1}\sigma(\xi_t^x)A^{\frac{1}{2}}\right)^*$$
$$v(t,z) = \left((\nabla\xi_t^x)^{-1}(\mathrm{Id}+\nabla g(\xi_{t-}^x,z))^{-1}g'(\xi_t^x,z)\right)^*$$

Then

$$\mathscr{A}^\theta = \int_0^t (\nabla\xi_{s-}^x)^{-1}B(\xi_{s-}^x)(\nabla\xi_{s-}^x)^{-1*}\mathrm{d}s$$
$$+ \iint_0^t (\nabla\xi_{s-}^x)^{-1}C(\xi_{s-}^x,z)(\nabla\xi_{s-}^x)^{-1*}\ N(\mathrm{d}z\mathrm{d}s)\ .$$

with

$$B(x) = \sigma(x)A\sigma(x)^*\ , \qquad C(x,z) = (\mathrm{Id}+\nabla g(x,z))^{-1}g'(x,z)g'(x,z)^*(\mathrm{Id}+\nabla g(x,z))^{-1*}$$

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It is enought to proof invertibility of

$$R_{0,t}^x = \int B(\xi_{s-}^x)\mathrm{d}s + \iint C(x,z)N(\mathrm{d}z\mathrm{d}s)\ !!$$

Particular shift

We want to define the perturbations

$$v_\varepsilon(t,z) = \mathbb{1}_{|z|\leqslant\varepsilon} \frac{zz^*}{\sigma_\nu(\varepsilon)} \Big((\nabla\xi^x_{t-})^{-1} (\mathrm{Id} + \nabla g(\xi^x_{t-}, z))^{-1} g'(\xi^x_t, 0) \Big)^*$$

Then

$$\hat{C}_\varepsilon(x,z) = (\mathrm{Id} + \nabla g(x,z))^{-1} g'(x,z) \frac{zz^*}{\sigma_\nu(\varepsilon)} g'(x,0)^* (\mathrm{Id} + \nabla g(x,z))^{-1*} \ ,$$

$$\hat{C}_0(x) = g'(x,0)\Sigma_\nu g'(x,0)^* = \sigma(x)\Sigma_\nu\sigma(x)^* \ .$$

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Note that

$$|g'(x,z)zz^*g'(x,0) - g'(x,0)zz^*g'(x,0)| \leqslant \sup_{|\vartheta|\leqslant\varepsilon} |g''(x,\vartheta)||z|^3$$

such that

$$\|\hat{C}_\varepsilon(x,z) - \hat{C}_0(x,z)\| = O(\varepsilon)$$

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such that

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Deduce that for ε small

$$\hat{R}^{x,\varepsilon}_{0,t} = \int \hat{B}(\xi^x_{s-})\mathrm{d}s + \iint \hat{C}_\varepsilon(\xi^x_{s-}, z) N(\mathrm{d}z\mathrm{d}s) > 0 \ .$$

Support theorem: **control set**

We introduce the quantity

$$\mu_\varepsilon := \int_{\varepsilon<|z|\leqslant 1} z\ \nu(\mathrm{d}z) \in \mathbb{R}^m\ , \tag{17}$$

and assume that either

(a) $\mu_0 = \lim_{\varepsilon\searrow 0} \mu_\varepsilon$ exists, or

(b) $\mu_\varepsilon \in A\mathbb{R}^m$ for any $0 < \varepsilon \leqslant \varepsilon_0$, and set $\mu_0 = 0$.

$$\begin{aligned}
\mathcal{U}^d &= \{\boldsymbol{u} = (u_n)_{n\in\mathbb{N}} = (t_n, z_n)_{n\in\mathbb{N}}\ , \quad 0 < t_n \nearrow \infty, (z_n)_{n\in\mathbb{N}} \subset \operatorname{supp}\nu\} \\
\mathcal{U}^c &= \left\{u^c \in C^1([0,T], A\mathbb{R}^m)\right\} \\
\mathcal{U}\ &= \mathcal{U}^c + \mu_0 t + \mathcal{U}^d\ .
\end{aligned}$$

[Kun99, Sim00]

Support theorem: **control equation**

For $\boldsymbol{u} \in \mathcal{U}$ let us consider the controlled differential equation

$$\phi_t^{\boldsymbol{u}} = x + \int_0^t X^{\boldsymbol{u}}(\phi_{s-}^{\boldsymbol{u}}, \mathrm{d}s) \ , \tag{18}$$

with a controlled generator given by

$$X^{\boldsymbol{u}}(x,t) = \tilde{\beta}(x)t + \int_0^t \sigma(x)\dot{u}(s)\mathrm{d}s + \sum_{t_n \leqslant t} \tilde{g}(x, z_n) \ . \tag{19}$$

For symmetry reasons we use $\tilde{\beta}$ to be the Stratonovich corrected drift

$$\tilde{\beta}(x) = \beta(x) + \frac{1}{2}\sum_j (D\sigma_{\cdot j})(x)\sigma_{\cdot j}(x) \ . \tag{20}$$

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Support Theorem

$$\mathrm{supp}(\xi) = \overline{\{\phi_\cdot^{\boldsymbol{u}} \ : \ \boldsymbol{u} \in \mathcal{U}\}}^{\mathbb{D}} \ ,$$ *cf.* [Kun99, Sim00]

Example: *" Lévy polar coordinates"*

$d = m = 2$ and $\nu(\mathbb{R}^d) < \infty$.

$$\mathrm{d}X_t = B_1 X_t \diamond \mathrm{d}Z_t^1 + B_2 X_t \diamond \mathrm{d}Z_t^2 \ .$$

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \qquad B_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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Then we have

$$q_j(\bar{x}) = \langle B_j \bar{x}, \bar{x} \rangle = \begin{cases} 0 \ , & j = 0 \ , \\ 1 \ , & j = 1 \end{cases} \qquad \text{and hence} \qquad \lambda^\diamond = \mathbb{E} Z^1 \ .$$

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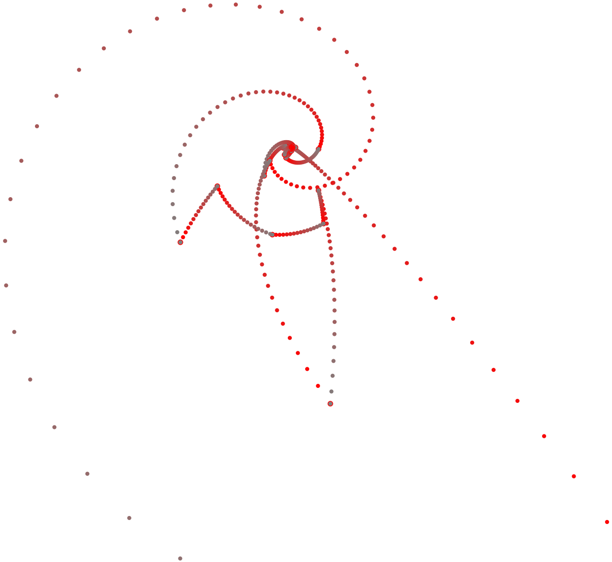
Then we have

$$q_j(\bar{x}) = \langle B_j \bar{x}, \bar{x} \rangle = \begin{cases} 0 \ , & j = 0 \ , \\ 1 \ , & j = 1 \end{cases} \qquad \text{and hence} \qquad \lambda^{\diamond} = \mathbb{E} Z^1 \ .$$

This is easily verified also by the Marcus solution $X_t = \exp(Z_t^1 B_1 + Z_t^2 B_2)x$. And thus

$$\ln |X_t|^2 = 2 Z_t^1 + \ln |x|^2 \ .$$

Example: *“ Lévy polar coordinates”*



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