Weak noise and non hyperbolic unstable fixed points

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Joint work with Mathieu Merle (Paris Diderot and LPMA)

Origin of the question

Consider the SDE

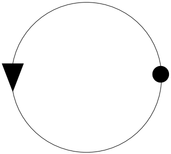
$$\mathrm{d}X_t^{\varepsilon} = -U'(X_t^{\varepsilon})\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t$$

where $\varepsilon \geq 0$, $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, and

$$U(x) = \sin(x) - x$$

0 is a saddle point if $\varepsilon = 0$

can go through 0 if $\varepsilon > 0$



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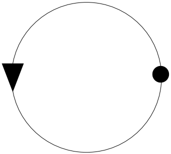

For $\varepsilon > 0$ (small) much of the time is spent at 0 $\Longrightarrow$ relation to case

$$U(x) = -\frac{x^3}{6}$$

Origin of the question

Easy to guess time scale:

$$X_t^\varepsilon = X_0^\varepsilon + \int_0^t (1 - \cos(X_s^\varepsilon))\, \mathrm{d}s + \varepsilon W_t$$

so if we set $Y_t = \varepsilon^{-2/3} X_{\varepsilon^{-2/3} t}$ (think of $\varepsilon^{-2/3} X_0 = O(1)$)

$$Y_t = \varepsilon^{-2/3} X_0 + \varepsilon^{-4/3} \int_0^t \left(1 - \cos\left(\varepsilon^{2/3} Y_s\right)\right) \mathrm{d}s + \varepsilon^{1/3} W_{\varepsilon^{-2/3} t}$$

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so we expect

$$\varepsilon^{2/3} \tau_\varepsilon \overset{\varepsilon \searrow 0}{\Longrightarrow} T_3 \text{ non degenerate r.v.}$$

About T_3

Universal character of T_3: for example [Sigeti, Horsthemke JSP 1989]

Pseudo-regular oscillations induced by external noise

They compute in particular:

$$\mathbb{E}[T_3] = 6\left(\frac{1}{3}\right)^{4/3}\Gamma^2(1/3) = 9.952\ldots$$

$$\mathrm{var}(T_3) = \frac{1}{3}\mathbb{E}[T_3]^2$$

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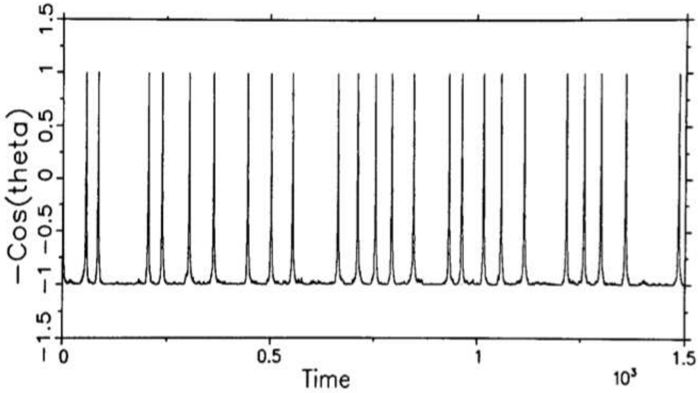
elementary but deep point

noise generates a time length ($\varepsilon^{-2/3}\mathbb{E}[T_3]$)

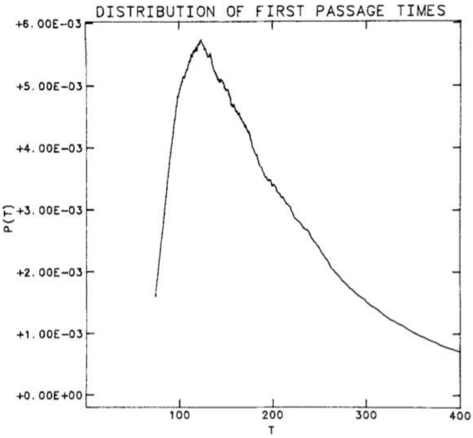
From Sigeti and Horsthemke 89

Pseudo-regular oscillations induced by external noise:

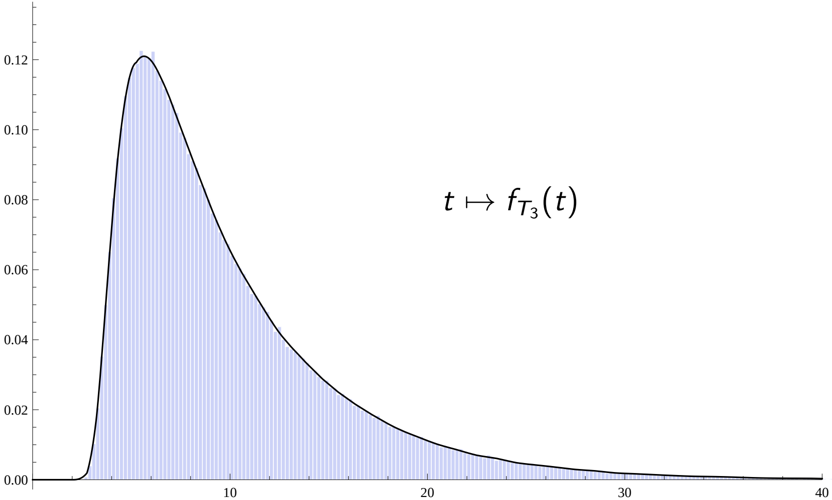


From Sigeti and Horsthemke 89

Law of T_3: *There is an effective cutoff time before which passage through the saddle node is highly improbable. A theoretical analysis indicates that this cutoff occurs at 0.3058 times the mean time.*

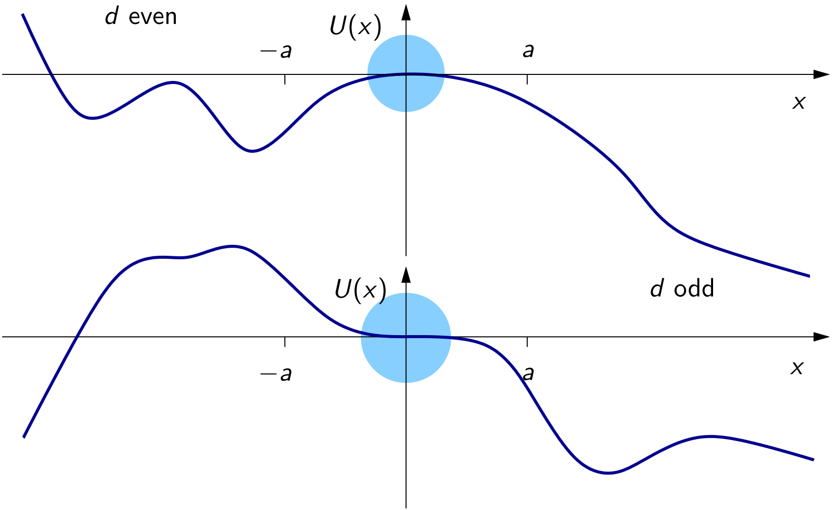


Same, in modern times



Other, more general, motivation: $U(x) \approx -x^d/(2d)$

Escape/Transit from unstable points:



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In the hyperbolic case ($d = 2$), $U''(0) < 0$ (i.e. *top of a hill*), the natural problem is $X_0^\varepsilon = 0$ and by solving the linearized equation one easily proves that the *escape time* $\tau_\varepsilon = \inf\{t > 0 : |X_t^\varepsilon| = \mathit{const}.\}$ is (in probability)

$$\tau_\varepsilon = \frac{1}{|U''(0)|} \log\left(\frac{1}{\varepsilon}\right) + O(1)$$

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Non hyperbolic cases:

- saddle points ($U(x) = -\frac{1}{2d}x^d + \ldots$, $d = 3, 5, \ldots$)
- subcritical pitchfork bifurcations ($U(x) = -\frac{1}{d}x^d + \ldots$, $d = 4, 6, \ldots$)

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Transit/escape from *flat* unstable points $\longrightarrow$ nonlinear fluctuations

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1. capture the scaling
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However

1. everything is very heuristical (often guesswork)
2. completely fail to capture tail behaviors

General set up

We consider the strong solution X to the stochastic differential equation

$$X_0 \, = \, x_0^{(\varepsilon)}, \qquad \mathrm{d}X_t = -U'(X_t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t$$

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We are after $U(x) \overset{x\to 0}{\sim} -\frac{x^d}{2d}$ $(+\ldots$ see previous figure!)

$$\tau_{a,\varepsilon}(X) := \begin{cases} \inf\{t : X_t = a\}, & \text{if } d \text{ is odd } (X_0 = -a), \\ \inf\{t : |X_t| = a\}, & \text{if } d \text{ is even } (X_0 = 0). \end{cases}$$

At last, the results

Theorem

We have

$$\mathcal{L} - \lim_{\varepsilon \searrow 0} \varepsilon^{2(d-2)/d} \tau_{a,\varepsilon}(X) =: T_d$$

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For a random variable $Z \geq 0$, we write

$$\Phi_Z(\lambda) := \mathbb{E}[\exp(\lambda Z)]$$

We set

$$\lambda_0 = \lambda_0(Z) := \sup\{\lambda : \Phi_Z(\lambda) < \infty\}$$

so $\Phi_Z(\cdot)$ is well defined and analytic in $\{\lambda \in \mathbb{C} : \Re(\lambda) < \lambda_0\}$.

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The Theorem is of course equivalent to

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\left[\exp\left(\lambda \varepsilon^{2(d-2)/d} \tau_a(X)\right)\right] = \Phi_{T_d}(\lambda) \quad \text{for every } \lambda < 0$$

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Is $\Phi_{T_d}(\lambda) < \infty$ for some $\lambda > 0$?

Tail behavior (d odd)

We introduce the Schrödinger operator L with domain $C^2(\mathbb{R};\mathbb{R})$:

$$Lu(x) = -\frac{1}{2}u''(x) + q_d(x)u(x).$$

where

$$q_d(x) := \frac{1}{2}\left(V'(x)\right)^2 - \frac{1}{2}V''(x) \quad \text{so} \quad q_3(x) = \frac{1}{2}x + \frac{1}{8}x^4$$

Classical deep results ensure that the equation $Lu = \lambda u$ has a solution that is in $\mathbb{L}^2(\mathbb{R};\mathbb{R})$ if and only if $\lambda = \widetilde{\lambda}_j = \widetilde{\lambda}_{j,d}$, with $\widetilde{\lambda}_0 < \widetilde{\lambda}_1 < \dots$

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Theorem

$\lambda_0 := \lambda_0(T_d)$ is (strictly) positive and it coincides with $\widetilde{\lambda}_0$. Moreover there exists a positive constant C_d such that

$$\Phi_{T_d}(\lambda) \overset{\lambda \nearrow \lambda_0}{\sim} \frac{C_d}{\lambda_0 - \lambda}$$

$\Phi_{T_d}(\cdot)$ extends to the whole of $\mathbb{C}$ as a meromorphic function.

The results ($d = 3$ at times)

Theorem

For $\lambda \to -\infty$ we have

$$\Phi_{T_3}(\lambda) = \left(1 + O(|\lambda|^{-1/4})\right) \exp\left(-C_{3/4}|\lambda|^{3/4}\right)$$

where $C_{3/4} = 3\Gamma\left(-(3/4)\right)^2 / (2^{9/4}\sqrt{2\pi})$.

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Corollary (via Tauberian argument)

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(T_d > t\right) = -\lambda_0(T_d)$$

$$\lim_{t\searrow 0} t^{d/(d-2)} \log \mathbb{P}\left(T_d < t\right) = -c_d < 0 \textit{ and explicit}$$

$$\textit{i.e.} \quad \mathbb{P}\left(T_3 < t\right) = \exp\left(-\frac{c_3 + o(1)}{t^3}\right)$$

The results

By studying the characteristic function $\varphi_X(s) := \mathbb{E}\exp(isX)$:

Proposition

$f_{T_d}(\cdot)$ is real analytic except at 0 and it can be extended to an analytic function in a cone containing the positive real axis. Moreover for $t \to \infty$

$$f_{T_d}(t) = C_d \exp\left(-\lambda_0(T_d)t\right) + O\left(\exp(-bt)\right) ,$$

for any choice of $b \in (\widetilde{\lambda}_0, \widetilde{\lambda}_1) = (\lambda_0(T_d), \widetilde{\lambda}_1)$.

Some ideas of the proofs (d odd)

Proofs can be separated into two (entangled!) blocks:

1. Convergence of $\varepsilon^{2(d-2)/d}\tau_{a,\varepsilon}(X)$ to a limit variable called T_d: probability arguments (martingale/stochastic analysis tools)
2. Analysis of the law of T_d: analytic arguments (WKB)

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so $V_\varepsilon(y) = \varepsilon^{-2} U_\varepsilon(\varepsilon^{2/d} y)$, $Y_0 = \varepsilon^{-2/d} y_0^{(\varepsilon)}$ and

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Itô formula: $\exp(\lambda t) f_\lambda(Y_t)$ is a local martingale if

$$\frac{1}{2} f_\lambda''(y) - V_\varepsilon'(y) f_\lambda'(y) + \lambda f_\lambda(y) = 0$$

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There must be something wrong because there are plenty of f_λ!

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1. introduce the *scale function*

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3. ... and the convergence issue is taken care of (;-)

A formula for the law of the limit variable

Important point

Once we have the convergence we can (and do) choose to work with $V_\varepsilon(y) = V(y) = -y^d/(2d)$, so the ODE to study does not contain ε

$$\frac{1}{2} f''_\lambda(y) - V'(y) f'_\lambda(y) + \lambda f_\lambda(y) = 0$$

and from now we stick to $d = 3$, i.e. $V(y) = -\frac{1}{6} y^3$.

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It is time to go back and try to really ask whether or not M_t

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is a true martingale. Sufficient condition:

$$\mathbb{E}\left[\int_0^{\tau_{\varepsilon^{-2/d}a}(Y)} \exp(2\lambda s)(f'_\lambda(Y_s))^2\, \mathrm{d}s\right] < \infty$$

and requiring $\sup_{y\in\mathbb{R}} |f'_\lambda(y)| < \infty$ does the job for $\lambda < 0$.

A formula for the law of the limit variable

We are therefore in front of the ODE with somewhat atypical b.c.'s

$$\begin{cases} \frac{1}{2} f_\lambda''(y) - V'(y) f_\lambda'(y) + \lambda f_\lambda(y) = 0 \\ \limsup_{y \to \pm\infty} |f_\lambda'(y)| < \infty \end{cases} \tag{ODE}$$

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and the formula (with $T := T_3$ and $\lambda \le 0$)

$$\Phi_T(\lambda) = \mathbb{E}\left[\exp(\lambda T)\right] = \frac{f_\lambda(-\infty)}{f_\lambda(+\infty)}$$

should hold.

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$$\begin{cases} \frac{1}{2} f_\lambda''(y) - V'(y) f_\lambda'(y) + \lambda f_\lambda(y) = 0 \\ \limsup_{y \to \pm\infty} |f_\lambda'(y)| < \infty \end{cases} \tag{ODE}$$

and the formula (with $T := T_3$ and $\lambda \leq 0$)

$$\Phi_T(\lambda) = \mathbb{E}\left[\exp(\lambda T)\right] = \frac{f_\lambda(-\infty)}{f_\lambda(+\infty)}$$

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The point is that...

the $y \to \pm\infty$ behaviors of solutions to (ODE) are limited!

Schrödinger and WKB enter the game

As a matter of fact a WKB analysis shows that

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Formal WKB analysis says that for every $\lambda \in \mathbb{C}$ there exist solutions such that

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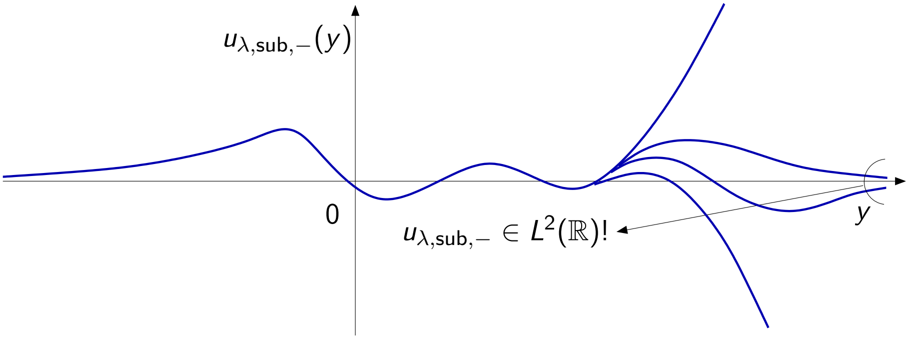
Notably there is a unique solution subdominant at $-\infty$ with the property:

$$u_{\lambda,\text{sub},-}(y) \overset{y \to -\infty}{\sim} \exp\left(\frac{y^3}{6} \right), \qquad \text{so } f_\lambda(-\infty) = 1$$

and $u_{\lambda,\text{sub},-}(y)$ is entire in λ (and y).

What does $u_{\lambda,\text{sub}}(y)$ do for $y \to \infty$?



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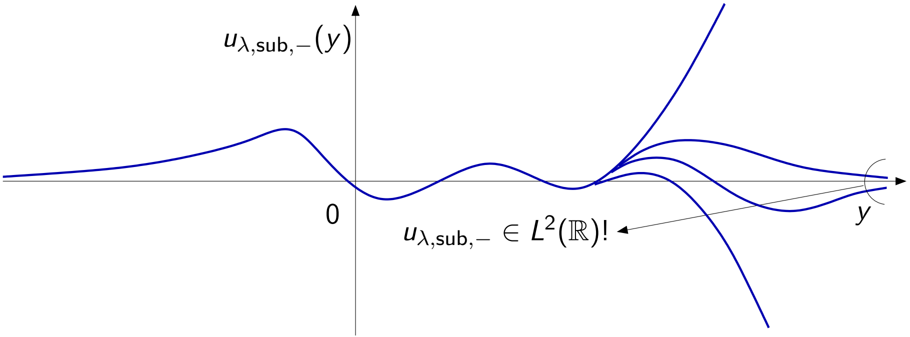

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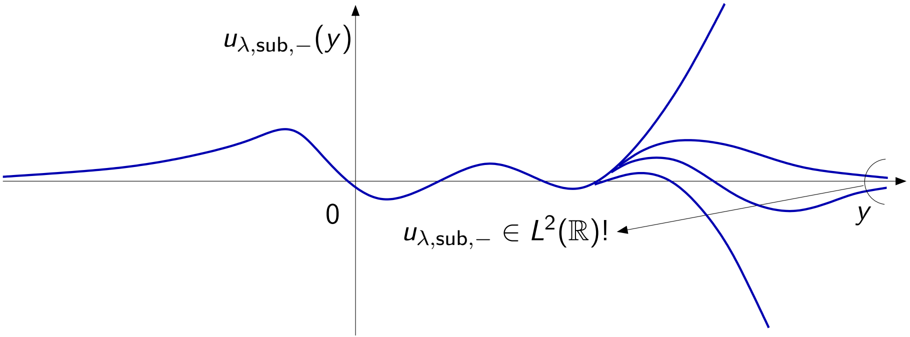

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i.e.

$$f_\lambda(y) = \exp\left(-\frac{y^3}{6}\right)\left(a(\lambda) u_{\lambda,\text{sub},+}(y) + b(\lambda) u_{\lambda,\text{dom},+}(y)\right) \overset{y\to\infty}{\sim} b(\lambda)c(\lambda)$$

Wrapping all up

Therefore now the formula

$$\Phi_T(\lambda) = \mathbb{E}\left[\exp(\lambda T)\right] = \frac{f_\lambda(-\infty)}{f_\lambda(+\infty)}$$

makes sense, a priori for $\Re(\lambda) \leq 0$.

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$$Lu(y) \,=\, -\frac{1}{2}u''(y) + q_d(y)u(y)\,, \quad q_3(y) \,=\, \frac{1}{2}y + \frac{1}{8}y^4$$

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Also the case $\lambda \to -\infty$ can be tackled by WKB techniques:

$$Lu(y) - \lambda u = -\frac{1}{2}u''(y) + (q_d(y) - \lambda)u(y) = 0$$

Conclusions and perspectives

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- Unimodality?
- Some higher dimensional cases can be tackled, but generalizations are not straightforward
- Potentially we can give a formula of the type

$$f_{T_d(t)} \sim \sum_j C_j \exp\left(-\widetilde{\lambda}_j t\right)$$