Stochastic Dynamical Systems and Climate Modeling

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Geometric singular perturbation theory: Application to simple stochastic climate models

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This will be a talk on mathematics. In case you're bored . . .


Seminar BINGO!
To play, simply print out this bingo sheet and attend a departmental seminar.
Mark over each square that occurs throughout the course of the lecture.
The first one to form a straight line (or all four corners) must yell out to win!
BINGO!!
SEMINAR
B I N G O
Speaker bashes previous work
Repeated use of "um…"
Speaker sucks up to host professor
Host Professor falls asleep
Speaker wastes 5 minutes explaining outline
Laptop malfunction
Work ties in to Cancer/HIV or War on Terror
"…et al."
You're the only one in your lab that bothered to show up
Blatant typo
Entire slide filled with equations
"The data *clearly* shows…"
FREE Speaker runs out of time
Use of Powerpoint template with blue background
References Advisor (past or present)
There's a Grad Student wearing same clothes as yesterday
Bitter Post-doc asks question
"That's an interesting question"
"Beyond the scope of this work"
Master's student bobs head fighting sleep
Speaker forgets to thank collaborators
Cell phone goes off
You've no idea what's going on
"Future work will…"
Results conveniently show improvement
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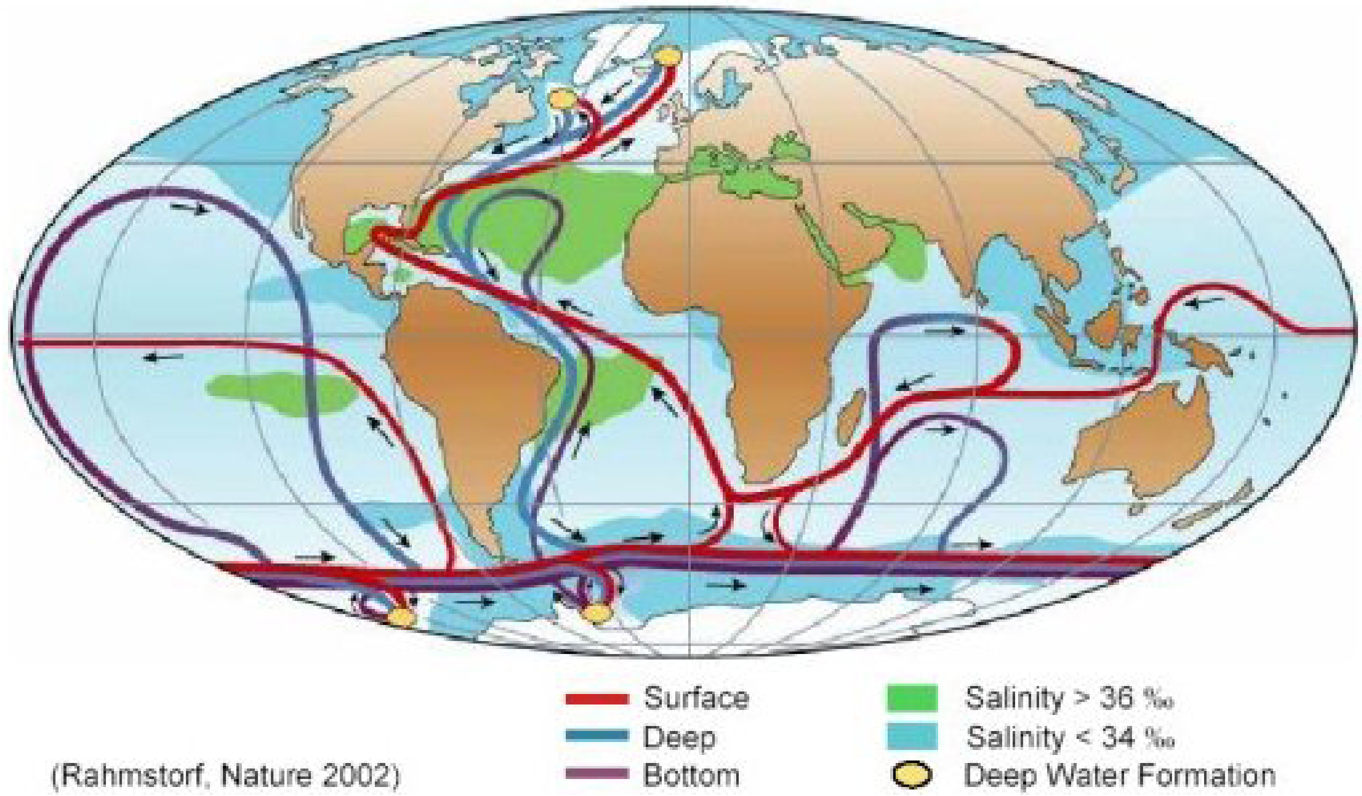
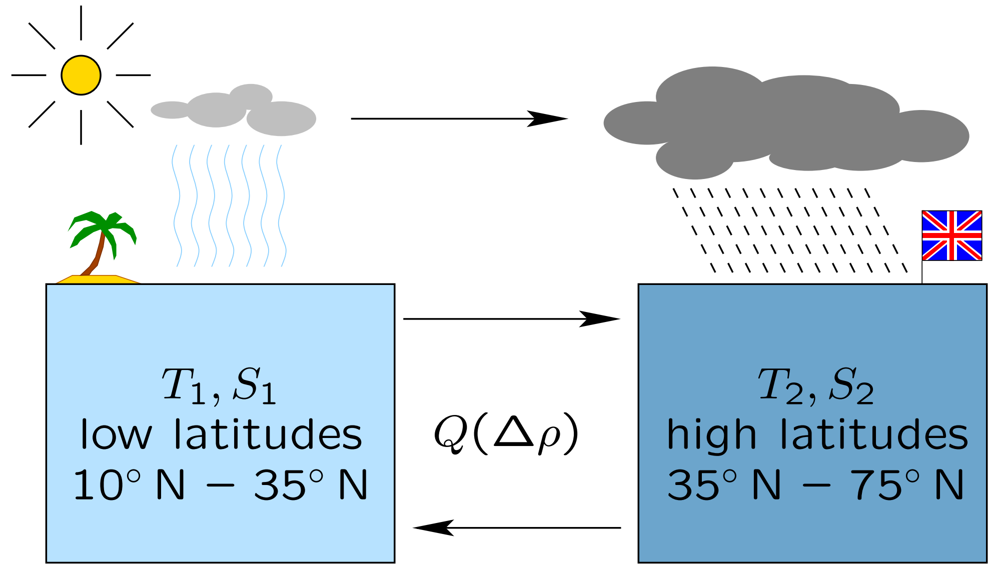
Thermohaline Circulation (THC)



- ▷ "Realistic" models (GCMs, EMICs): Numerical analysis
- ▷ Simple conceptual models: Analytical results
- ▷ In particular: Box models

North-Atlantic THC: Stommel's Box Model ('61)

T_i: Temperatures
S_i: Salinities
F: Freshwater flux
$Q(\Delta\rho)$: Mass exchange
$\Delta\rho = \alpha_S \Delta S - \alpha_T \Delta T$
$\Delta T = T_1 - T_2$
$\Delta S = S_1 - S_2$



$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\Delta T = -\dfrac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\Delta S = \dfrac{S_0}{H}F - Q(\Delta\rho)\Delta S \end{cases}$$

Model for Q [Cessi '94]: $Q(\Delta\rho) = \dfrac{1}{\tau_d} + \dfrac{q}{V}(\Delta\rho)^2$

Stommel's box model as a slow–fast system

Separation of time scales: $\tau_r \ll \tau_d$

Rescaling: $x = \Delta_T/\theta$, $y = (\alpha_S/\alpha_T)(\Delta S/\theta)$, $s = \tau_d t$

$$\begin{cases} \varepsilon\dot{x} = -(x-1) - \varepsilon x[1+\eta^2(x-y)^2] \\ \dot{y} = \mu - y[1+\eta^2(x-y)^2] \end{cases}$$

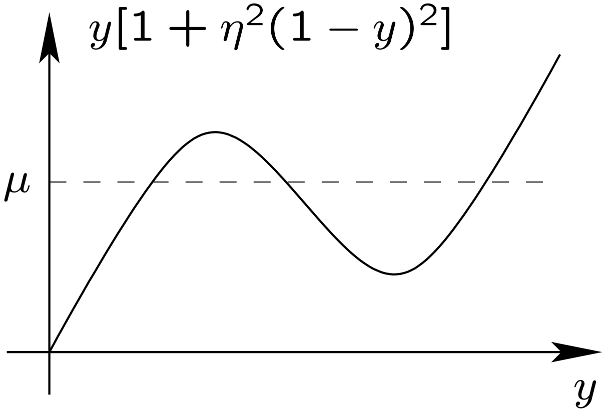
$\varepsilon = \tau_r/\tau_d \ll 1$

Slow manifold ($\varepsilon\dot{x} = 0$):

$$x = x^\star(y) = 1 + \mathcal{O}(\varepsilon)$$

Reduced equation on slow manifold:

$$\dot{y} = \mu - y[1+\eta^2(1-y)^2 + \mathcal{O}(\varepsilon)]$$



1 or 2 stable equilibria, depending on freshwater flux μ (and η)

Geometric singular perturbation theory

General slow–fast system

$$\begin{cases} \varepsilon \dot{x} = f(x,y) & \text{(fast variables } \in \mathbb{R}^n) \\ \dot{y} = g(x,y) & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

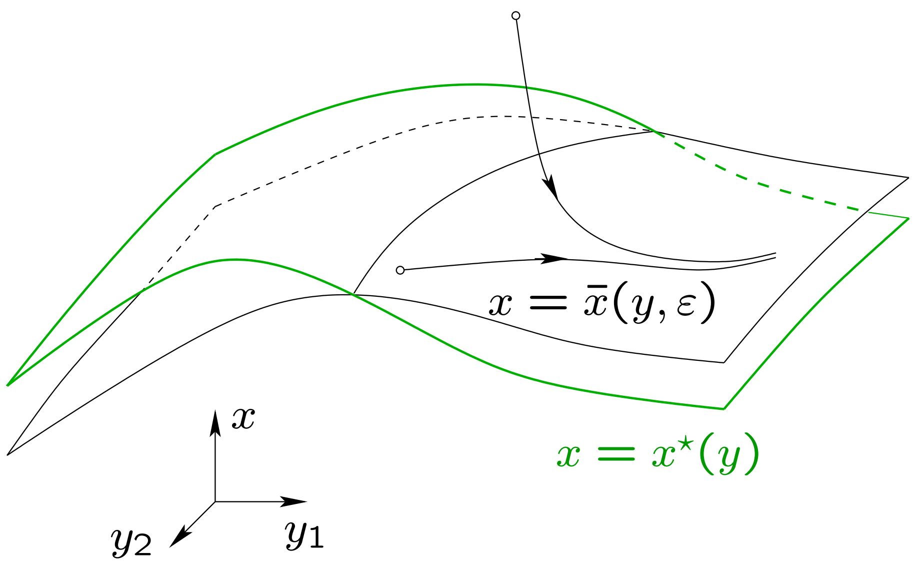
- ▷ Slow manifold: $f = 0$ for $x = x^\star(y)$
- ▷ Stability: e.v. of $\partial_x f(x^\star(y), y)$ have real parts $\Re(\lambda_i(y)) < 0$

Assume $\Re(\lambda_i(y)) \leq -\delta < 0 \quad \forall y$

Theorem [Tihonov '52, Fenichel '79]

$\exists$ *adiabatic manifold* $x = \bar{x}(y,\varepsilon)$
s.t.

- ▷ $\bar{x}(y,\varepsilon)$ is invariant
- ▷ $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- ▷ $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



Random dynamical systems

Random perturbations: One-dim. slowly driven systems

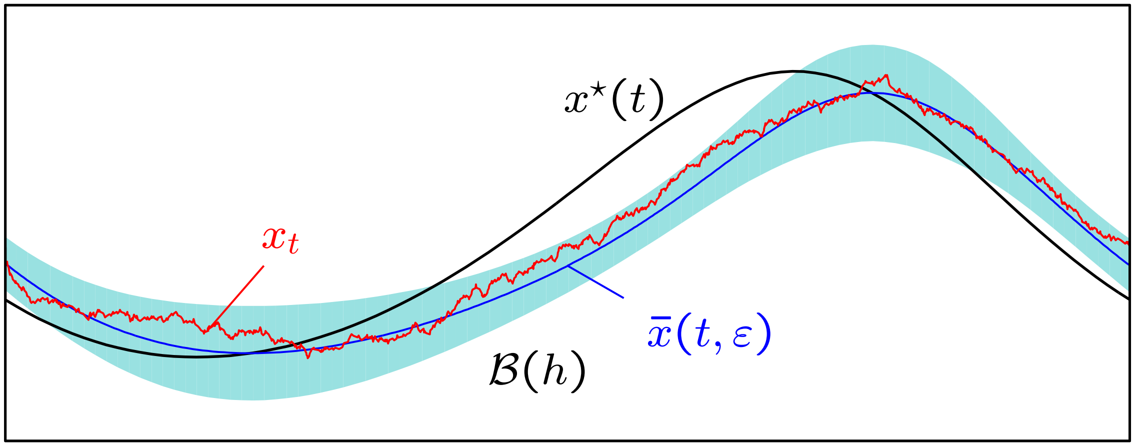
$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t,t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Stable slow manifold / stable equilibrium branch $x^\star(t)$:

$$f(x^\star(t),t) = 0\;, \qquad a^\star(t) = \partial_x f(x^\star(t),t) \leqslant -a_0$$

Adiabatic solution:
$\bar{x}(t,\varepsilon) = x^\star(t) + \mathcal{O}(\varepsilon)$

$\mathcal{B}(h)$: strip around $\bar{x}(t,\varepsilon)$
of width $\simeq h/|a^\star(t)|$



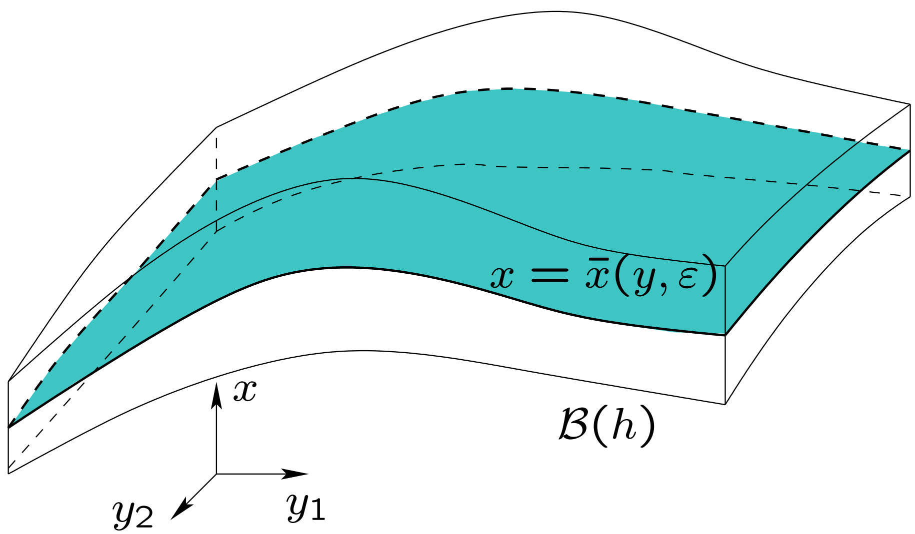
Theorem [Berglund & G '02], [Berglund & G '05]

$$\mathbb{P}\Bigl\{x_t \text{ leaves } \mathcal{B}(h) \text{ before time } t\Bigr\} \simeq \sqrt{\frac{2}{\pi}}\,\frac{1}{\varepsilon}\biggl|\int_0^t a^\star(s)\,\mathrm{d}s\biggr|\,\frac{h}{\sigma}\;\mathrm{e}^{-h^2/2\sigma^2}$$

Random perturbations: General slow–fast systems

$$\begin{cases} dx_t = \frac{1}{\varepsilon} f(x_t, y_t)\,dt + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,dW_t & \text{(fast variables } \in \mathbb{R}^n\text{)} \\ dy_t = \; g(x_t, y_t)\,dt + \sigma' \, G(x_t, y_t)\,dW_t & \text{(slow variables } \in \mathbb{R}^m\text{)} \end{cases}$$

Stable slow manifold: $f(x^\star(y), y) = 0$, $A^\star(y) = \partial_x f(x^\star(y), y)$ stable



$$\mathcal{B}(h) := \Big\{ (x, y) \colon \Big\langle \big[x - \bar{x}(y, \varepsilon)\big], X^\star(y)^{-1} \big[x - \bar{x}(y, \varepsilon)\big] \Big\rangle < h^2 \Big\}$$

$X^\star(y)$ sol. of $A^\star(y) X^\star + X^\star A^\star(y)^\mathsf{T} + F(x^\star(y), y) F(x^\star(y), y)^\mathsf{T} = 0$

Random perturbations: General slow–fast systems

$$\begin{cases} \mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n) \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \sigma'\, G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

Theorem [Berglund & G '03]

▷ $\mathbb{P}\Big\{(x_t, y_t)$ leaves $\mathcal{B}(h)$ before time $t\Big\} \simeq C_{n,m}(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$
with $\kappa = 1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)$
(provided y_t does not drive the system away from the region where assumptions are satisfied)

▷ Reduction to adiabatic manifold $\bar{x}(y, \varepsilon)$:

$$\mathrm{d}y_t^0 = g(\bar{x}(y_t^0, \varepsilon), y_t^0)\,\mathrm{d}t + \sigma' G(\bar{x}(y_t^0, \varepsilon), y_t^0)\,\mathrm{d}W_t$$

y_t^0 approximates y_t to order $\sigma\sqrt{\varepsilon}$ up to Lyapunov time of $\dot{y}^{\text{det}} = g(\bar{x}(y^{\text{det}}, \varepsilon) y^{\text{det}})$

Ex. of inertial manifolds for slow–fast RDS [Schmalfuß & Schneider '06]

Stommel's box model with Ornstein–Uhlenbeck noise

$$\begin{aligned}
\mathrm{d}x_t &= \frac{1}{\varepsilon}\Big[-(x_t-1)-\varepsilon x_t Q(x_t-y_t)\Big]\,\mathrm{d}t + \mathrm{d}\xi^1_t \\
\mathrm{d}\xi^1_t &= -\frac{\gamma_1}{\varepsilon}\xi^1_t\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W^1_t \\
\mathrm{d}y_t &= \Big[\mu - y_t Q(x_t-y_t)\Big]\,\mathrm{d}t + \mathrm{d}\xi^2_t \\
\mathrm{d}\xi^2_t &= -\gamma_2\xi^2_t\,\mathrm{d}t + \sigma'\,\mathrm{d}W^2_t
\end{aligned}$$

Cross section of $\mathcal{B}(h)$ is controlled by matrix

$$X^\star(y) = \begin{pmatrix} \dfrac{1}{2(1+\gamma_1)} & \dfrac{1}{2(1+\gamma_1)} \\[2ex] \dfrac{1}{2(1+\gamma_1)} & \dfrac{1}{2\gamma_1} \end{pmatrix} + \mathcal{O}(\varepsilon)$$

- ▷ Variance of $x_t - 1 \simeq \sigma^2/(2(1+\gamma_1))$
- ▷ Reduced system for (y_t, ξ^2_t) is bistable (for suitable choice of μ)

Modelling the freshwater flux

$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta T = -\frac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta S = \frac{S_0}{H}F(s) - Q(\Delta\rho)\Delta S$$

- ▷ Feedback: F or $\dot{F}$ depending on ΔT and ΔS
  - $\Rightarrow$ relaxation oscillations, excitability
- ▷ External periodic forcing
  - $\Rightarrow$ stochastic resonance, hysteresis
- ▷ Internal periodic forcing of ocean–atmosphere system
  - $\Rightarrow$ stochastic resonance, hysteresis

Case I: Feedback (with Gaussian white noise)

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\Big[-(x_t - 1) - \varepsilon x_t Q(x_t - y_t)\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^0$$

$$\mathrm{d}y_t = \Big[\mu_t - y_t Q(x_t - y_t)\Big]\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_t^1$$

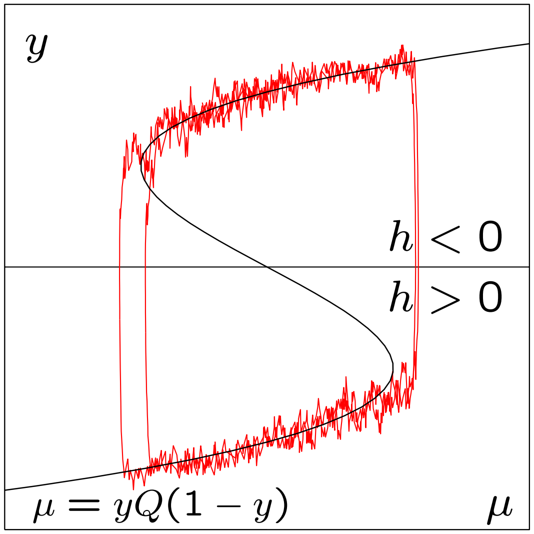
$$\mathrm{d}\mu_t = \tilde{\varepsilon} h(x_t, y_t, \mu_t)\,\mathrm{d}t + \sqrt{\tilde{\varepsilon}}\sigma_2\,\mathrm{d}W_t^2$$ (slow change in freshwater flux)

Reduced equation (after time change $t \mapsto \tilde{\varepsilon}t$)

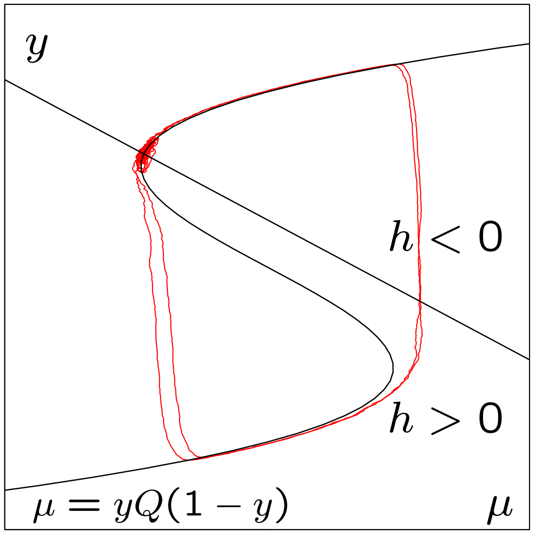
$$\mathrm{d}y_t = \frac{1}{\tilde{\varepsilon}}\Big[\mu_t - y_t Q(1 - y_t)\Big]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\tilde{\varepsilon}}}\,\mathrm{d}W_t^1$$

$$\mathrm{d}\mu_t = h(1, y_t, \mu_t)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^2$$

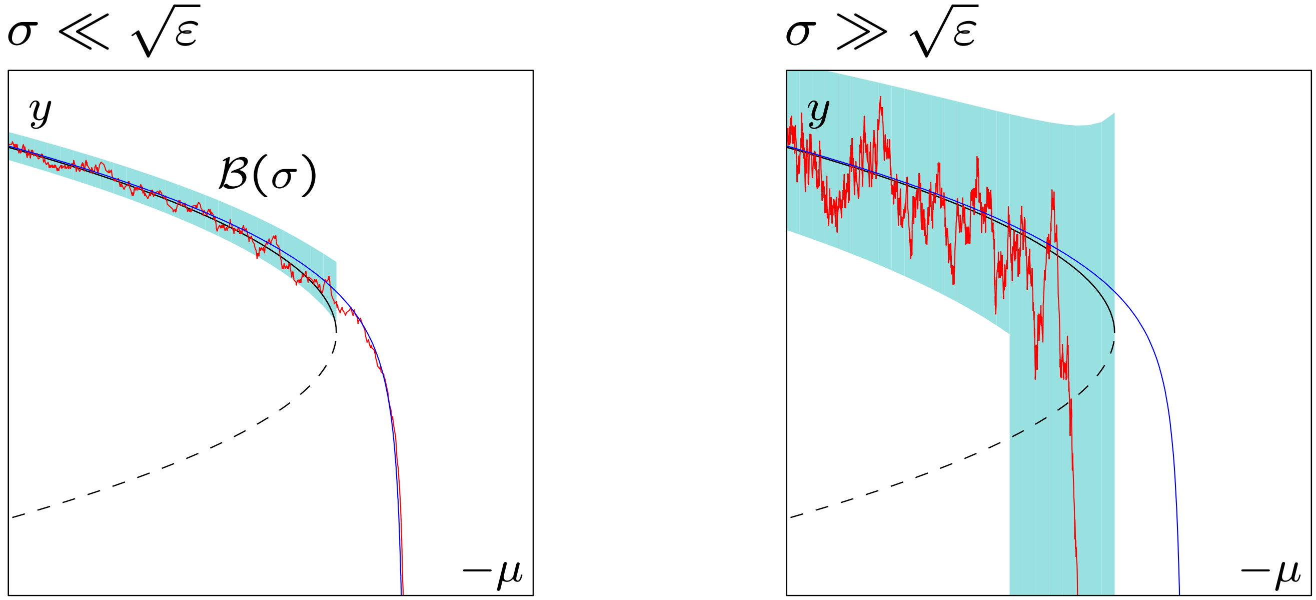
Relaxation oscillations



Excitability



Saddle–node bifurcation



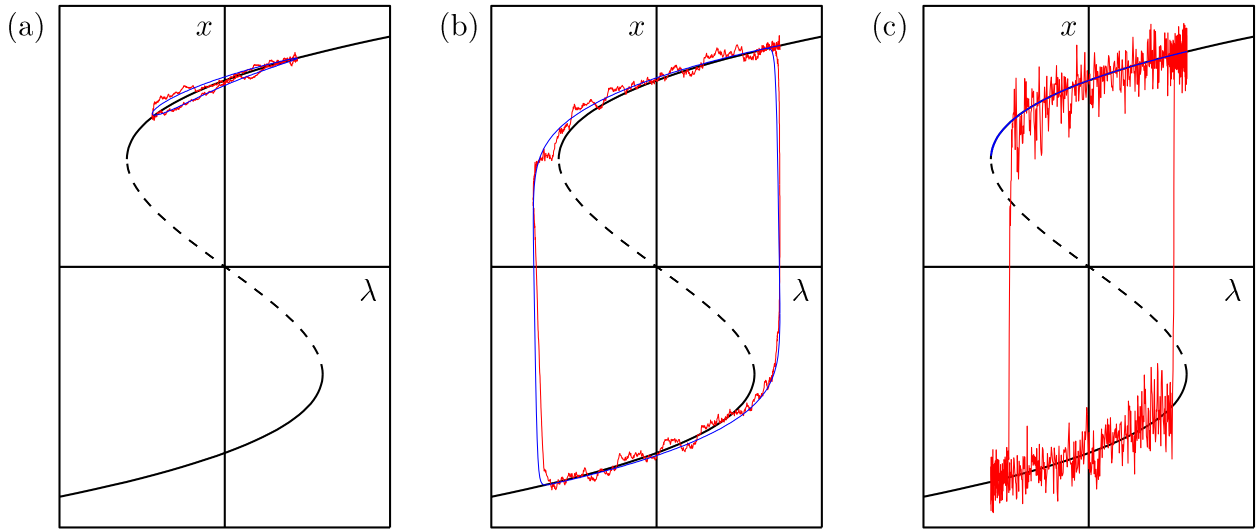
Deterministic solutions stay at distance $\varepsilon^{1/3}$ *above* the bifurcation point $(-\widehat{\mu}, \widehat{y})$ until time $-\mu = -\widehat{\mu} + \varepsilon^{2/3}$

Theorem [Berglund & G '02]

- $\sigma \ll \sqrt{\varepsilon}$: Paths likely to remain in $\mathcal{B}(\sigma)$ until time $\varepsilon^{2/3}$ after bifurcation, with maximal spreading $\sigma/\varepsilon^{1/6}$
- $\sigma \gg \sqrt{\varepsilon}$: Paths likely to escape at time $\sigma^{4/3}$ *before* bifurcation

Case II: Periodic forcing

Assume periodic freshwater flux $\mu(t)$ (centred w.r.t. bifurcation diagram)



Theorem [Berglund & G '02]

- Small amplitude, small noise: Transitions unlikely during one cycle (However: Concentration of transition times within each period)
- Large amplitude, small noise: Hysteresis cycles
  Area = static area + $\mathcal{O}(\varepsilon^{2/3})$ (as in deterministic case)
- Large noise: Stoch. resonance / noise-induced synchronization
  Area = static area − $\mathcal{O}(\sigma^{4/3})$ (reduced due to noise)

Density of the first-passage time through the unstable branch

Theorem [Berglund & G '05], work in progress

After a model-dependent time change:

$$p(t,t_0) = \frac{1}{\mathcal{N}}\, Q_{\lambda T}\big(t - |\log\sigma|\big) \frac{1}{\lambda T_{\mathsf{K}}(\sigma)}\, \mathrm{e}^{-(t-t_0)\,/\,\lambda T_{\mathsf{K}}(\sigma)}\, f_{\text{trans}}(t,t_0)$$

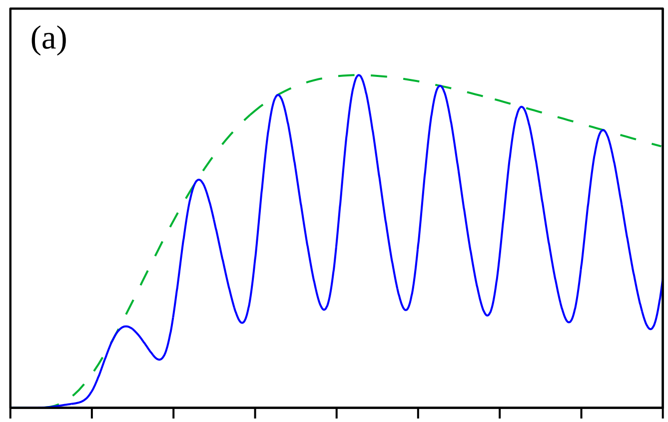
- ▷ $\mathcal{N}$ is the normalization
- ▷ $T_{\mathsf{K}}(\sigma)$ is the analogue of Kramers' time: $T_{\mathsf{K}}(\sigma) = \dfrac{C}{\sigma}\, e^{\overline{V}/\sigma^2}$
- ▷ f_{trans} grows from 0 to 1 in time $t - t_0$ of order $|\log\sigma|$
- ▷ $Q_{\lambda T}(y)$ is a *universal* λT-periodic function

Periodic dependence on $|\log\sigma|$: Peaks rotate as σ decreases

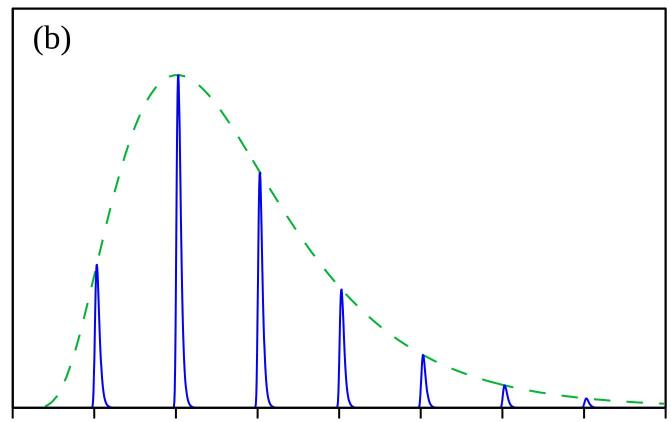
Rate of escape (in quasistat. regime) does not converge for $\sigma \to 0$!

$$Q_{\lambda T}(y) = 2\lambda T \sum_{k=-\infty}^{\infty} P(y - k\lambda T)$$

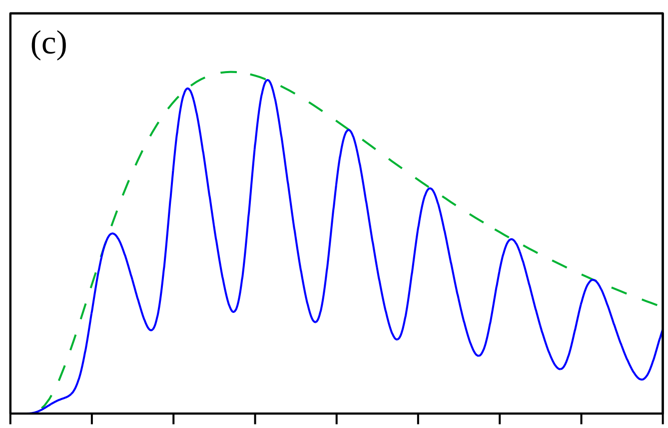
with double-exponential (Gumbel) peaks $P(z) = \frac{1}{2}\,\mathrm{e}^{-2z} \exp\left\{-\frac{1}{2}\,\mathrm{e}^{-2z}\right\}$



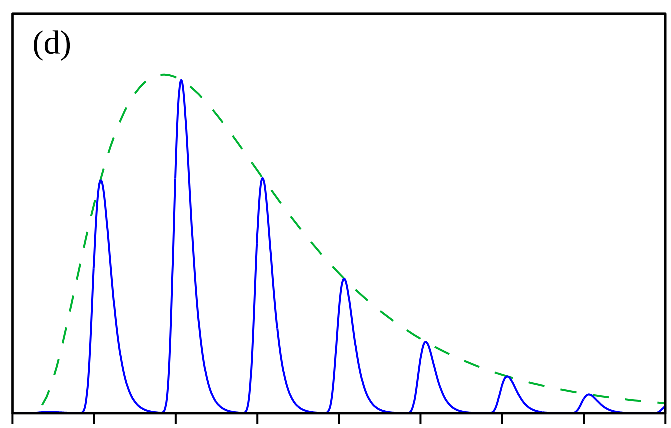
$\sigma = 0.4,\ T = 2$



$\sigma = 0.4,\ T = 20$



$\sigma = 0.5,\ T = 2$



$\sigma = 0.5,\ T = 5$

At approximately $78°55'$ N

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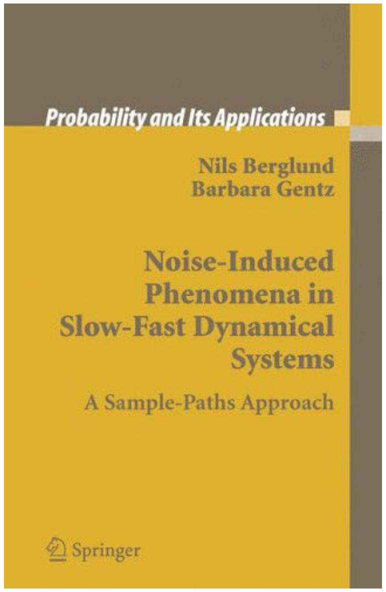

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