Asymptotic Problems in Stochastic Processes and PDEs

University of Maryland, 20–24 May 2013

Small eigenvalues and mean transition times for irreversible diffusions

Barbara Gentz

University of Bielefeld, Germany

Joint work with Nils Berglund (Université d'Orléans, France)

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Outline

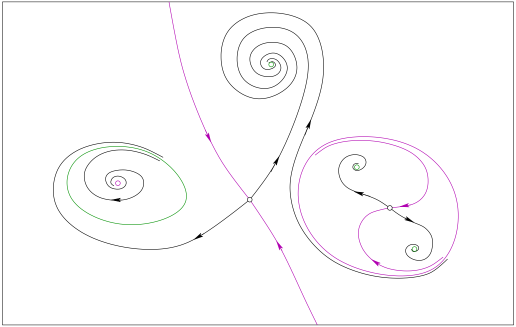
- Random perturbations of ODEs: A brief reminder
- The irreversible case
- Exit through an unstable periodic orbit
- Transitions between stable periodic orbits

Random Perturbations of ODEs:
A brief reminder

Dynamics of ODEs

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}g(x_t)\,\mathrm{d}W_t\;, \qquad x \in \mathbb{R}^n$$

Phase portrait
$(\varepsilon = 0)$



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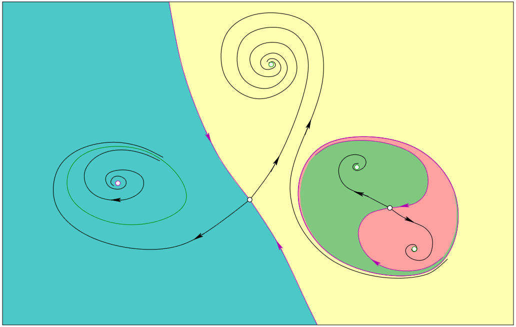
Phase portrait
($\varepsilon = 0$)

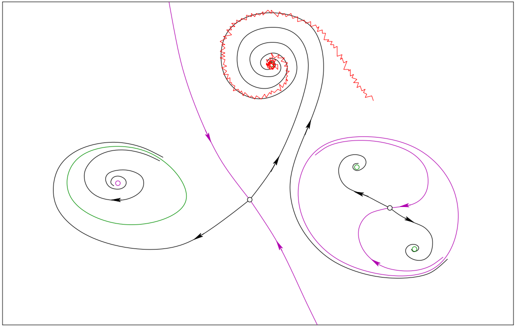
with basins of
attraction



Random perturbations of ODEs

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}g(x_t)\,\mathrm{d}W_t\ , \qquad x \in \mathbb{R}^n$$

Sample path
$(0 < \varepsilon \ll 1)$



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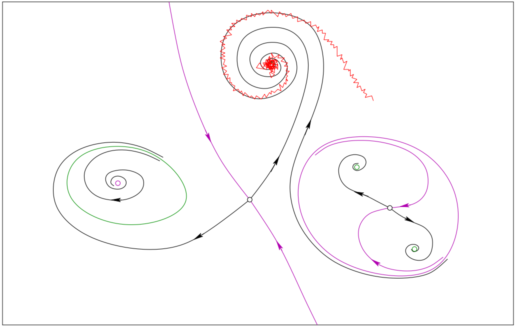

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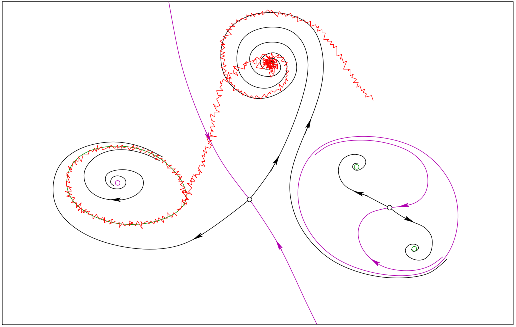

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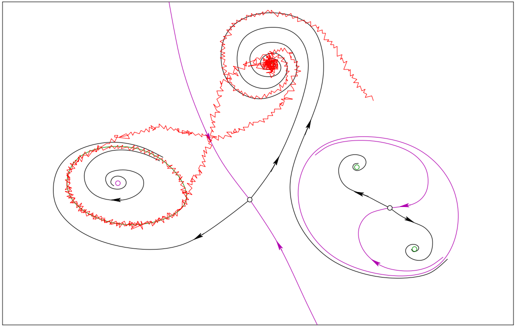

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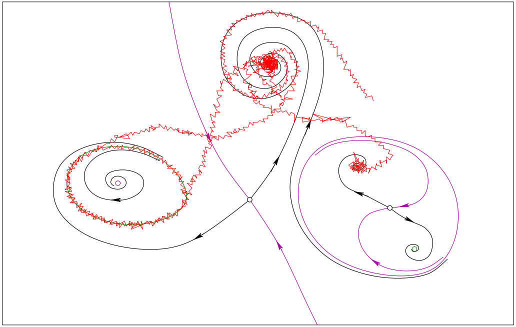

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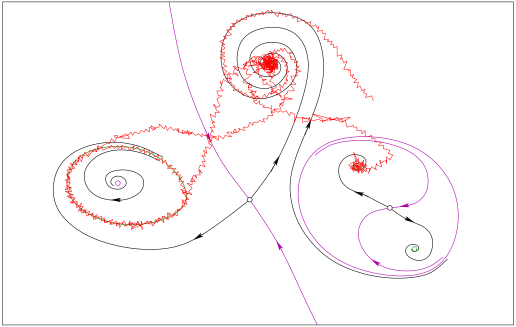


- Transitions between attractors are rare events
- Optimal transition paths described by Wentzell–Freidlin theory
- Jumps between attractors described by Markov process

Transition probabilities and generators

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}g(x_t)\,\mathrm{d}W_t\;, \qquad x \in \mathbb{R}^n$$

- Transition probability density $p_t(x,y)$
- Markov semigroup T_t: For measurable $\varphi \in L^\infty$,

$$(T_t\varphi)(x) = \mathbb{E}^x\{\varphi(x_t)\} = \int p_t(x,y)\varphi(y)\,\mathrm{d}y$$

- Generator: $L\varphi = \frac{\mathrm{d}}{\mathrm{d}t} T_t\varphi|_{t=0}$

$$(L\varphi)(x) = \sum_i f_i(x)\frac{\partial\varphi}{\partial x_i} + \varepsilon\sum_{i,j}(gg^T)_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}$$

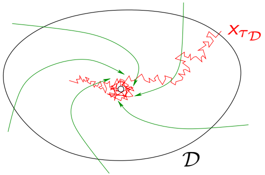
- Adjoint semigroup: For probabilty measures μ

$$(\mu T_t)(y) = \mathbb{P}^\mu\{x_t = \mathrm{d}y\} = \int p_t(x,y)\,\mu(\mathrm{d}x)$$

with generator L^*

Stochastic exit problem

- ▹ $\mathcal{D} \subset \mathbb{R}^n$ bounded domain
- ▹ First-exit time $\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$
- ▹ First-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$
- ▹ Harmonic measure $\mu(A) = \mathbb{P}^x\{x_{\tau_{\mathcal{D}}} \in A\}$



Facts (following from Dynkin's formula)

- ▹ $u(x) = \mathbb{E}^x\{\tau_{\mathcal{D}}\}$ satisfies

$$\begin{cases} Lu(x) = -1 & \text{for } x \in \mathcal{D} \\ u(x) = 0 & \text{for } x \in \partial\mathcal{D} \end{cases}$$

- ▹ For $\varphi \in L^\infty(\partial\mathcal{D}, \mathbb{R})$, $h(x) = \mathbb{E}^x\{\varphi(x_{\tau_{\mathcal{D}}})\}$ satisfies

$$\begin{cases} Lh(x) = 0 & \text{for } x \in \mathcal{D} \\ h(x) = \varphi(x) & \text{for } x \in \partial\mathcal{D} \end{cases}$$

Wentzell–Freidlin theory

$$dx_t = f(x_t)\,dt + \sqrt{2\varepsilon} g(x_t)\,dW_t\ , \qquad x \in \mathbb{R}^n$$

- Large-deviation rate function / action funtional

$$I(\gamma) = \frac{1}{2}\int_0^T [\dot\gamma_t - f(\gamma_t)]^T D(\gamma_t)^{-1}[\dot\gamma_t - f(\gamma_t)]\,dt\ , \qquad \text{where } D = gg^T$$

- Large-deviation principle: For a set Γ of paths $\gamma : [0, T] \to \mathbb{R}^n$

$$\mathbb{P}\{(x_t)_{0\leqslant t\leqslant T} \in \Gamma\} \simeq e^{-\inf_\Gamma I/2\varepsilon}$$

Consider first exit from $\mathcal{D}$ contained in basin of attraction of an attractor $\mathcal{A}$

- Quasipotential

$$V(y) = \inf\{I(\gamma) \colon \gamma \text{ connects } \mathcal{A} \text{ to } y \text{ in arbitrary time}\}\ , \qquad y \in \partial\mathcal{D}$$

Wentzell–Freidlin theory

$$V(y) = \inf\{I(\gamma) \colon \gamma \text{ connects } \mathcal{A} \text{ to } y \text{ in arbitrary time}\}\;, \qquad y \in \partial\mathcal{D}$$

Facts

- $\lim\limits_{\varepsilon\to0} 2\varepsilon \log \mathbb{E}\{\tau_{\mathcal{D}}\} = \overline{V} = \inf\limits_{y\in\partial\mathcal{D}} V(y)$ [Wentzell, Freidlin '69]
- If infimum is attained in a single point $y^* \in \mathcal{D}$ then
  $\lim\limits_{\varepsilon\to0} \mathbb{P}\{\|x_{\tau_{\mathcal{D}}} - y^*\| > \delta\} = 0 \quad \forall\delta > 0$ [Wentzell, Freidlin '69]
- Limiting distribution of $\tau_{\mathcal{D}}$ is exponential
  $\lim\limits_{\varepsilon\to0} \mathbb{P}\{\tau_{\mathcal{D}} > s\mathbb{E}\{\tau_{\mathcal{D}}\}\} = \mathrm{e}^{-s}$ [Day '83; Bovier *et al* '05]

The reversible case

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t\,, \qquad x \in \mathbb{R}^n$$

- $L = \varepsilon\Delta - \nabla V(x)\cdot\nabla = \varepsilon\, e^{V/\varepsilon}\,\nabla\cdot e^{-V/\varepsilon}\,\nabla$ is self-adjoint in $L^2(\mathbb{R}^n, e^{-V/\varepsilon}\,dx)$
- Reversibility (detailed balance): $e^{-V(x)/\varepsilon}\,p_t(x,y) = e^{-V(y)/\varepsilon}\,p_t(y,x)$
- $\widetilde{L} = e^{-V/2\varepsilon}\,L\,e^{V/2\varepsilon}$ is self-adjoint in $L^2(\mathbb{R}^n, dx)$

Results

Assume V has N local minima

- $-L$ has N exponentially small ev's $0 = \lambda_0 < \dots < \lambda_{N-1}$ + spectral gap
- Precise expressions for the λ_i (Kramers' law)
- λ_i^{-1} are the expected transition times between neighbourhoods of minima, $i = 1, \dots, N-1$ (in specific order)

Methods

- Large deviations [Wentzell, Freidlin, Sugiura, ...]
- Semiclassical analysis [Mathieu, Miclo, Kolokoltsov, ...]
- Potential theory [Bovier, Gayrard, Eckhoff, Klein]
- Witten Laplacian [Helffer, Nier, Le Peutrec, Viterbo]

The irreversible case

Irreversible case

If f is *not* of the form $-\nabla V$

- Large-deviation techniques still work, but ...
- L not self-adjoint, analytical approaches harder
- not reversible, standard potential theory does not work

Nevertheless,

- Results exist on the Kramers–Fokker–Planck operator

$$L = \varepsilon y \frac{\partial}{\partial x} - \varepsilon V'(x) \frac{\partial}{\partial y} + \frac{\gamma}{2}\Bigl(y - \varepsilon \frac{\partial}{\partial y}\Bigr)\Bigl(y + \varepsilon \frac{\partial}{\partial y}\Bigr)$$

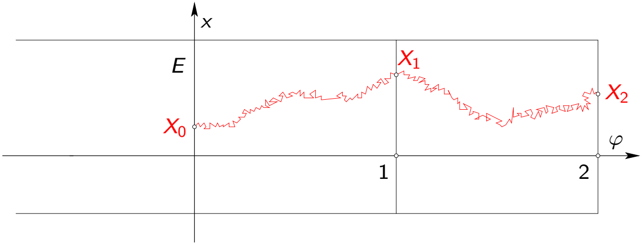
  [Hérau, Hitrik, Sjöstrand, ...]
- Here we consider two questions involving periodic orbits, namely
  - What is the harmonic measure for the exit through an unstable periodic orbit?
  - What can we say on exponentially small eigenvalues for systems admitting N stable periodic orbits?

Random Poincaré maps

Near a periodic orbit, in appropriate coordinates

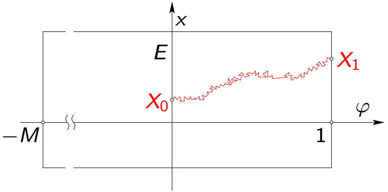
$$\begin{aligned} d\varphi_t &= f(\varphi_t, x_t)\, dt + \sigma F(\varphi_t, x_t)\, dW_t & \varphi &\in \mathbb{R} \\ dx_t &= g(\varphi_t, x_t)\, dt + \sigma G(\varphi_t, x_t)\, dW_t & x &\in E \subset \mathbb{R}^{n-1} \end{aligned}$$

- All functions periodic in φ (e.g. period 1)
- $f \geqslant c > 0$ and σ small $\Rightarrow \varphi_t$ likely to increase
- Process may be killed when x leaves E



Random variables $X_0, X_1, \dots$ form (substochastic) Markov chain

Random Poincaré map and harmonic measures



- First-exit time τ of $z_t = (\varphi_t, x_t)$ from $\mathcal{D} = (-M, 1) \times E$
- $\mu_z(A) = \mathbb{P}^z\{z_\tau \in A\}$ is harmonic measure (w.r.t. generator L)
- μ_z admits (smooth) density $h(z, y)$ w.r.t. arclength on $\partial\mathcal{D}$ (under hypoellipticity condition) [Ben Arous, Kusuoka, Stroock '84]
- Remark: $Lh(\cdot, y) = 0$ (kernel is harmonic)
- For Borel sets $B \subset E$

$$\mathbb{P}^{X_0}\{X_1 \in B\} = K(X_0, B) := \int_B K(X_0, \mathrm{d}y)$$

where $K(x, \mathrm{d}y) = h((0, x), (1, y))\,\mathrm{d}y =: k(x, y)\,\mathrm{d}y$

Fredholm theory

Consider integral operator K acting

- on L^∞ via $f \mapsto (Kf)(x) = \displaystyle\int_E k(x,y)f(y)\,\mathrm{d}y = \mathbb{E}^x\{f(X_1)\}$
- on L^1 via $m \mapsto (mK)(A) = \displaystyle\int_E m(x)k(x,y)\,\mathrm{d}x = \mathbb{P}^\mu\{X_1 \in A\}$

[Fredholm 1903]

- If $k \in L^2$, then K has eigenvalues λ_n of finite multiplicity
- Eigenfunctions $Kh_n = \lambda_n h_n$, $h_n^* K = \lambda_n h_n^*$ form a complete ONS

[Perron; Frobenius; Jentzsch 1912; Krein–Rutman '50; Birkhoff '57]

- Principal eigenvalue λ_0 is real, simple, $|\lambda_n| < \lambda_0 \quad \forall n \geqslant 1$ and $h_0 > 0$

Spectral decomposition: $\quad k\ (x,y) = \lambda_0 h_0(x) h_0^*(y) + \lambda_1 h_1(x) h_1^*(y) + \dots$

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Spectral decomposition: $\quad k^n(x,y) = \lambda_0^n h_0(x) h_0^*(y) + \lambda_1^n h_1(x) h_1^*(y) + \dots$

$$\Rightarrow \quad \mathbb{P}^x\{X_n \in \mathrm{d}y | X_n \in E\} = \pi_0(\mathrm{d}y) + \mathcal{O}((|\lambda_1|/\lambda_0)^n)$$

where $\pi_0 = h_0^* \,/ \int_E h_0^*$ is the quasistationary distribution (QSD)

How to estimate the principal eigenvalue

- Trivial bounds: $\forall A \subset E$ with Lebesgue$(A) > 0$,

$$\inf_{x\in A} K(x,A) \leqslant \lambda_0 \leqslant \sup_{x\in E} K(x,E)$$

Proof

$$x^* = \operatorname{argmax} h_0 \Rightarrow \lambda_0 = \int_E k(x^*,y)\frac{h_0(y)}{h_0(x^*)}\,\mathrm{d}y \leqslant K(x^*,E)$$

$$\lambda_0 \int_A h_0^*(y)\,\mathrm{d}y = \int_E h_0^*(x)K(x,A)\,\mathrm{d}x \geqslant \inf_{x\in A} K(x,A)\int_A h_0^*(y)\,\mathrm{d}y$$

- Donsker–Varadhan-type bound:

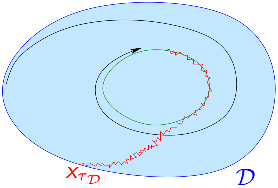
$$\lambda_0 \leqslant 1 - \frac{1}{\sup_{x\in E}\mathbb{E}^x\{\tau_\Delta\}} \qquad \text{where} \quad \tau_\Delta = \inf\{n > 0 \colon X_n \notin E\}$$

- Bounds using Laplace transforms (see below)

Exit through a characteristic boundary: The case of an unstable periodic orbit

Application: Exit through an unstable periodic orbit

- Planar SDE
  $dx_t = f(x_t)\,dt + \sigma g(x_t)\,dW_t$
- $\mathcal{D} \subset \mathbb{R}^2$: interior of unstable periodic orbit
- First-exit time $\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$



Law of first-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$?

- Large-deviation principle with rate function
  $$I(\gamma) = \frac{1}{2}\int_0^T [\dot\gamma_t - f(\gamma_t)]^T D(\gamma_t)^{-1}[\dot\gamma_t - f(\gamma_t)]\,dt\,, \qquad \text{where } D = gg^T$$
- Quasipotential
  $V(y) = \inf\{I(\gamma) \colon \gamma \text{ connects } \mathcal{A} \text{ to } y \text{ in arbitrary time}\}$

Theorem [Freidlin, Wentzell '69]
If V attains its min at a unique $y^* \in \partial\mathcal{D}$, then $x_{\tau_{\mathcal{D}}}$ concentrates in y^* as $\sigma \to 0$

Problem: V is constant on $\partial\mathcal{D}$!

Most probable exit paths

Minimizers of I obey Hamilton equations with Hamiltonian

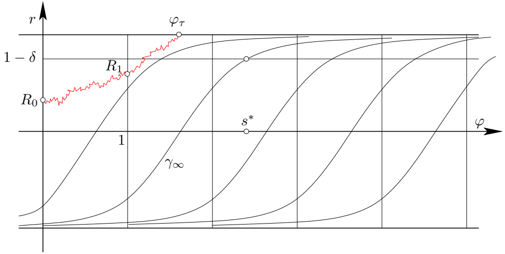
$$H(\gamma, \psi) = \frac{1}{2}\psi^T D(\gamma)\psi + f(\gamma)^T\psi$$

where $\psi = D(\gamma)^{-1}(\dot{\gamma} - f(\gamma))$

Generically optimal path (for infinite time) is isolated

Random Poincaré map

In polar-type coordinates (r, φ)



$$\mathbb{P}^{R_0}\{R_n \in A\} = \lambda_0^n h_0(R_0) \int_A h_0^*(y)\,\mathrm{d}y \left[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\right]$$

If $t = n + s$,

$$\mathbb{P}^{R_0}\{\varphi_\tau \in \mathrm{d}t\} = \lambda_0^n h_0(R_0) \int h_0^*(y) \mathbb{P}^y\{\varphi_\tau \in \mathrm{d}s\}\,\mathrm{d}y \left[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\right]$$

Periodically modulated exponential distribution

Main result: Cycling [Day '90]

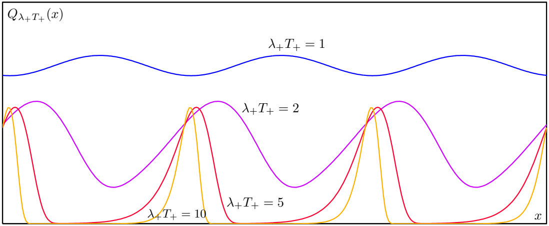
Theorem [Berglund & G, 2012 (submitted)]

$\forall\,\Delta > 0\;\forall\,\delta > 0\;\exists\,\sigma_0 > 0\;\forall\,0 < \sigma < \sigma_0$

$$\mathbb{P}^{r_0,0}\{\varphi_\tau \in [\varphi, \varphi + \Delta]\} = C(\sigma)(\lambda_0)^\varphi \chi_\Delta(\varphi) Q_{\lambda_+ T_+}\left(\frac{|\log\sigma| - \theta(\varphi) + \mathcal{O}(\delta)}{\lambda_+ T_+}\right) \times \left[1 + \mathcal{O}(\mathrm{e}^{-c\varphi/|\log\sigma|}) + \mathcal{O}(\delta|\log\delta|)\right]$$

- Cycling profile, periodicised Gumbel distribution

$$Q_{\lambda T}(x) = \sum_{n=-\infty}^{\infty} A(\lambda T(n - x)) \quad \text{with} \quad A(x) = \tfrac{1}{2}\exp\{-2x - \tfrac{1}{2}\,\mathrm{e}^{-2x}\}$$



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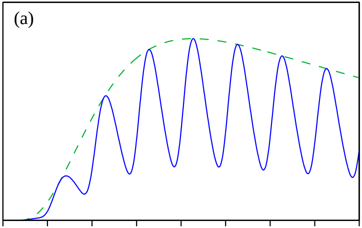
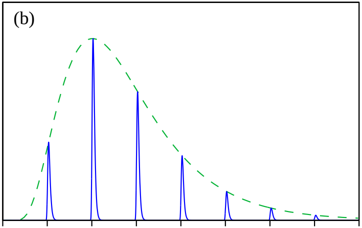
- $\theta(\varphi)$ explicit function of $D_{rr}(1,\varphi)$, $\theta(\varphi+1) = \theta(\varphi) + \lambda_+ T_+$ ($\lambda_+ =$ Lyapunov exponent, $T_+ =$ period of unstable orbit)
- λ_0 principal eigenvalue, $\lambda_0 = 1 - \mathrm{e}^{-\tilde{V}/\sigma^2}$
- $C(\sigma) = \mathcal{O}(\mathrm{e}^{-\tilde{V}/\sigma^2})$
- $\chi_\Delta(\varphi) \sim \mathbb{P}^{\pi_0^{\mathrm{u}}}\{\varphi_\tau \in [\varphi, \varphi+\Delta]\}$, period 1 (in linear case $\chi_\Delta(\varphi) \simeq \theta'(\varphi)\Delta$)

Cycling: Periodic dependence on $|\log\sigma|$ [Day '90, Maier & Stein '96]

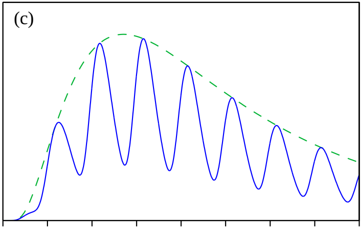
Density of the first-passage time (for $V = 0.5$, $\lambda_+ = 1$)



$\sigma^2 = 0.4,\ T_+ = 2$



$\sigma^2 = 0.4,\ T_+ = 20$



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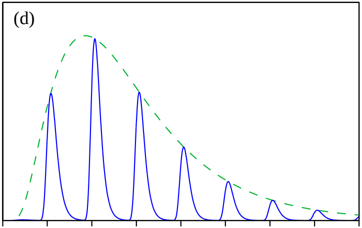
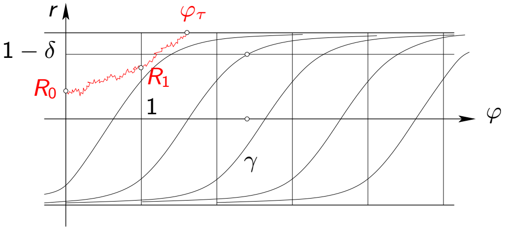


$\sigma^2 = 0.5,\ T_+ = 5$

Sketch of the proof



Split into two Markov chains:

- First chain killed upon r reaching $1-\delta$ in $\varphi = \varphi_{\tau_-}$

$$\mathbb{P}^0\{\varphi_{\tau_-} \in [\varphi_1, \varphi_1 + \Delta]\} \simeq (\lambda_0^{\mathrm{s}})^{\varphi_1} \,\mathrm{e}^{-J(\varphi_1)/\sigma^2}$$

- Second chain killed at $r = 1 - 2\delta$ and on unstable orbit $r = 1$
  - Principal eigenvalue: $\lambda_0^{\mathrm{u}} = \mathrm{e}^{-2\lambda_+ T_+}(1 + \mathcal{O}(\delta))$
    $\lambda_+ =$ Lyapunov exponent, $T_+ =$ period of unstable orbit
  - Using LDP

$$\mathbb{P}^{\varphi_1}\{\varphi_\tau \in [\varphi, \varphi + \Delta]\} \simeq (\lambda_0^{\mathrm{u}})^{\varphi - \varphi_1} \,\mathrm{e}^{-[I_\infty + c(\mathrm{e}^{-2\lambda_+ T_+(\varphi - \varphi_1)})]/\sigma^2}$$

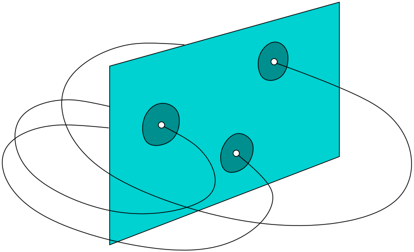
Open questions

- Proof involving only spectral theory, without using large-deviation principle
- More precise estimates on spectrum and eigenfunctions of K
- Link between spectra of K and of L (with Dirichlet b.c.)
- Origin of Gumbel distribution

Transitions between stable periodic orbits

Systems with several stable periodic orbits

[Joint work with Nils Berglund & Christian Kuehn, in progress]



- ▹ Consider system of dim $\geqslant 3$ with several stable periodic orbits
- ▹ We want to quantify transitions between these orbits
- ▹ Define again a Poincaré section and associated Markov process
- ▹ Exponentially small eigenvalues of this process?

Laplace transforms

Given $A \subset E$, $B \subset E \cup \{\Delta\}$, $A \cap B = \emptyset$, $x \in E$ and $u \in \mathbb{C}$, define

$$\tau_A = \inf\{n \geqslant 1 \colon X_n \in A\} \qquad G^u_{A,B}(x) = \mathbb{E}^x\{\mathrm{e}^{u\tau_A} 1_{\{\tau_A < \tau_B\}}\}$$

$$\sigma_A = \inf\{n \geqslant 0 \colon X_n \in A\} \qquad H^u_{A,B}(x) = \mathbb{E}^x\{\mathrm{e}^{u\sigma_A} 1_{\{\sigma_A < \sigma_B\}}\}$$

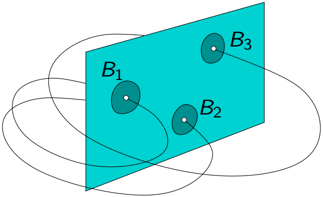
- $G^u_{A,B}(x)$ is analytic for $|\mathrm{e}^u| < \left[\sup_{x \in (A\cup B)^c} K(x, (A \cup B)^c)\right]^{-1}$
- $G^u_{A,B} = H^u_{A,B}$ in $(A \cup B)^c$, $H^u_{A,B} = 1$ in A and $H^u_{A,B} = 0$ in B
- Feynman–Kac-type relation

$$KH^u_{A,B} = \mathrm{e}^{-u}\, G^u_{A,B}$$

(Proof: Split events according to $X_1 \in A$ or $X_1 \notin A$)

Conclusion: If $G^u_{A,B}$ varies little in $A \cup B$, it is close to an eigenfunction.

Heuristics (inspired by [Bovier, Eckhoff, Gayrard, Klein '04])



- Stable periodic orbits in $x_1, \dots, x_N$
- B_i small ball around x_i, $B = \bigcup_{i=1}^N B_i$
- Eigenvalue equation $(Kh)(x) = \mathrm{e}^{-u}\, h(x)$
- Assume $h(x) \simeq h_i$ in B_i

Ansatz: $$h(x) = \sum_{j=1}^N h_j H^u_{B_j, B \setminus B_j}(x) + r(x)$$

- $x \in B_i$: $h(x) = h_i + r(x)$
- $x \in B^c$: eigenvalue equation satisfied (by Feynman–Kac)
- $x = x_i$: eigenvalue equation requires (by Feynman–Kac)

$$h_i = \sum_{j=1}^N h_j M_{ij}(u) \qquad M_{ij}(u) = G^u_{B_j, B\setminus B_j}(x_i) = \mathbb{E}^{x_i}\{\mathrm{e}^{u\tau_B}\, 1_{\{\tau_B = \tau_{B_j}\}}\}$$

$\Rightarrow$ condition $\det(M - \mathbb{1}) = 0 \Rightarrow N$ eigenvalues exp close to 1

If $\mathbb{P}\{\tau_B > 1\} \ll 1$ then $M_{ij}(u) \simeq \mathrm{e}^u\, \mathbb{P}^{x_i}\{\tau_B = \tau_{B_j}\} =: \mathrm{e}^u\, P_{ij}$ and $Ph \simeq \mathrm{e}^{-u}\, h$

Control of the error term

The error term satisfies the boundary value problem

$$\begin{aligned} (Kr)(x) &= \mathrm{e}^{-u}\, r(x) && x \in B^c \\ r(x) &= h(x) - h_i && x \in B_i \end{aligned}$$

Lemma: For u s.t. G^u_{B,E^c} exists, the unique solution of

$$\begin{aligned} (K\psi)(x) &= \mathrm{e}^{-u}\, \psi(x) && x \in B^c \\ \psi(x) &= \theta(x) && x \in B \end{aligned}$$

is given by $\psi(x) = \mathbb{E}^x\{\mathrm{e}^{u\tau_B}\, \theta(X_{\tau_B})\}$.

Proof

- Show that $\mathcal{T}f(x) = \mathbb{E}^x\{\mathrm{e}^u\, \theta(X_1) 1_{\{X_1 \in B\}}\} + \mathbb{E}^x\{\mathrm{e}^u\, f(X_1) 1_{\{X_1 \in B^c\}}\}$ is a contraction on $L^\infty(B^c)$
- Set $\psi_0(x) = 0$, $\psi_{n+1}(x) = \mathcal{T}\psi_n(x) \quad \forall n \geqslant 0$
- Show by induction that $\psi_n(x) = \mathbb{E}^x\{\mathrm{e}^{u\tau_B}\, \theta(X_{\tau_B}) 1_{\{\tau_B \leqslant n\}}\}$
- $\psi(x) = \lim_{n\to\infty} \psi_n(x)$ is fixed point of $\mathcal{T}$ ⇒ satisfies the boundary value problem

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is given by $\psi(x) = \mathbb{E}^x\{\mathrm{e}^{u\tau_B}\, \theta(X_{\tau_B})\}$.

$\Rightarrow r(x) = \mathbb{E}^x\{\mathrm{e}^{u\tau_B}\, \theta(X_{\tau_B})\}$ where $\theta(x) = \sum_j [h(x) - h_j] 1_{\{x \in B_j\}}$

To show that $h(x) - h_j$ is small in B_j: use Harnack inequalities

Conclusions

- Reduction to an N-state process in the sense that

$$\mathbb{P}^{x}\{X_n \in B_i\} = \sum_{j=1}^{N} \lambda_j^n h_j(x) h_j^*(B_i) + \mathcal{O}(|\lambda_{N+1}|^n)$$

- Residence times are approximately exponentially distributed (provided system can relax to QSD)
- Generically, eigenvalues λ_j are determined by "metastable hierarchy" of periodic orbits

Open problems

- How to determine efficiently the M_{ij} or $P_{ij} = \mathbb{P}^{x_i}\{\tau_B = \tau_{B_j}\}$?
  Large deviations – but not easy to implement and not always precise enough
- How to approximate left eigenfunctions (QSDs)?
- Chaotic orbits?

Thank you for your attention!