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The Effect of Gaussian White Noise on Dynamical Systems: Reduced Dynamics

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General slow–fast systems

General slow–fast systems

Fully coupled SDEs on well-separated time scales

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables} \in \mathbb{R}^n\text{)} \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \ \sigma' \ G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables} \in \mathbb{R}^m\text{)} \end{cases}$$

- $\{W_t\}_{t\geq 0}$ k-dimensional (standard) Brownian motion
- $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$
- $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^m$ drift coefficients, $\in \mathcal{C}^2$
- $F : \mathcal{D} \to \mathbb{R}^{n\times k}$, $G : \mathcal{D} \to \mathbb{R}^{m\times k}$ diffusion coefficients, $\in \mathcal{C}^1$

Small parameters

- $\varepsilon > 0$ adiabatic parameter (*no quasistatic* approach)
- $\sigma, \sigma' \geq 0$ noise intensities; may depend on ε:
  $\sigma = \sigma(\varepsilon)$, $\sigma' = \sigma'(\varepsilon)$ and $\sigma'(\varepsilon)/\sigma(\varepsilon) = \varrho(\varepsilon) \leq 1$

Singular limits for deterministic slow–fast systems

In slow time t		In fast time $s = t/\varepsilon$
$\varepsilon\dot{x} = f(x,y)$ $\dot{y} = g(x,y)$	$\overset{t\mapsto s}{\Longleftrightarrow}$	$x' = f(x,y)$ $y' = \varepsilon g(x,y)$
$\downarrow \varepsilon\to 0$		$\downarrow \varepsilon\to 0$
Slow subsystem $0 = f(x,y)$ $\dot{y} = g(x,y)$	$\not\Longleftrightarrow$	**Fast subsystem** $x' = f(x,y)$ $y' = 0$
Study slow variable y on slow manifold $f(x,y) = 0$		Study fast variable x for frozen slow variable y

Near slow manifolds: Assumptions on the fast variables

- Existence of a slow manifold
  $\exists \mathcal{D}_0 \subset \mathbb{R}^m \quad \exists x^\star : \mathcal{D}_0 \to \mathbb{R}^n$
  s.t. $(x^\star(y), y) \in \mathcal{D}$ and $f(x^\star(y), y) = 0$ for $y \in \mathcal{D}_0$

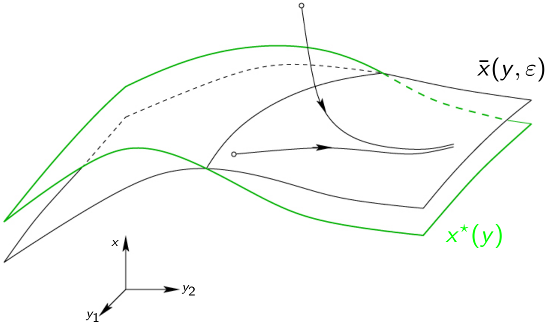
- Slow manifold is attracting
  Eigenvalues of $A^\star(y) := \partial_x f(x^\star(y), y)$ satisfy $\operatorname{Re} \lambda_i(y) \leq -a_0 < 0$
  (uniformly in $\mathcal{D}_0$)

Fenichel's theorem

Theorem ([Tihonov '52, Fenichel '79])

There exists an *adiabatic manifold*:
$\exists\, \bar{x}(y,\varepsilon)$ s.t.

- $\bar{x}(y,\varepsilon)$ is invariant manifold for deterministic dynamics
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,0) = x^\star(y)$
- $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



Consider now *stochastic* system under these assumptions

Random slow–fast systems: Slowly driven systems

Typical neighbourhoods for the stochastic fast variable

Special case: One-dim. slowly driven systems

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Stable slow manifold / stable equilibrium branch $x^\star(t)$:

$$f(x^\star(t), t) = 0\,, \qquad a^\star(t) = \partial_x f(x^\star(t), t) \leqslant -a_0 < 0$$

Linearize SDE for deviation $x_t - \bar{x}(t,\varepsilon)$ from adiabatic solution $\bar{x}(t,\varepsilon) \approx x^\star(t)$

$$\mathrm{d}z_t = \frac{1}{\varepsilon} a(t) z_t\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

We can solve the non-autonomous SDE for z_t

$$z_t = z_0 e^{\alpha(t)/\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha(t,s)/\varepsilon}\,\mathrm{d}W_s$$

where $\alpha(t) = \int_0^t a(s)\,\mathrm{d}s$, $\alpha(t,s) = \alpha(t) - \alpha(s)$ and $a(t) = \partial_x f(\bar{x}(t,\varepsilon), t)$

Typical spreading

$$z_t = z_0 e^{\alpha(t)/\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha(t,s)/\varepsilon} \, dW_s$$

z_t is a Gaussian r.v. with variance

$$v(t) = \mathrm{Var}(z_t) = \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\alpha(t,s)/\varepsilon} \, ds \approx \frac{\sigma^2}{|a(t)|}$$

For any fixed time t, z_t has a typical spreading of $\sqrt{v(t)}$, and a standard estimate shows

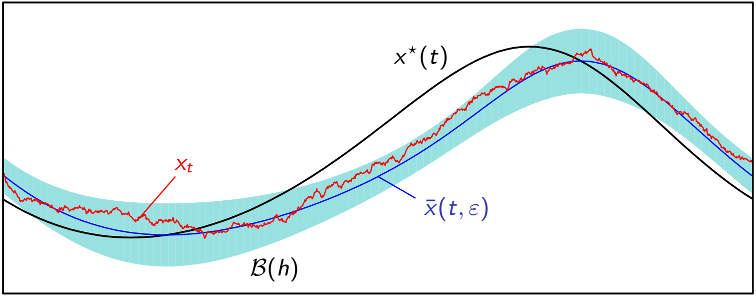
$$\mathbb{P}\{|z_t| \geq h\} \leq e^{-h^2/2v(t)}$$

Goal: Similar concentration result for the whole sample path

Define a strip $\mathcal{B}(h)$ around $\bar{x}(t,\varepsilon)$ of width $\simeq h/\sqrt{|a(t)|}$

$$\mathcal{B}(h) = \{(x,t)\colon |x - \bar{x}(t,\varepsilon)| < h/\sqrt{|a(t)|}\}$$

Concentration of sample paths



Theorem [Berglund & G '02, '06]

$$\mathbb{P}\{x_t \text{ leaves } \mathcal{B}(h) \text{ before time } t\} \simeq \sqrt{\frac{2}{\pi}}\,\frac{1}{\varepsilon}\left|\int_0^t a(s)\,\mathrm{d}s\right|\,\frac{h}{\sigma}\,\mathrm{e}^{-h^2[1-\mathcal{O}(\varepsilon)-\mathcal{O}(h)]/2\sigma^2}$$

Fully coupled random slow–fast systems

Typical spreading in the general case

$$\begin{cases} dx_t = \frac{1}{\varepsilon} f(x_t, y_t)\, dt + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\, dW_t & \text{(fast variables} \in \mathbb{R}^n) \\ dy_t = \quad g(x_t, y_t)\, dt + \sigma' \; G(x_t, y_t)\, dW_t & \text{(slow variables} \in \mathbb{R}^m) \end{cases}$$

- Consider det. process $(x_t^{\text{det}} = \bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}})$ on adiabatic manifold
- Deviation $\xi_t := x_t - x_t^{\text{det}}$ of fast variables from adiabatic manifold
- Linearize SDE for ξ_t ; resulting process ξ_t^0 is Gaussian

Key observation

$\frac{1}{\sigma^2} \operatorname{Cov} \xi_t^0$ is a particular solution of the deterministic slow–fast system

$$(*) \begin{cases} \varepsilon \dot{X}(t) = A(y_t^{\text{det}}) X(t) + X(t) A(y^{\text{det}})^{\mathrm{T}} + F_0(y^{\text{det}}) F_0(y^{\text{det}})^{\mathrm{T}} \\ \dot{y}_t^{\text{det}} \; = g(\bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}}) \end{cases}$$

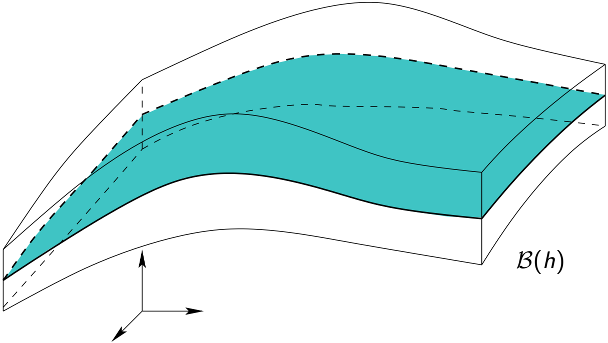
with $A(y) = \partial_x f(\bar{x}(y, \varepsilon), y)$ and F_0 $0th$-order approximation to F

Typical neighbourhoods in the general case

Typical neighbourhoods

$$\mathcal{B}(h) := \left\{ (x,y) \colon \left\langle \left[x - \bar{x}(y,\varepsilon) \right], \bar{X}(y,\varepsilon)^{-1} \left[x - \bar{x}(y,\varepsilon) \right] \right\rangle < h^2 \right\}$$

where $\bar{X}(y,\varepsilon)$ denotes the adiabatic manifold for the system $(*)$



Concentration of sample paths

Define (random) first-exit times

$\tau_{\mathcal{D}_0} \quad := \inf\{s > 0 \colon y_s \notin \mathcal{D}_0\}$

$\tau_{\mathcal{B}(h)} := \inf\{s > 0 \colon (x_s, y_s) \notin \mathcal{B}(h)\}$

Theorem [Berglund & G, JDE 2003]

Assume $\|\bar{X}(y,\varepsilon)\|$, $\|\bar{X}(y,\varepsilon)^{-1}\|$ uniformly bounded in $\mathcal{D}_0$

Then $\quad \exists\, \varepsilon_0 > 0 \quad \exists\, h_0 > 0 \quad \forall\, \varepsilon \leqslant \varepsilon_0 \quad \forall\, h \leqslant h_0$

$$\mathbb{P}\bigl\{\tau_{\mathcal{B}(h)} < \min(t, \tau_{\mathcal{D}_0})\bigr\} \leqslant C_{n,m}(t)\, \exp\biggl\{-\frac{h^2}{2\sigma^2}\bigl[1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)\bigr]\biggr\}$$

where $C_{n,m}(t) = \bigl[C^m + h^{-n}\bigr]\biggl(1 + \dfrac{t}{\varepsilon^2}\biggr)$

Reduced dynamics

Reduction to adiabatic manifold $\bar{x}(y,\varepsilon)$:

$$\mathrm{d}y_t^0 = g(\bar{x}(y_t^0,\varepsilon),y_t^0)\,\mathrm{d}t + \sigma' G(\bar{x}(y_t^0,\varepsilon),y_t^0)\,\mathrm{d}W_t$$

Theorem – informal version [Berglund & G '06]

y_t^0 approximates y_t to order $\sigma\sqrt{\varepsilon}$ up to Lyapunov time of $\dot{y}^{\mathrm{det}} = g(\bar{x}(y^{\mathrm{det}},\varepsilon)y^{\mathrm{det}})$

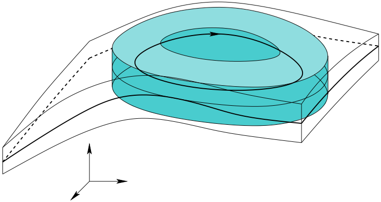
Remark

For $\dfrac{\sigma'}{\sigma} < \sqrt{\varepsilon}$, the deterministic reduced dynamics provides a better approximation

Longer time scales

Behaviour of g or behaviour of y_t and y_t^{det} becomes important

Example:
y_t^{det} following a stable periodic orbit



- $y_t \sim y_t^{\text{det}}$ for $t \leqslant \dfrac{const}{\sigma \vee \varrho^2 \vee \varepsilon}$

  linear coupling $\to$ ε
  nonlinear coupling $\to$ σ
  noise acting on slow variable $\to$ ϱ

- On longer time scales: Markov property allows for restarting
  y_t stays exponentially long in a neighbourhood of the periodic orbit (with probability close to 1)

The main idea of deterministic averaging

Which timescale should be studied?

Simple example

$$\dot{y}_s^\varepsilon = \varepsilon b(y_s^\varepsilon, \xi_s) , \quad y_0^\varepsilon = y \in \mathbb{R}^m$$

- $b : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$
- $\xi : [0, \infty) \to \mathbb{R}^n$
- $0 \leq \varepsilon \ll 1$

If b is not increasing too fast then

$y_s^\varepsilon \to y_s^0 \equiv y$ as $\varepsilon \to 0$ uniformly on any finite time interval $[0, T]$

Not the relevant timescale! ...need to look at time intervals of length $\geq 1/\varepsilon$

- Introduce slow time $t = \varepsilon s$
- Note that $t \in [0, T] \Leftrightarrow s \in [0, T/\varepsilon]$
- Rewrite equation

$$\dot{y}_t^\varepsilon = b(y_t^\varepsilon, \xi_{t/\varepsilon}) , \qquad y_0^\varepsilon = y \in \mathbb{R}^m$$

Deterministic averaging

Assumptions (simplest setting)

- $\|b(y_1,\xi) - b(y_2,\xi)\| \leqslant K\|y_1 - y_2\|$ for all $\xi \in \mathbb{R}^n$ (Lipschitz condition)
- $\lim_{T\to\infty} \frac{1}{T}\int_0^T b(y,\xi_t)\,\mathrm{d}t = \overline{b}(y)$ uniformly in $y \in \mathbb{R}^m$ (e.g., periodic ξ_t)

Can we obtain an autonomous equation for y_t^ε? Can we replace b by $\overline{b}$?

For small time steps Δ

$$y_\Delta^\varepsilon - y = \int_0^\Delta b(y_t^\varepsilon,\xi_{t/\varepsilon})\,\mathrm{d}t = \int_0^\Delta b(y,\xi_{t/\varepsilon})\,\mathrm{d}s + \int_0^\Delta \left[b(y_t^\varepsilon,\xi_{t/\varepsilon}) - b(y,\xi_{t/\varepsilon})\right]\mathrm{d}t$$

1. integral $= \Delta\frac{\varepsilon}{\Delta}\int_0^{\Delta/\varepsilon} b(y,\xi_s)\,\mathrm{d}s \approx \Delta\overline{b}(y)$ as $\varepsilon/\Delta \to 0$
2. integral $= \mathcal{O}(\Delta^2)$ (using Lipschitz continuity and leading order)

With a little work: y_t^ε converges uniformly on $[0,T]$ towards solution of $\dot{\overline{y}}_t = \overline{b}(\overline{y}_t)$

Averaging principle

Slow variable y_t^ε and fast variable ξ_t^ε (now depending on y_t^ε)

$$\begin{aligned} \dot{y}_t^\varepsilon &= b_1(y_t^\varepsilon, \xi_t^\varepsilon) , & y_0^\varepsilon &= y \in \mathbb{R}^m \\ \dot{\xi}_t^\varepsilon &= \frac{1}{\varepsilon} b_2(y_t^\varepsilon, \xi_t^\varepsilon) , & \xi_0^\varepsilon &= \xi \in \mathbb{R}^n \end{aligned}$$

Freeze slow variable y and consider

$$\dot{\xi}_t(y) = b_2(y, \xi_t(y)) , \qquad \xi_0(y) = \xi$$

Assume $\displaystyle\lim_{T\to\infty} \frac{1}{T} \int_0^T b_1(y, \xi_t(y))\,\mathrm{d}t = \overline{b}_1(y)$ exists (and is independent of ξ)

Averaging principle

The slow variable y_t^ε is well approximated by $\quad \dot{\overline{y}}_t = \overline{b}_1(\overline{y}_t) , \quad \overline{y}_0 = y$

Random fast motion:

The main idea of stochastic averaging

Random fast motion

Consider again assumption form last slide

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T b_1(y,\xi_t(y))\,\mathrm{d}t = \overline{b}_1(y) \quad \text{exists}$$

Convergence of time averages: Resembles *Law of Large Numbers*!

Our goal: Consider ξ_t given by a random motion

The general setting

$$\dot{y}_t^\varepsilon = b(\varepsilon, t, y_t^\varepsilon, \omega) , \qquad y_0^\varepsilon = y \in \mathbb{R}^m$$

$\omega \in \Omega$ indicates the random influence; underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Assumptions

- $(t, y) \mapsto b(\varepsilon, t, y, \omega)$ is continuous for almost all ω and all ε
- $\sup_{\varepsilon>0} \sup_{t\geq 0} \mathbb{E}\|b(\varepsilon, t, y, \omega)\|^2 < \infty$
- $\|b(\varepsilon, t, x, \omega) - b(\varepsilon, t, y, \omega)\| \leq K\|x - y\|$
  for almost all ω, all $x, y \in \mathbb{R}^m$, all $t \geq 0$ and $\varepsilon > 0$
- There exists $\overline{b}(y, t)$, continuous in (y, t), s.t. $\forall\delta > 0 \; \forall T > 0 \; \forall y \in \mathbb{R}^m$

$$\lim_{\varepsilon\to 0} \mathbb{P}\biggl\{\biggl\|\int_{t_0}^{t_0+T} b(\varepsilon, t, y, \omega)\,\mathrm{d}t - \int_{t_0}^{t_0+T} \overline{b}(t, y)\,\mathrm{d}t\biggr\| \geq \delta\biggr\} = 0$$

uniformly in $t_0 \geq 0$

Stochastic averaging

Theorem (c.f. [WF '84])

Under the assumptions on the previous slide,

$$\dot{\overline{y}}_t = \overline{b}(t, \overline{y}_t) , \qquad \overline{y}_0 = y$$

has a unique solution, and

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big\{ \max_{t \in [0,T]} \|y_t^\varepsilon - \overline{y}_t\| \geq \delta \Big\} = 0$$

for all $T > 0$ and all $\delta > 0$.

Remarks

- Convergence in probability is a rather weak notion
- Stronger assumptions yield stronger result

Idea of the proof I

$$\|y_t^\varepsilon - \overline{y}_t\| \leq \int_0^t \|b(\varepsilon, s, y_s^\varepsilon, \omega) - b(\varepsilon, s, \overline{y}_s, \omega)\| \,\mathrm{d}s + \left\| \int_0^t [b(\varepsilon, s, \overline{y}_s, \omega) - \overline{b}(s, \overline{y}_s)] \,\mathrm{d}s \right\|$$

Using Lipschitz condition

$$m(t) := \sup_{s\in[0,t]} \|y_s^\varepsilon - \overline{y}_s\| \leq K \int_0^t m(s) \,\mathrm{d}s + \sup_{s\in[0,t]} \left\| \int_0^s [b(\varepsilon, u, \overline{y}_u, \omega) - \overline{b}(u, \overline{y}_u)] \,\mathrm{d}s \right\|$$

Gronwall's lemma: sufficient to estimate

$$\mathbb{P}\left\{ \sup_{s\in[0,T]} \left\| \int_0^s [b(\varepsilon, u, \overline{y}_u, \omega) - \overline{b}(u, \overline{y}_u)] \,\mathrm{d}s \right\| \geq \tilde{\delta} \right\}$$

Idea of the proof II

- b Lipschitz continuous $\Rightarrow \overline{b}$ Lipschitz continuous
- On short time intervals $[kT/n, (k+1)T/n]$ replace $\overline{y}_u$ by $\overline{y}_{kT/n}$
- Total error accumulated over all time intervals is still $\mathcal{O}(1/n)$
- Apply assumption on $\overline{b}$ to

$$\int_{kT/n}^{(k+1)T/n} [b(\varepsilon, u, \overline{y}_{kT/n}, \omega) - \overline{b}(u, \overline{y}_{kT/n})] \, ds$$

- It remains to deal with upper integration limits *not* of the form $(k+1)T/n$
- Use: interval short, Tchebyschev's inequality, assumption on second moment

Deviation from the averaged process

Deviations of order $\sqrt{\varepsilon}$

If b is sufficiently smooth & other conditions ...

$$\frac{1}{\sqrt{\varepsilon}}(y_t^\varepsilon - \overline{y}_t) \quad \Rightarrow \quad \text{Gaussian Markov process}$$

(Convergence in distribution on $[0, T]$)

Averaging for stochastic differential equations

$$\begin{cases} \mathrm{d}y_t^\varepsilon = b(y_t^\varepsilon, \xi_t^\varepsilon)\,\mathrm{d}t + \sigma(y_t^\varepsilon)\,\mathrm{d}W_t & \text{(slow variable} \in \mathbb{R}^m) \\ \mathrm{d}\xi_t^\varepsilon = \dfrac{1}{\varepsilon} f(y_t^\varepsilon, \xi_t^\varepsilon)\,\mathrm{d}t + \dfrac{1}{\sqrt{\varepsilon}} F(y_t^\varepsilon, \xi_t^\varepsilon)\,\mathrm{d}W_t & \text{(fast variable} \in \mathbb{R}^n) \end{cases}$$

$\sigma = \sigma(y_t^\varepsilon, \xi^\varepsilon i_t)$ depending also on ξ_t^ε can be considered
(we refrain from doing so since this would require to introduce additional notations)

Introduce Markov process $\xi_t^{y.\xi}$ for frozen slow variable y

$$\mathrm{d}\xi_t^{y,\xi} = f(y, \xi_t^{y,\xi})\,\mathrm{d}t + F(y, \xi_t^{y,\xi})\,\mathrm{d}W_t\,, \qquad \xi_0^{y,\xi} = \xi$$

Averaging Theorem for SDEs

Assume there exist functions $\bar{b}(y)$ and $\kappa(T)$ s.t. for all $t_0 \geq 0$, $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}^m$:

$$\mathbb{E}\left(\left\| \frac{1}{T} \int_{t_0}^{t_0+T} b(y, \xi_s^{y,\xi})\, \mathrm{d}s - \bar{b}(y) \right\|\right) \leq \kappa(T) \to 0 \quad \text{as } T \to \infty$$

Let $\bar{y}_t$ denote the solution of

$$\mathrm{d}\bar{y}_t = \bar{b}(\bar{y}_t) + \sigma(\bar{y}_t)\, \mathrm{d}W_t\ , \qquad \bar{y}_0 = y$$

Theorem

For all $T > 0$, $\delta > 0$ and all initial conditions $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

$$\lim_{\varepsilon \to 0} \mathbb{P}\left\{ \sup_{0 \leq t \leq T} \|y_t^\varepsilon - \bar{y}_t\| > \delta \right\} = 0$$

(convergence in probability)

References

Deterministic slow–fast systems

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Averaging

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