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The Effect of Gaussian White Noise on Dynamical Systems: Bifurcations in Slow–Fast Systems

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Slowly driven systems in dimension $n = 1$

Slowly driven systems

Recall from yesterday's lecture

Parameter dependent ODE, perturbed by Gaussian white noise

$$\mathrm{d}x_s = \tilde{f}(x_s, \lambda)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s \qquad (x_s \in \mathbb{R}^1)$$

Assume parameter varies slowly in time: $\lambda = \lambda(\varepsilon s)$

$$\mathrm{d}x_s = \tilde{f}(x_s, \lambda(\varepsilon s))\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$$

Rewrite in slow time $t = \varepsilon s$

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Assumptions yesterday

Existence of a uniformly asymptotically stable equlibrium branch $x^\star(t)$

$$\exists!\, x^\star : I \to \mathbb{R} \text{ s.t. } f(x^\star(t), t) = 0$$

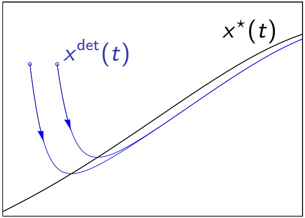
and

$$a^\star(t) = \partial_x f(x^\star(t), t) \leqslant -a_0 < 0$$

Then there exists an adiabatic solution $\bar{x}(t, \varepsilon)$

$$\bar{x}(t, \varepsilon) = x^\star(t) + \mathcal{O}(\varepsilon)$$

and $\bar{x}(t, \varepsilon)$ attracts nearby solutions exp. fast



Defining the strip describing the typical spreading

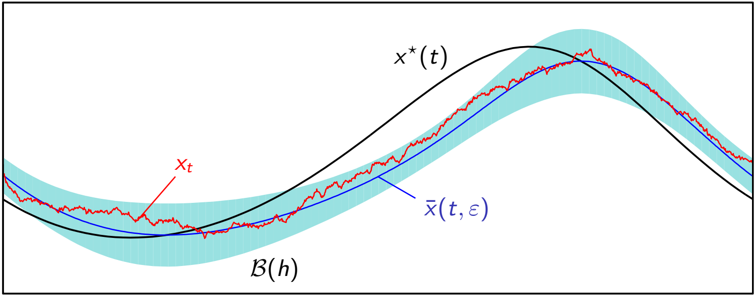
- Let $v(t)$ be the variance of the solution $z(t)$ of the linearized SDE for the deviation $x_t - \bar{x}(t,\varepsilon)$
- $v(t)/\sigma^2$ is solution of a deterministic slowly driven system admitting a uniformly asymptotically stable equilibrium branch
- Let $\zeta(t)$ be the adiabatic solution of this system
- $\zeta(t) \approx 1/|a(t)|$, where $a(t) = \partial_x f(\bar{x}(t,\varepsilon),t) \leq -a_0/2 < 0$

Define a strip $\mathcal{B}(h)$ around $\bar{x}(t,\varepsilon)$ of width $\simeq h\sqrt{\zeta(t)}$ and the first-exit time $\tau_{\mathcal{B}(h)}$

$$\mathcal{B}(h) = \{(x,t)\colon |x - \bar{x}(t,\varepsilon)| < h\sqrt{\zeta(t)}\}$$

$$\tau_{\mathcal{B}(h)} = \inf\{t > 0\colon (x_t,t) \notin \mathcal{B}(h)\}$$

Concentration of sample paths



Theorem [Berglund & G '02, '05]

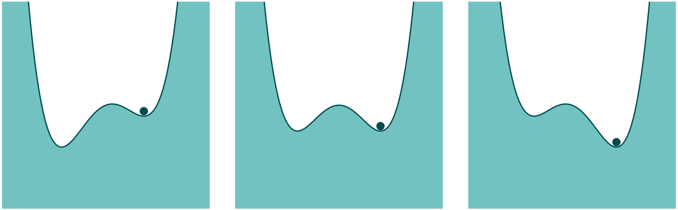
$$\mathbb{P}\bigl\{\tau_{\mathcal{B}(h)} < t\bigr\} \leq const\,\frac{1}{\varepsilon}\biggl|\int_0^t a(s)\,\mathrm{d}s\biggr|\,\frac{h}{\sigma}\;\mathrm{e}^{-h^2[1-\mathcal{O}(\varepsilon)-\mathcal{O}(h)]/2\sigma^2}$$

Avoided bifurcation: Stochastic Resonance

Overdamped motion of a Brownian particle in a periodically modulated potential

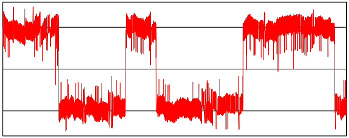
$$\mathrm{d}x_t = -\frac{1}{\varepsilon}\frac{\partial}{\partial x}V(x_t, t)\,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

$$V(x,t) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + (\lambda_{\mathrm{c}} - a_0)\cos(2\pi t)x$$

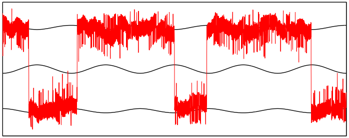


Sample paths

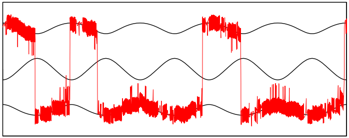
Amplitude of modulation $A = \lambda_{\mathrm{c}} - a_0$
Speed of modulation ε
Noise intensity σ



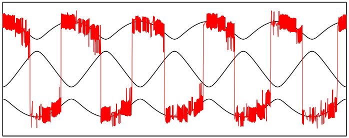
$A = 0.00,\ \sigma = 0.30,\ \varepsilon = 0.001$



$A = 0.10,\ \sigma = 0.27,\ \varepsilon = 0.001$



$A = 0.24,\ \sigma = 0.20,\ \varepsilon = 0.001$



$A = 0.35,\ \sigma = 0.20,\ \varepsilon = 0.001$

Different parameter regimes and stochastic resonance

Synchronisation I

- For matching time scales: $2\pi/\varepsilon = T_{\text{forcing}} = 2\,T_{\text{Kramers}} \asymp e^{2H/\sigma^2}$
- Quasistatic approach: Transitions twice per period likely (physics' literature; [Freidlin '00], [Imkeller *et al*, since '02])
- Requires exponentially long forcing periods

Synchronisation II

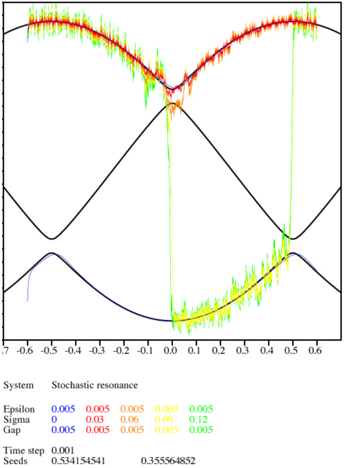
- For intermediate forcing periods: $T_{\text{relax}} \ll T_{\text{forcing}} \ll T_{\text{Kramers}}$ and close-to-critical forcing amplitude: $A \approx A_c$
- Transitions twice per period with high probability
- Subtle dynamical effects: Effective barrier heights [Berglund & G '02]

SR outside synchronisation regimes

- Only occasional transitions
- But transition times localised within forcing periods

Synchronisation regime II

Characterised by 3 small parameters: $0 < \sigma \ll 1$, $0 < \varepsilon \ll 1$, $0 < a_0 \ll 1$



Effective barrier heights and scaling of small parameters

Theorem [Berglund & G '02] (informal version; exact formulation uses first-exit times)

$$\exists \;\; \text{threshold value} \;\; \sigma_c = (a_0 \vee \varepsilon)^{3/4}$$

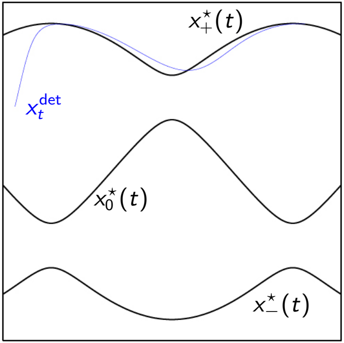
Below: $\sigma \leq \sigma_c$

- Transitions unlikely; sample paths concentrated in one well
- Typical spreading $\asymp \dfrac{\sigma}{\left(|t|^2 \vee a_0 \vee \varepsilon\right)^{1/4}} \asymp \dfrac{\sigma}{(\text{curvature})^{1/2}}$
- Probability to observe a transition $\leq e^{-const\, \sigma_c^2/\sigma^2}$

Above: $\sigma \gg \sigma_c$

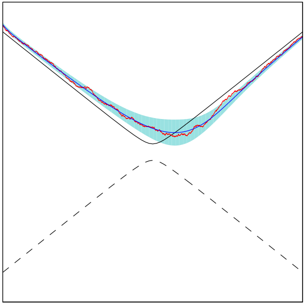
- 2 transitions per period likely (back and forth)
- with probabilty $\geq 1 - e^{-const\, \sigma^{4/3}/\varepsilon|\log \sigma|}$
- Transitions occur near instants of minimal barrier height; window $\asymp \sigma^{2/3}$

Deterministic dynamics



- For $t \leq -const$:
  x^{det}_t reaches ε-nbhd of $x^\star_+(t)$
  in time $\asymp \varepsilon|\log\varepsilon|$ (Tihonov '52)
- For $-const \leq t \leq -(a_0 \vee \varepsilon)^{1/2}$:
  $x^{\text{det}}_t - x^\star_+(t) \asymp \varepsilon/|t|$
- For $|t| \leq (a_0 \vee \varepsilon)^{1/2}$:
  $x^{\text{det}}_t - x^\star_0(t) \asymp (a_0 \vee \varepsilon)^{1/2} \geq \sqrt{\varepsilon}$
  (effective barrier height)
- For $(a_0 \vee \varepsilon)^{1/2} \leq t \leq +const$:
  $x^{\text{det}}_t - x^\star_+(t) \asymp -\varepsilon/|t|$
- For $t \geq +const$:
  $|x^{\text{det}}_t - x^\star_+(t)| \asymp \varepsilon$

Below threshold: $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



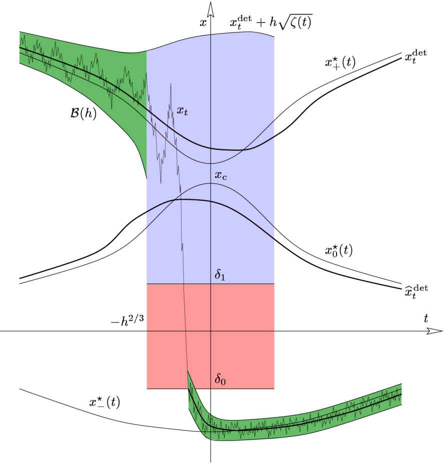
$$v(t) \sim \frac{\sigma^2}{\text{curvature}} \sim \frac{\sigma^2}{(|t|^2 \vee a_0 \vee \varepsilon)^{1/2}}$$

Approach for stable case can still be used

$$C(h/\sigma, t, \varepsilon)\, e^{-\kappa_- h^2/2\sigma^2} \leq \mathbb{P}\{\tau_{\mathcal{B}(h)} < t\} \leq C(h/\sigma, t, \varepsilon)\, e^{-\kappa_+ h^2/2\sigma^2}$$

with $\kappa_+ = 1 - \mathcal{O}(\varepsilon) - \mathcal{O}(h)$, $\kappa_- = 1 + \mathcal{O}(\varepsilon) + \mathcal{O}(h) + \mathcal{O}(e^{-c_2 t/\varepsilon})$

Above threshold: $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



- Typical paths stay below $x_t^{\text{det}} + h\sqrt{\zeta(t)}$
- For $t \ll -\sigma^{2/3}$:

  Transitions unlikely; as below threshold
- At time $t = -\sigma^{2/3}$:

  Typical spreading is $\sigma^{2/3} \gg x_t^{\text{det}} - x_0^\star(t)$

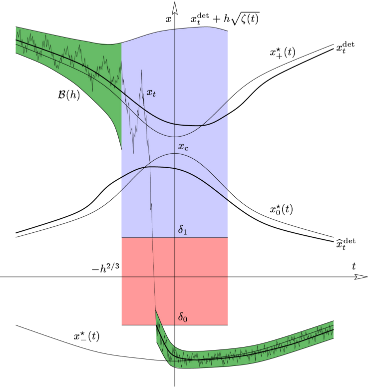
  Transitions become likely
- Near saddle:

  Diffusion dominated dynamics
- $\delta_1 > \delta_0$ with $f \asymp -1$;

  δ_0 in domain of attraction of $x_-^\star(t)$

  Drift dominated dynamics
- Below δ_0: behaviour as for small σ

Above threshold: $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



Idea of the proof

With probability $\geq \delta > 0$, in time $\asymp \varepsilon|\log\sigma|/\sigma^{2/3}$, the path reaches

- x_t^{det} if above
- then the saddle
- finally the level δ_1

In time $\sigma^{2/3}$ there are $\dfrac{\sigma^{4/3}}{\varepsilon|\log\sigma|}$ attempts possible

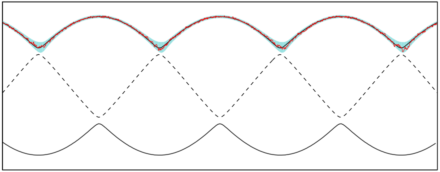
During a subsequent timespan of length ε, level δ_0 is reached (with probability $\geq \delta$)

Finally, the path reaches the new well

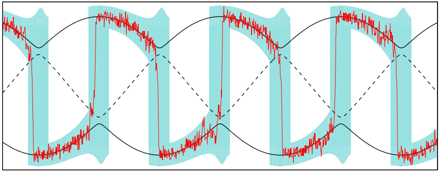
Result

$$\mathbb{P}\{x_s > \delta_0 \quad \forall s \in [-\sigma^{2/3}, t]\} \leq \mathrm{e}^{-const\ \sigma^{4/3}/\varepsilon|\log\sigma|} \qquad (t \geq -\gamma\sigma^{2/3},\ \gamma \text{ small})$$

Space–time sets for stochastic resonance



Below threshold

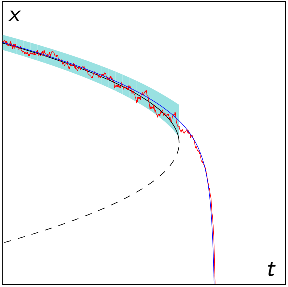


Above threshold

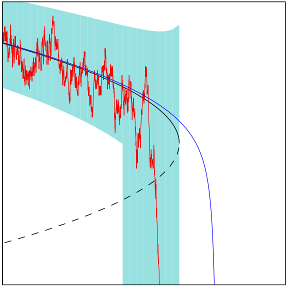
Saddle–node bifurcation

Saddle–node bifurcation (e.g. $f(x,t) = -t - x^2$)

$\sigma \ll \sigma_{\rm c} = \varepsilon^{1/2}$



$\sigma \gg \sigma_{\rm c} = \varepsilon^{1/2}$



$\sigma = 0$: Solutions stay at distance $\varepsilon^{1/3}$ above bif. point until time $\varepsilon^{2/3}$ after bif.

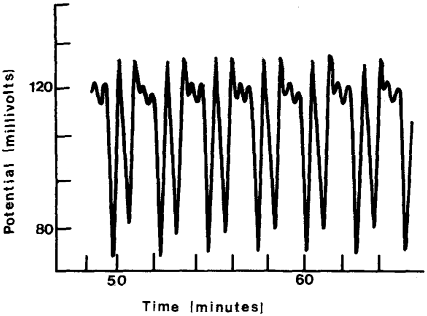
Theorem

- If $\sigma \ll \sigma_{\rm c}$: Paths likely to stay in $\mathcal{B}(h)$ until time $\varepsilon^{2/3}$ after bifurcation; maximal spreading $\sigma/\varepsilon^{1/6}$.
- If $\sigma \gg \sigma_{\rm c}$: Transition typically for $t \asymp -\sigma^{4/3}$; transition probability $\geqslant 1 - e^{-c\sigma^2/\varepsilon|\log\sigma|}$

Mixed-mode oscillations

Mixed-Mode Oscillations (MMOs)

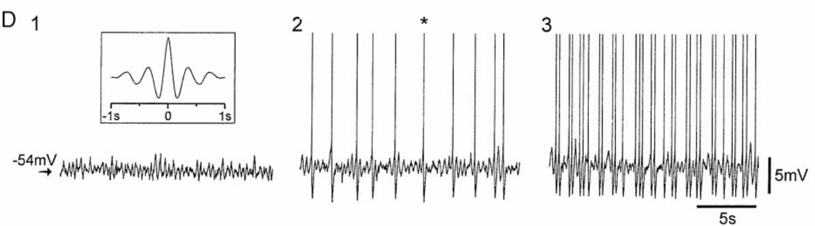
Belousov–Zhabotinsky reaction



Recording from bromide ion electrode; T=25° C; flow rate = 3.99 ml/min; Ce^{+3} catalyst [Hudson, Hart, Marinko '79]

MMOs in Biology

Layer II Stellate Cells



D: subthreshold membrane potential oscillations (1 and 2) and spike clustering (3) develop at increasingly depolarized membrane potential levels positive to about –55 mV. Autocorrelation function (*inset* in 1) demonstrates the rhythmicity of the subthreshold oscillations [Dickson *et al* '00]

Questions: Origin of small-amplitude oscillations?
Source of irregularity in pattern?

MMOs & Slow–Fast Systems

Observation

MMOs can be observed in slow–fast systems undergoing a folded-node bifurcation (1 fast, 2 slow variables)

Normal form of folded-node (bif) [Benoît, Lobry '82; Szmolyan, Wechselberger '01]

$$\begin{aligned} \epsilon\dot{x} &= y - x^2 \\ \dot{y} &= -(\mu+1)x - z \\ \dot{z} &= \frac{\mu}{2} \end{aligned}$$

Questions: Dynamics for small $\varepsilon > 0$?
Effect of noise?

MMOs & Slow–Fast Systems

Observation

MMOs can be observed in slow–fast systems undergoing a folded-node bifurcation (1 fast, 2 slow variables)

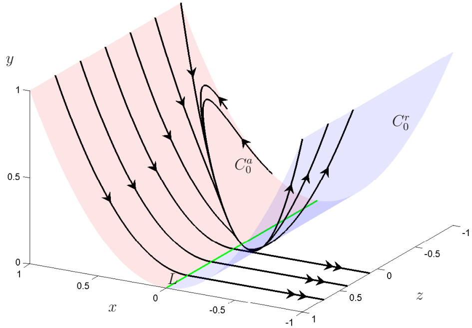
Normal form of folded-node (bif) [Benoît, Lobry '82; Szmolyan, Wechselberger '01]

$$\begin{aligned} \epsilon\dot{x} &= y - x^2 + \text{noise} \\ \dot{y} &= -(\mu+1)x - z + \text{noise} \\ \dot{z} &= \frac{\mu}{2} \end{aligned}$$

Questions: Dynamics for small $\varepsilon > 0$?
Effect of noise?

Folded-Node Bifurcation: Slow Manifold

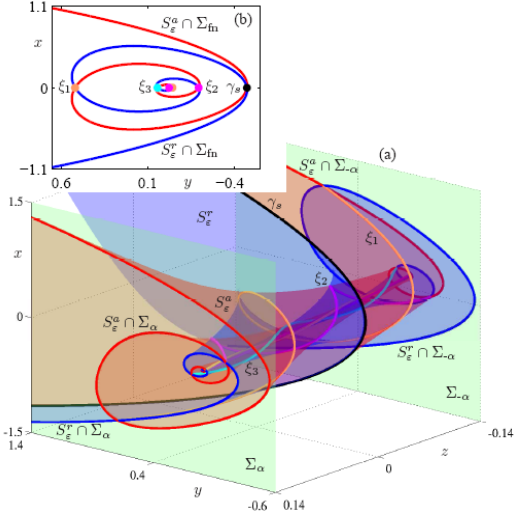
$$\begin{aligned}\epsilon\dot{x} &= y - x^2 \\ \dot{y} &= -(\mu+1)x - z \\ \dot{z} &= \frac{\mu}{2}\end{aligned}$$



Slow manifold has a decomposition

$$\mathcal{C}_0 = \{(x,y,z) \in \mathbb{R}^3 \colon y = x^2\} = \mathcal{C}_0^a \cup L \cup \mathcal{C}_0^r$$

Folded-Node: Adiabatic Manifolds and Canard Solutions



[Desroches *et al* '12]

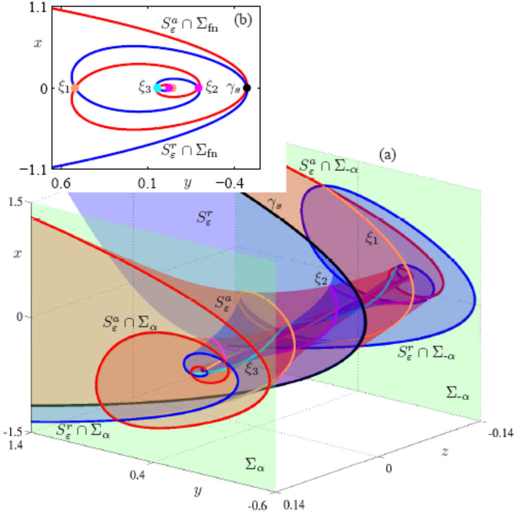
Assume

- ε sufficiently small
- $\mu \in (0,1)$, $\mu^{-1} \notin \mathbb{N}$

Theorem

[Benoît, Lobry '82;
Szmolyan, Wechselberger '01;
Wechselberger '05;
Brøns, Krupa, Wechselberger '06]

Folded-Node: Adiabatic Manifolds and Canard Solutions



[Desroches *et al* '12]

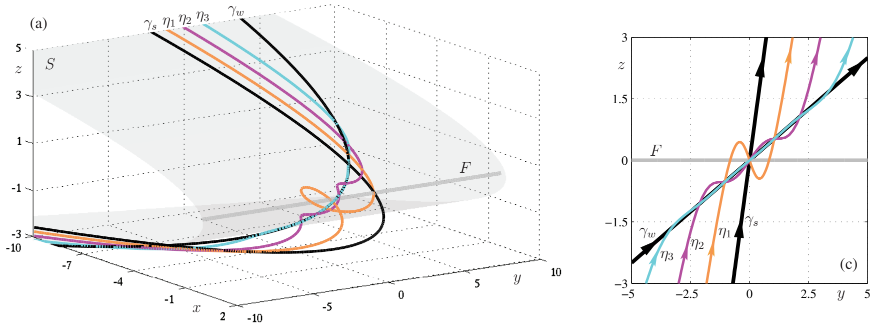
Assume

- ε sufficiently small
- $\mu \in (0,1)$, $\mu^{-1} \notin \mathbb{N}$

Theorem

- Existence of *strong* and *weak* (maximal) canard $\gamma_\varepsilon^{s,w}$
- $2k+1 < \mu^{-1} < 2k+3$: $\exists\ k$ *secondary* canards γ_ε^j
- γ_ε^j makes $(2j+1)/2$ oscillations around γ_ε^w

Folded-Node: Canard Spacing



[Desroches, Krauskopf, Osinga '08]

Lemma

For $z = 0$: Distance between canards γ_ε^k and γ_ε^{k+1} is $\mathcal{O}(\mathrm{e}^{-c_0(2k+1)^2\mu})$

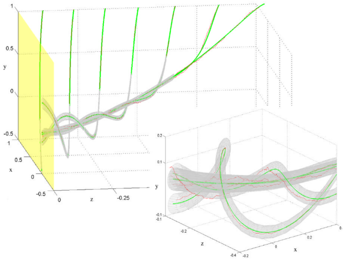
Stochastic Folded Nodes: Concentration of Sample Paths

Theorem [Berglund, G & Kuehn, JDE '12]

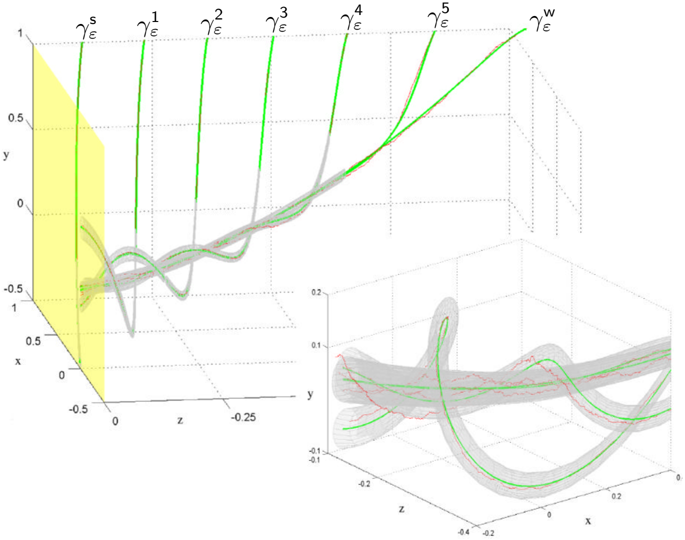
$$\mathbb{P}\{\tau_{\mathcal{B}(h)} < z\} \leqslant C(z_0, z) \exp\Big\{-\kappa \frac{h^2}{2\sigma^2}\Big\} \qquad \forall z \in [z_0, \sqrt{\mu}]$$

For $z = 0$:

- Distance between canards γ_ε^k and γ_ε^{k+1} is $\mathcal{O}(\mathrm{e}^{-c_0(2k+1)^2\mu})$
- Section of $\mathcal{B}(h)$ is close to circular with radius $\mu^{-1/4}h$
- Noisy canards become indistinguishable when typical radius $\mu^{-1/4}\sigma \approx$ distance



Canards or Pasta . . . ?

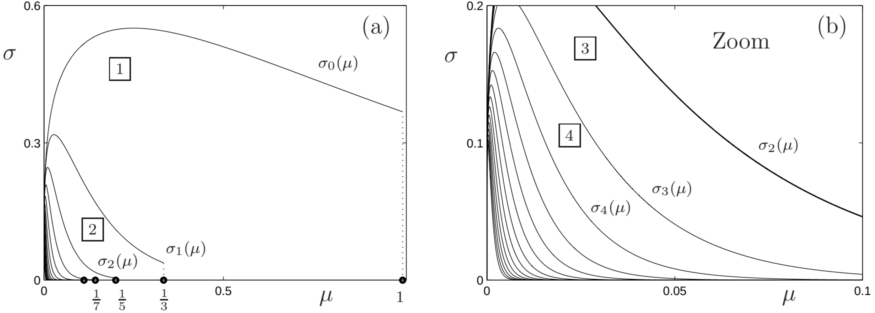


Noisy Small-Amplitude Oscillations

Theorem

Canards with $\frac{2k+1}{2}$ oscillations become indistinguishable from noisy fluctuations for

$$\sigma > \sigma_k(\mu) = \mu^{1/4}\,\mathrm{e}^{-(2k+1)^2\mu}$$



Early Escape

Model allowing for global returns

- Consider $z > \sqrt{\mu}$
- $\mathcal{D}_0 =$ neighbourhood of γ^{w}, growing like $\sqrt{z}$

Theorem [Berglund, G & Kuehn '10]

$\exists \kappa, \kappa_1, \kappa_2, C > 0$

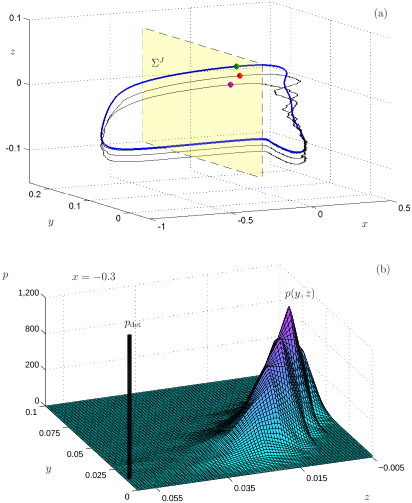
s.t.

for $\sigma|\log\sigma|^{\kappa_1} \leqslant \mu^{3/4}$

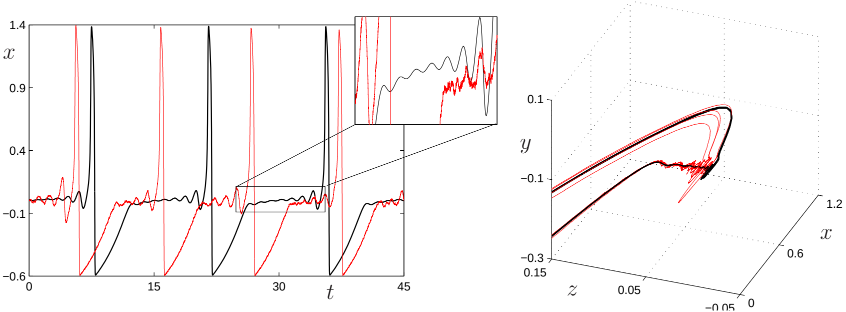
$$\mathbb{P}\{\tau_{\mathcal{D}_0} > z\} \leqslant C|\log\sigma|^{\kappa_2}\, \mathrm{e}^{-\kappa(z^2-\mu)/(\mu|\log\sigma|)}$$

Note:

r.h.s. small for $z \gg \sqrt{\mu|\log\sigma|/\kappa}$



Mixed-Mode Oscillations in the Presence of Noise



Observations

- Noise smears out small-amplitude oscillations
- Early transitions modify the mixed-mode pattern

Collaborators

Nils Berglund, Orléans

Christian Kuehn, Vienna

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