Weierstraß-Institut für Angewandte Analysis und Stochastik

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Noise-Induced Phenomena in Slow–Fast Dynamical Systems

Joint work with Nils Berglund (CPT–CNRS Marseille)

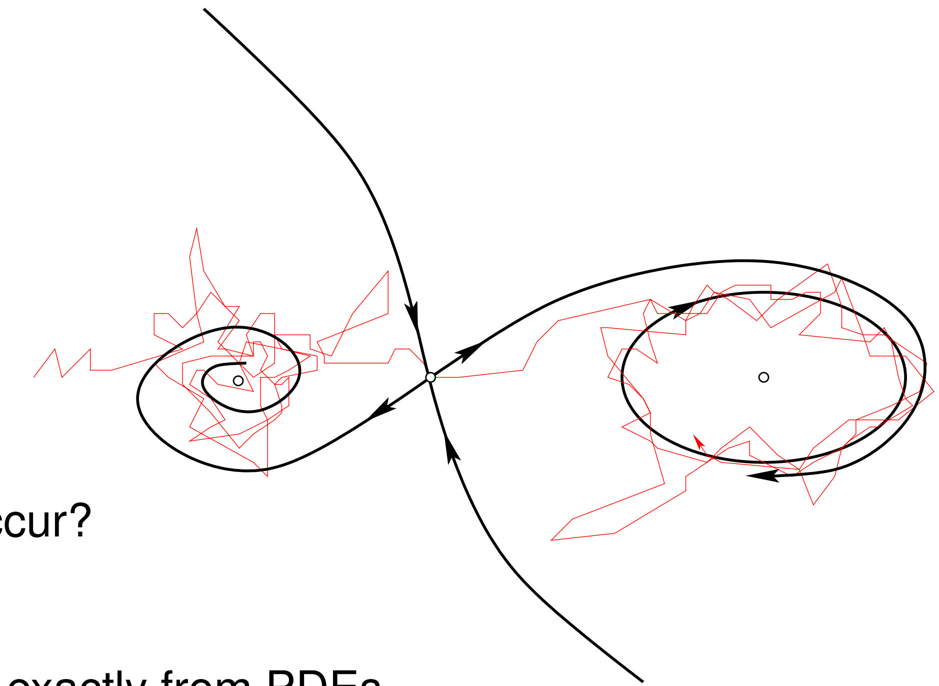
Leibniz Gemeinschaft

Deterministic ODE $\dot{x}_t^{\text{det}} = f(x_t^{\text{det}})$ with $x_0^{\text{det}} \in \mathbb{R}^n$

Small random perturbation $\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma F(x_t)\,\mathrm{d}W_t$ with $x_0 = x_0^{\text{det}}$

where

- $f : \mathbb{R}^n \to \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^{k \times n}$
- $\{W_t\}_{t \geqslant 0}$ k-dim. standard Brownian motion
- $\sigma > 0$ small



Noise enables transitions between basins of attraction

Questions

- Transition times? Where do transitions typically occur?

Answers

- *Mean* first-exit times and locations can be obtained exactly from PDEs
  (via infinitesimal generators of the diffusion)
- Exponential asymptotics provided by Wentzell–Freidlin theory (via variational principle)
- Rigorous proof of subexponential asymptotics for reversible diffusions is recent $(n > 1)$
  [Bovier, Eckhoff, Gayrard & Klein, 2004/05]

Fully coupled stochastic differential equations on two well-separated time scales

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n) \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \sigma'\; G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

where

- ▷ $\{W_t\}_{t \geqslant 0}$ k-dimensional standard Brownian motion
- ▷ $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$
- ▷ $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^m$ drift coefficients, $\in \mathcal{C}^2$
- ▷ $F : \mathcal{D} \to \mathbb{R}^{n \times k}$, $G : \mathcal{D} \to \mathbb{R}^{m \times k}$ diffusion coefficients, $\in \mathcal{C}^1$

Note diffusive scaling: $\mathrm{Var}\left(\dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t F(x_s, y_s)\,\mathrm{d}W_s\right) = \dfrac{\sigma^2}{\varepsilon} \displaystyle\int_0^t \mathbb{E}\big(F(x_s, y_s)^2\big)\,\mathrm{d}s$ (for $k = n = 1$)

Small parameters

- ▷ $\varepsilon > 0$ adiabatic parameter (*no quasistatic* approach)
- ▷ $\sigma, \sigma' \geqslant 0$ noise intensities; may depend on ε : $\sigma = \sigma(\varepsilon)$, $\sigma' = \sigma'(\varepsilon)$ and $\sigma'(\varepsilon)/\sigma(\varepsilon) = \varrho(\varepsilon) \leqslant 1$

Time scales

- ▷ Aiming at regime $T_{\text{relax}} = \mathcal{O}(\varepsilon) \ll T_{\text{driving}} = 1 \ll T_{\text{Kramers}} = \varepsilon\, \mathrm{e}^{\overline{V}/\sigma^2}$ (in slow time)

Existence of a slow manifold

$\exists\, \mathcal{D}_0 \subset \mathbb{R}^m \quad \exists\, x^\star : \mathcal{D}_0 \to \mathbb{R}^n$ s.t. $(x^\star(y), y) \in \mathcal{D}$ and $f(x^\star(y), y) = 0$ on $\mathcal{D}_0$

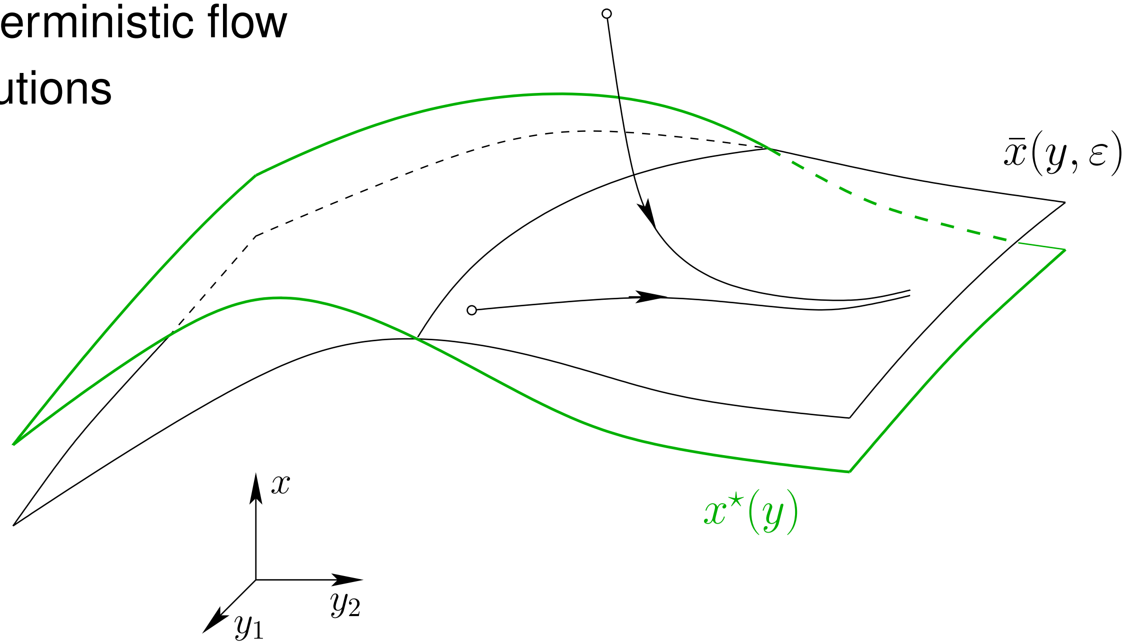
Slow manifold is attracting

Eigenvalues of $A^\star(y) := \partial_x f(x^\star(y), y)$ satisfy $\operatorname{Re} \lambda_i(y) \leqslant -a_0 < 0$, uniformly on $\mathcal{D}_0$

Theorem [Tihonov 1952, Fenichel 1979]

There exists an *adiabatic manifold* : $\exists\, \bar{x}(y, \varepsilon)$ s.t.

- ▷ $\bar{x}(y, \varepsilon)$ is an invariant manifold for the deterministic flow
- ▷ $\bar{x}(y, \varepsilon)$ attracts nearby (deterministic) solutions
- ▷ $\bar{x}(y, 0) = x^\star(y)$ and $\bar{x}(y, \varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



Consider now *stochastic* system
under these assumptions

- Consider deterministic process $(x_t^{\text{det}} = \bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}})$ on (invariant) adiabatic manifold
- Deviation $\xi_t := x_t - x_t^{\text{det}}$ of random fast variables from adiabatic manifold
- Linearize SDE for ξ_t
- Resulting process ξ_t^0 is Gaussian

Key observation

$\dfrac{1}{\sigma^2} \operatorname{Cov} \xi_t^0$ is a particular solution of the deterministic slow–fast system

$$\begin{cases} \varepsilon \dot{X}(t) = A(y_t^{\text{det}})X(t) + X(t)A(y^{\text{det}})^T + F_0(y^{\text{det}})F_0(y^{\text{det}})^T \\ \dot{y}_t^{\text{det}} \quad = g(\bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}}) \end{cases}$$

where $A(y) = \partial_x f(\bar{x}(y, \varepsilon), y)$ and F_0 is zeroth-order approximation to F

- System admits an adiabatic manifold $\overline{X}(y, \varepsilon)$

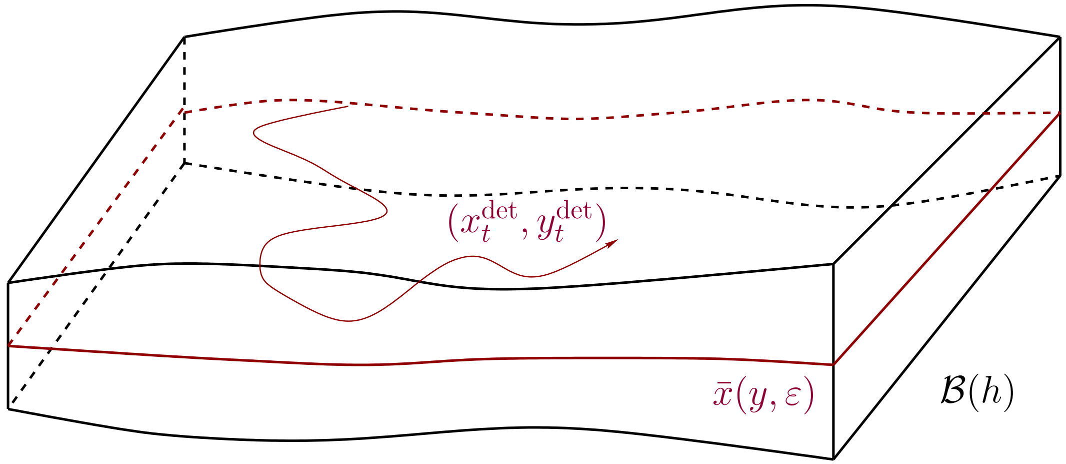
Typical neighbourhoods

$\mathcal{B}(h) := \big\{ (x, y) \colon \big\langle \big[x - \bar{x}(y, \varepsilon)\big], \overline{X}(y, \varepsilon)^{-1} \big[x - \bar{x}(y, \varepsilon)\big] \big\rangle < h^2 \big\}$

Define (random) first-exit times

$\tau_{\mathcal{D}_0} \;:= \inf\{s > 0\colon y_s \notin \mathcal{D}_0\}$

$\tau_{\mathcal{B}(h)} := \inf\{s > 0\colon (x_s, y_s) \notin \mathcal{B}(h)\}$



Theorem [Berglund & G, J. Differential Equations, 2003]

Assume: $\|\overline{X}(y,\varepsilon)\|$, $\|\overline{X}(y,\varepsilon)^{-1}\|$ are uniformly bounded in $\mathcal{D}_0$

Then: $\exists\, \varepsilon_0 > 0 \quad \exists\, h_0 > 0 \quad \forall\, \varepsilon \leqslant \varepsilon_0 \quad \forall\, h \leqslant h_0$

$$\mathbb{P}\bigl\{\tau_{\mathcal{B}(h)} < \min(t, \tau_{\mathcal{D}_0})\bigr\} \leqslant C_{n,m}(t)\, \exp\biggl\{-\frac{h^2}{2\sigma^2}\bigl[1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)\bigr]\biggr\}$$

where $C_{n,m}(t) = \bigl[C^m + h^{-n}\bigr]\biggl(1 + \dfrac{t}{\varepsilon^2}\biggr)$

Remarks

▷ Bound is sharp: Lower bound similar

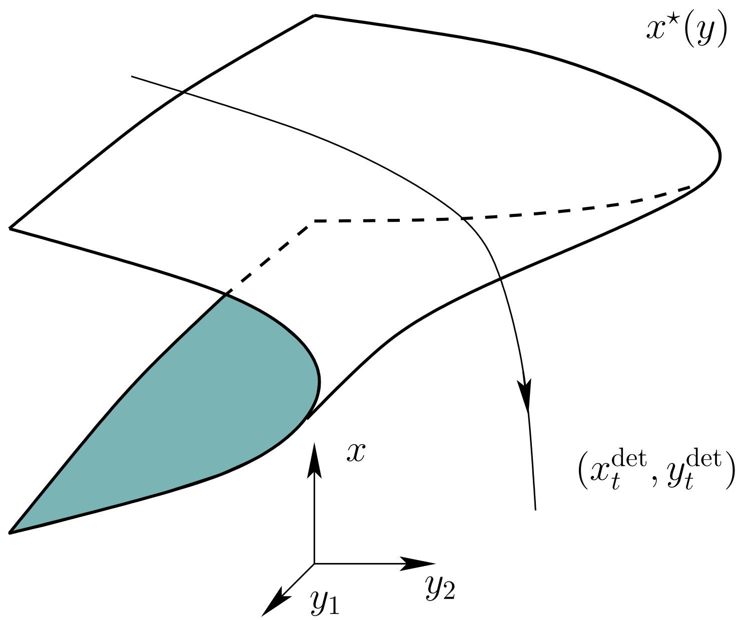
▷ If initial condition not on $\bar{x}(y,\varepsilon)$: additional transitional phase

▷ On longer time scales: Behaviour of slow variables becomes crucial (Assumptions on g)

Question

What happens if (x_t, y_t) approaches a bifurcation point $(\hat{x}, \hat{y})$ for the deterministic dynamics?

Example: Saddle–node bifurcation



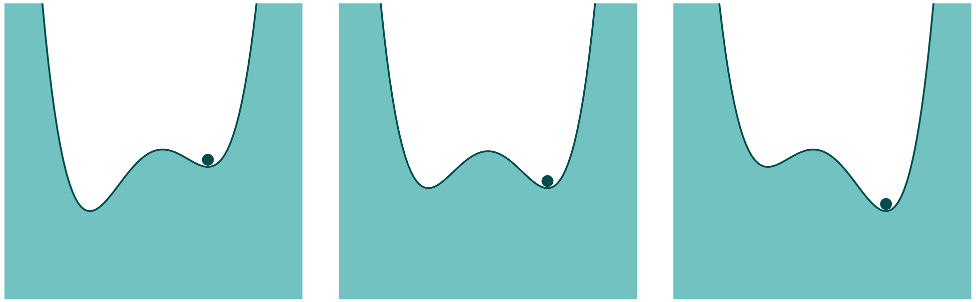
General approach

- ▷ Apply centre-manifold theorem
- ▷ Concentration results for deviation from centre manifold [Berglund & G, 2003]
- ▷ Consider reduced dynamics on centre manifold
- ▷ Concentration results for deviation of reduced system from original variables [Berglund & G, 2003]

Interesting phenomena for one-dimensional reduced dynamics

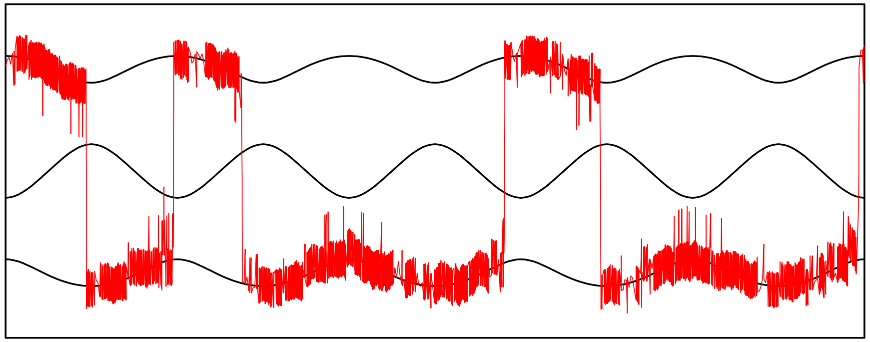
- ▷ Reduction of bifurcation delay due to noise
- ▷ Stochastic resonance
- ▷ Effect of noise on the size of hysteresis cycles

Overdamped motion of a Brownian particle in a periodically modulated double-well potential

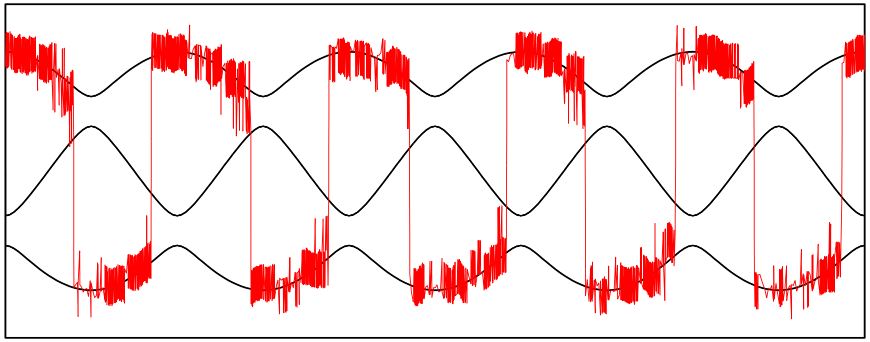


$$\mathrm{d}x_t = -\frac{1}{\varepsilon}\frac{\partial}{\partial x}V(x_t,t)\;\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\;\mathrm{d}W_t \qquad \text{with} \qquad V(x,t) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - A\cos(t)x \qquad \text{(where } A < A_{\mathrm{c}}\text{)}$$

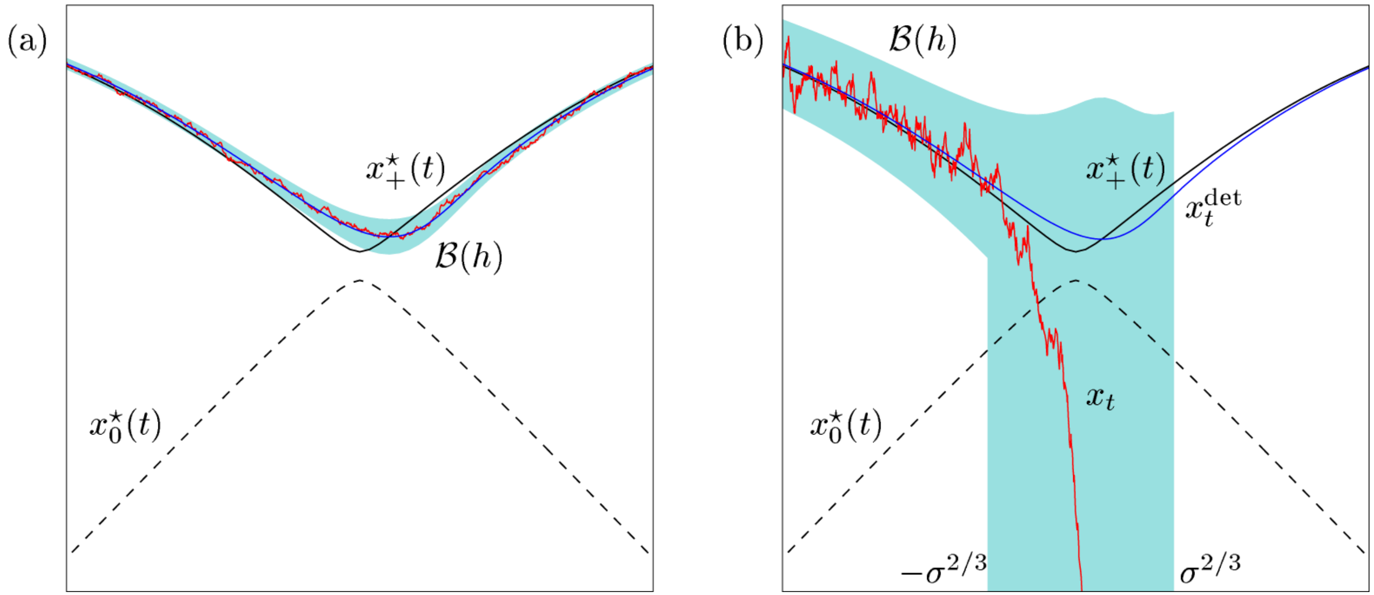
Simulations



$A = 0.24$, $\sigma = 0.20$, $\varepsilon = 0.001$



$A = 0.35$, $\sigma = 0.20$, $\varepsilon = 0.001$



Theorem [Berglund & G, 2002]

There exists a threshold value $\sigma_c = (a_0 \vee \varepsilon)^{3/4}$ s.t.

Below threshold $\sigma \ll \sigma_c = (a_0 \vee \varepsilon)^{3/4}$*:*

- Transitions unlikely, probability for transition $\leqslant e^{-\mathit{const}\,\sigma_c^2/\sigma^2}$
- Sample paths concentrated near bottom of well; typical spreading depends on local curvature

Above threshold $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$*:*

- 2 transitions per period likely (back and forth) with probability $\geqslant 1 - e^{-\mathit{const}\,\sigma^{4/3}/\varepsilon|\log\sigma|}$
- Transitions likely when barrier low; transition window has width $\asymp \sigma^{2/3}$

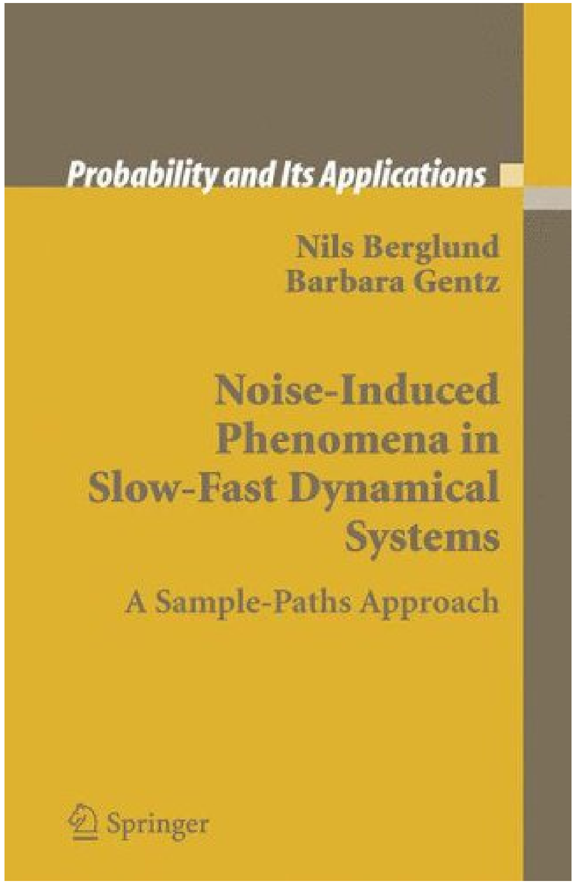
References

General results on sample-path behaviour in slow–fast systems

- *Noise-Induced Phenomena in Slow–Fast Dynamical Systems. A Sample-Paths Approach,* "Probability and its Applications", Springer, London, 2005
- *Geometric singular perturbation theory for stochastic differential equations*, J. Differential Equations **191**, 1–54 (2003)

Case studies: Bifurcations in slowly driven systems

- *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probab. Theory Related Fields **122**, 341–388 (2002)
- *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Appl. Probab. **12**, 1419–1470 (2002)
- *The effect of additive noise on dynamical hysteresis*, Nonlinearity **15**, 605–632 (2002)



Passage through an unstable periodic orbit

- *On the noise-induced passage through an unstable periodic orbit I: Two-level model*, J. Statist. Phys. **114**, 1577–1618 (2004)
- *Universality of first-passage- and residence-time distributions in non-adiabatic stochastic resonance*, Europhys. Lett. **70**, 1–7 (2005)