ALGEBRAIC K-THEORY

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To Mary

PREFACE

This book is based on a course I gave at Columbia University in 1966-67. Its writing was greatly facilitated by the notes for that course which were taken by Tsit-Yuen Lam, M. Pavaman Murthy, and Charles Small. I am extremely grateful to them for their assistance and criticism.

I had originally hoped to make the exposition here more or less self-contained, modulo a first year algebra course. Because of the variety of techniques employed, however, this ambition threatened to lead to an infinite regress. Thus, Part 1 on preliminaries still contains, despite its length, a few results which are merely quoted without proof.

Time prevented me from including here a treatment of the "K-theory of symplectic modules," which I hope to publish in the near future. For the theory of "quadratic modules" there is so far only a discussion of the formalism (construction of the classical invariants) in my Tata lectures [4], and only partial results are known at present in the way of general stability theorems. It is worth noting, however, that the discussion in Chapter VII has been deliberately arranged so that it can be applied directly to a variety of contexts. Thus, for example, one has Mayer–Vietoris sequences and excision isomorphisms for the theories of symplectic, quadratic, and Hermitian forms, for the Brauer group, and for various other theories (roughly speaking, for those based on projective modules supplied with some type of tensor).

An important feature of algebraic K-theory, and one which has led to genuinely new insights in pure algebra, is its ability to exploit the techniques of a highly developed branch of topology—the homotopy theory of vector bundles. In turn, and for entirely different reasons, which go back to J.H.C. Whitehead's theory of simple homotopy types, the topologists are active patrons of the subject, providing an abundant supply of interesting and difficult questions with which the theory can be tested and expanded.

Under these circumstances it seemed worthwhile to make available a reasonably comprehensive and systematic treatment of the main ideas of the subject, as so far developed. I have written these notes with that intention. I hope they may be useful, as a reference to topologists, and as an invitation to an area of new techniques and problems to algebraists. Finally, I have tried to organize the notes so that they might serve as the basis for a second-year graduate algebra course, such as the one from which they originated.

HYMAN BASS

New York, New York
October 1967

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PART 3. *ALGEBRAIC K-THEORY*

PART 4. *K-THEORY OF PROJECTIVE MODULES*

INTRODUCTION

The "algebraic K-theory" presented here is, essentially, a part of general linear algebra. It is concerned with the structure theory of projective modules, and of their automorphism groups. Thus, it is a generalization, in the most naive sense, of the theorem asserting the existence and uniqueness of bases for vector spaces, and of the group theory of the general linear group over a field. One witnesses here the evolution of these theorems as the base ring becomes more general than a field. There is a satisfactory "stable form" in which the above theorems survive (Part 2). In a stricter sense these theorems fail in the general case, and the Grothendieck groups (K_0) and Whitehead groups (K_1) which we study can be viewed as providing a measure of their failure.

A topologist can similarly seek such a generalization of the structure theorems of linear algebra. He views a vector space as a special case of a vector bundle. The homotopy theory of vector bundles, and topological K-theory, then provide a completely satisfactory framework within which to treat such questions. It is remarkable that there exists, in algebra, nothing of remotely comparable depth or generality, even though many of these questions are algebraic in character.

The techniques used here are, therefore, topologically inspired. They are based on the philosophy, supported by theorems of Swan (Chapter XIV) and Serre (cf. Chapter IV), that a projective module should be thought of as the module of sections of a vector bundle. This dictates the choice of projective modules (rather than some wider class of modules) as the objects of the theory. This point of view further exhibits the stability theorems (Part 2) as direct imitations of their topological precursors (cf. Chapter XIV). It was Serre [1] who originated the techniques for proving such stability theorems in a purely algebraic setting.

The formalism of K-theory originated with Grothendieck's proof of the generalized Riemann–Roch theorem. The ideas were then quickly developed in topology by Atiyah and Hirzebruch, who made the Grothendieck groups, K(X), part of a generalized cohomology theory, using the suspension functor. While our point of view leads to an obvious translation of K(X), there is no clear algebraic counterpart for suspension. As a result our algebraic K-theory in Part 3 is far from complete, and the treatment here should be regarded as a provisional one, albeit sufficient for a number of applications in later chapters.

The development in Part 3 is axiomatic so that the results can be usefully applied to many categories other than those of projective modules. The exposition there is substantially influenced by ideas of Milnor. It was he who first called attention to the existence and importance of the Mayer–Vietoris sequence of a Cartesian square, and this has become a cornerstone of the whole theory. In particular, it leads to a very general analog of the excision isomorphisms. Otherwise the results of Part 3 are taken largely from a paper of Heller [1]. The latter contains another major tool of the theory, the exact sequence of a localizing functor, which does not seem to have any familiar topological counterpart. Chapter VIII also contains a striking new theorem of Leslie Roberts, with which he has computed K_1 for nonsingular projective algebraic varieties.

There has been some recent progress in finding satisfactory definitions of higher algebraic K's. For example, Milnor has defined a K_2, on which some work has been done by Gersten [2]. From a quite different point of view, A. Nóbile and O. Villamayor [1] have constructed an algebraic K-theory with functors K_n for all $n \geqq 0$. Other (unpublished) definitions have been proposed as well. However, in none of these cases are the new functors yet very well understood. It therefore seemed premature to attempt an excursion in that direction in these notes.

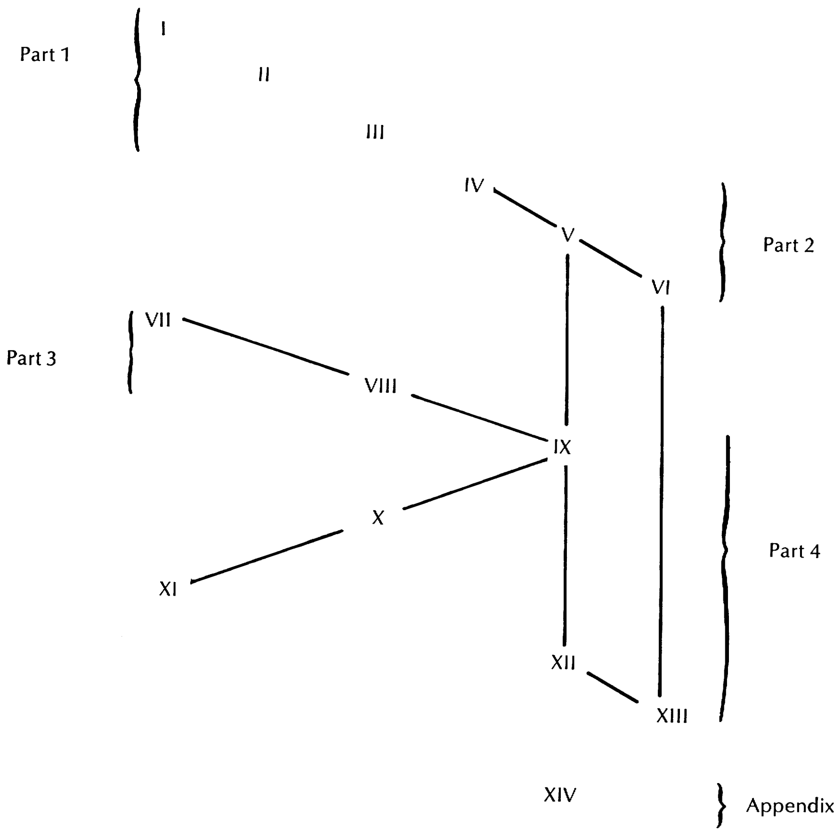
In Part 4 the general results of Parts 2 and 3 are assembled and applied to the computation of Grothendieck groups $K_0(A)$ and Whitehead groups $K_1(A)$ for a variety of rings A. Special emphasis is given to the case of group rings $A = \underline{\underline{Z}}\pi$ because of the interest of the groups $K_i(\underline{\underline{Z}}\pi)$ to topologists. In particular, the long Chapter XI is devoted to a new exposition of techniques, developed by Swan and Lam, which are based on the theory of induced representations for finite groups.

There are two unanticipated, and mathematically interesting, high points in the theory. The first is the fact that when A is a Dedekind ring, the group theory of $SL_n(A)$, as formulated in terms of K_1, is intimately connected with certain "reciprocity laws" in A. The latter include the classical power reciprocity laws in totally imaginary number fields as well as certain geometric reciprocity laws on algebraic curves. This phenomenon was first witnessed in the recent papers of C. Moore [1] and of Bass–Milnor–Serre [1]. The discussion of this in Chapter VI is an axiomatization, based the latter reference. I am further indebted here to T.-Y. Lam for a number of suggestions. The upshot of this theory is that known reciprocity laws can be used to compute K_1. Conversely, using the machinery developed in later chapters, we can sometimes compute K_1 directly, and in turn use these calculations to exhibit new reciprocity laws. Examples of both of these procedures occur in the text (cf. Chapters VI and XII).

The other surprise is the "Fundamental Theorem" in Chapter XII, §7, which computes $K_1(A[t, t^{-1}])$. Its principal feature is that $K_0(A)$ appears as a natural direct summand of $K_1(A[t, t^{-1}])$. This is surprising because, at least algebraically, K_0 and K_1 look like rather different kinds of animals. The surprise disappears, however, if one interprets the theorem topologically, whereupon it is seen to be an algebraic analog of Bott's complex periodicity theorem (cf. Chapter XIV, §6). This theorem first appeared (in a less precise form) in the paper of Bass–Heller–Swan [1]. A new feature, which emerged only at the end of the writing of these notes, is that the fundamental theorem has a built-in iteration procedure, which can be used to manufacture a whole sequence of functors K_{-n} $(n \geqq 0)$ with which to extend the (K_1, K_0)—exact sequence to the right. They help to clarify some calculations made in Bass–Murthy [1], but their significance is otherwise still unclear (to me).

LOGICAL DEPENDENCE OF CHAPTERS

The following diagram is a rough indication of the logical interdependence of the chapters. If Chapter B depends logically on Chapter A then A is placed above B; the converse is not necessarily true. In some cases this dependence is rather peripheral, so a line joining A and B appears only when the contents of A are an essential prerequisite for the reading of B.



SOME GENERAL NOTATION

Let A be a ring. We write

mod-A and A-mod

for the categories of right and left A-modules, respectively. We have the full subcategories

$$\underline{\underline{P}}(A) \subset \underline{\underline{H}}(A) \subset \underline{\underline{M}}(A) \subset \text{mod-}A$$

defined as follows: $M \in \underline{\underline{M}}(A) \Leftrightarrow M$ is a finitely generated A-module, and $M \in \underline{\underline{P}}(A) \Leftrightarrow M$ is also projective. Finally, $M \in \underline{\underline{H}}(A) \Leftrightarrow M$ has a finite resolution by objects of $\underline{\underline{P}}(A)$ (see Chapter III, §6).

Let R be a commutative ring and suppose A is an R-algebra. Let S be a multiplicative set in R and let $\underline{\underline{C}}$ be a subcategory of mod-A. Then $\underline{\underline{C}}_S$ denotes the full subcategory of all $M \in \underline{\underline{C}}$ such that $S^{-1} M = 0$.

The ring of n by n matrices over A is denoted $M_n(A)$, and its invertible elements constitute the group $GL_n(A)$. We often identify $M_n(A)$ with the A-endomorphisms of the right A-module A^n. When $n = 1$ we write

$$U(A) = GL_1(A)$$

so that $GL_n(A) = U(M_n(A))$.

If $\underline{\underline{C}}$ is any category we write

$$\Sigma \underline{\underline{C}}$$

for the category of pairs (M, α) $(M \in \underline{\underline{C}}, \alpha \in \mathrm{Aut}_{\underline{\underline{C}}}(M))$ (see Chapter VII, §1), i.e., the category of automorphisms of objects of $\underline{\underline{C}}$.

Part 1

PRELIMINARIES

Chapter I
SOME CATEGORICAL ALGEBRA

This chapter introduces some of the basic language of categories and functors. It should be used mainly for reference, rather than being read outright. The first sections lead up to the notion of an Abelian category, in §4. In §§5-6 we assemble some basic facts about homology and projective resolutions which will be used extensively in the following sections. In §8 we prepare some less standard results on direct limits, which are needed in Chapter VII.

Essentially all of the material of this chapter can be found in the books of MacLane [1] and Mitchell [1].

§1. CATEGORIES AND FUNCTORS

Recall that a category $\underline{\underline{A}}$ consists of objects, ob $\underline{\underline{A}}$, a set of morphisms, $\underline{\underline{A}}(A, B)$, for each $A, B \in$ ob $\underline{\underline{A}}$, and a composition

$$\underline{\underline{A}}(B, C) \times \underline{\underline{A}}(A, B) \longrightarrow \underline{\underline{A}}(A, C), \quad (a, b) \longrightarrow ab$$

The latter is associative, and there are identities $1_A \in \underline{\underline{A}}(A, A)$ with the usual properties. The dual category $\underline{\underline{A}}^0$ has the same objects, $\underline{\underline{A}}^0(A, B) = \underline{\underline{A}}(B, A)$, and composition is reversed. The dual of a statement about categories is the same statement but interpreted in $\underline{\underline{A}}^0$. In this sense, general theorems about categories have duals, and the latter are also theorems.

The notion of subcategory is obvious. Similarly, we can form the Cartesian product of categories, in a naive way, to obtain new categories.

We shall often confuse $\underline{\underline{A}}$ with ob $\underline{\underline{A}}$, and write $A \varepsilon \underline{\underline{A}}$ in place of $A \varepsilon$ ob $\underline{\underline{A}}$. The class of all morphisms in $\underline{\underline{A}}$ is denoted mor $\underline{\underline{A}}$.

$$\alpha\colon A \longrightarrow B \quad \text{means} \quad \alpha \varepsilon \underline{\underline{A}}(A, B)$$

as usual. We call α an <u>isomorphism</u> if there exists $b \varepsilon \underline{\underline{A}}(B, A)$ such that $\alpha b = 1_B$ and $b\alpha = 1_A$, i.e. if α is invertible. We call α a monomorphism (resp., epimorphism) if $\alpha b = \alpha c \Rightarrow b = c$ (resp., $b\alpha = c\alpha \Rightarrow b = c$), whenever the indicated compositions are defined. Note that an isomorphism is both a monomorphism and an epimorphism. The converse fails in general. For example, in the category of topological groups and continuous homomorphisms, an inclusion of a dense subgroup is an epimorphism and a monomorphism.

We shall commonly use the following alternative notations:

$$\mathrm{Hom}_{\underline{\underline{A}}}(A, B) = \underline{\underline{A}}(A, B)$$

$$\mathrm{End}_{\underline{\underline{A}}}(A) = \underline{\underline{A}}(A, A)$$

$$\mathrm{Aut}_{\underline{\underline{A}}}(A) = \text{the group of automorphisms of } A \text{ (in } \underline{\underline{A}}\text{)}.$$

A <u>functor</u> $T\colon \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ consists of a map on objects, $A \longmapsto TA$, and maps on morphisms

$$T(=T_{A, B})\colon \underline{\underline{A}}(A, B) \longrightarrow \underline{\underline{B}}(TA, TB)$$

which preserve composition and identities. T is called <u>faithful</u> (resp., <u>full</u>) if $T_{A, B}$ is injective (resp., surjective) for all $A, B \varepsilon \underline{\underline{A}}$. Note that a faithful functor might carry nonismorphic objects to ismorphic ones (e.g., the functor (topological groups) $\xrightarrow[\text{topology}]{\text{ignore the}}$ (groups)) but this cannot happen if it is also full. A <u>contravariant functor</u> $\underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ is a functor $\underline{\underline{A}}^0 \longrightarrow \underline{\underline{B}}$. <u>Functors of several variables</u> are just functors on product categories.

In practice a category will often be specified by naming only its objects. Such license will be allowed when either the morphisms and composition are clear from the context, or, if there is some ambiguity, it is of no consequence for the discussion at hand. Similarly, we shall often define functors by specifying their effect on objects when their effect on morphisms is then clear from the context.

The functors from $\underline{\underline{A}}$ to $\underline{\underline{B}}$ are themselves the objects of a category, denoted $\underline{\underline{B}}^{\underline{\underline{A}}}$. The morphisms are sometimes called natural transformations, so we write

$$\text{Nat. Tran.}(T, S) = \underline{\underline{B}}^{\underline{\underline{A}}}(T, S)$$

A natural transformation $\alpha: T \longrightarrow S$ is a family, $\alpha = (\alpha_A)_{A \in \underline{\underline{A}}}$, of $\underline{\underline{B}}$ - morphisms $\alpha_A : TA \longrightarrow SA$ such that $Sf\,\alpha_A = \alpha_B\,Tf$ whenever $f : A \longrightarrow B$ in $\underline{\underline{A}}$. (Rather innocent assumptions on $\underline{\underline{A}}$ and $\underline{\underline{B}}$ will guarantee that $\underline{\underline{B}}^{\underline{\underline{A}}}(T, S)$ is a set; this will always be so in the examples we treat.) Composition is defined in the obvious way. Suppose we are given functors

$$\underline{\underline{A}} \xrightarrow{S} \underline{\underline{B}} \overset{T_1}{\underset{T_2}{\rightrightarrows}} \underline{\underline{C}} \xrightarrow{U} \underline{\underline{D}}$$

and a morphism $\alpha: T_1 \longrightarrow T_2$. Then we have the composite functors, T_iS, UT_i, etc., and we also have morphisms

$$\alpha S : T_1S \longrightarrow T_2S \qquad (\alpha S)_A = \alpha_{SA} \quad (A \in \underline{\underline{A}})$$

and

$$U\alpha : UT_1 \longrightarrow UT_2 \qquad (U\alpha)_B = U(\alpha_B) \quad (B \in \underline{\underline{B}})$$

If $S^1: \underline{\underline{A}}^1 \longrightarrow \underline{\underline{A}}$ and $U^1: \underline{\underline{D}} \longrightarrow \underline{\underline{D}}^1$ are functors, and if $\alpha^1: T_2 \longrightarrow T_3$ is a morphism of functors then we have the following easily verified rules:

$$\alpha(SS^1) = (\alpha S)S^1 \quad , \qquad (U^1U)\alpha = U^1(U\alpha)$$
$$1_{T_i}S = 1_{T_iS} \quad , \qquad U\,1_{T_i} = 1_{UTi}$$
$$(\alpha^1\alpha)S = (\alpha^1S)(\alpha S) \quad , \qquad U(\alpha^1\alpha) = (U\alpha^1)(U\alpha)$$

The latter show that composition with S and U defines functors $\cdot S : \underline{\underline{C}}^{\underline{\underline{B}}} \longrightarrow \underline{\underline{C}}^{\underline{\underline{A}}}$ and $U\cdot : \underline{\underline{C}}^{\underline{\underline{B}}} \longrightarrow \underline{\underline{D}}^{\underline{\underline{B}}}$, respectively.

A functor $T : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ is an isomorphism if there is a functor $S : \underline{\underline{B}} \longrightarrow \underline{\underline{A}}$ such that $TS = 1_{\underline{\underline{B}}}$ and $ST = 1_{\underline{\underline{A}}}$.

A more natural notion is that of an *equivalence*; for an equivalence we require only that $TS \simeq 1_{\underline{\underline{B}}}$ and $ST \simeq 1_{\underline{\underline{A}}}$. An equivalence preserves all of the properties of interest to us in a category except size. In particular an equivalence is full and faithful, so it is bijective on isomorphism classes of objects.

We shall have frequent occasion to use the following:

(1.1) PROPOSITION. (Criterion for equivalence). *A functor* $T : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ *is an equivalence if and only if*: (a) T *is full and faithful; and* (b) *every object of* $\underline{\underline{B}}$ *is isomorphic to* TA *for some* $A \in \underline{\underline{A}}$.

Clearly (a) and (b) are necessary for an equivalence. We prove sufficiency and construct $S : \underline{\underline{B}} \longrightarrow \underline{\underline{A}}$ by *choosing*, for each $B \in \underline{\underline{B}}$, an $SB \in \underline{\underline{A}}$ together with an isomorphism $\beta_B : B \longrightarrow TSB$. Then the cummutative triangle

$$\begin{array}{ccc} \underline{\underline{B}}(B, B') & \xrightarrow{\quad S_{B,B'} \quad} & \underline{\underline{A}}(SB, SB') \\ {\scriptstyle \underline{\underline{B}}(\beta_B^{-1},\, \beta_{B'})} \searrow & & \swarrow {\scriptstyle T_{SB,\, SB'}} \\ & \underline{\underline{B}}(TSB, TSB') & \end{array}$$

gives the effect of S on morphisms. It is easily seen then that S is a functor and that $\beta = (\beta_B) : 1_{\underline{\underline{B}}} \longrightarrow TS$ is an isomorphism of functors. Since $\beta_{TA} : TA \longrightarrow TSTA$ is an isomorphism it follows from (a) that $\beta_{TA} = T(\alpha_A)$ for a unique isomorphism $\alpha_A : A \longrightarrow STA$. It is easily checked now that $\alpha = (\alpha_A) : 1_{\underline{\underline{A}}} \longrightarrow ST$ is an isomorphism of functors.

We shall close this section with some basic examples of categories, and the notation to be used for them.

(1.2) CATEGORIES OF MODULES. Let A be a ring. We shall write

mod-A (resp., A-mod)

for the category of right (resp., left) A-modules and A-linear maps. If A° is the opposite ring of A there is a canonical isomorphism A-mod $\longrightarrow$ mod-A°. We shall deal extensively with the following heirarchy of full subcategories:

$$\underline{\underline{P}}(A) \subset \underline{\underline{H}}(A) \subset \underline{\underline{M}}(A) \subset \text{mod-}A$$

Here $M \in \underline{\underline{M}}(A) \iff M$ is finitely generated, $M \in \underline{\underline{P}}(A) \iff M$ is also a projective A-module. Finally $M \in \underline{\underline{H}}(A) \iff M$ has a finite resolution by objects in $\underline{\underline{P}}(A)$ (see §6 below).

(1.3) CATEGORIES OF ENDOMORPHISMS AND AUTOMORPHISMS. A monoid G(e.g., a group) can be viewed as (the morphisms of) a category with a single object. As such a functor from G to a category $\underline{\underline{A}}$ is just a monoid homomorphism $r : G \longrightarrow \text{End}_{\underline{\underline{A}}}(A)$ for some object $A \in \underline{\underline{A}}$. These are the "representations" of G in $\underline{\underline{A}}$, and $\underline{\underline{A}}^G$ is like a category of "G-modules in $\underline{\underline{A}}$." For if $r' : G \longrightarrow \text{End}_{\underline{\underline{A}}}(A')$ then a morphism $f : r \longrightarrow r'$ of functors is just an $\underline{\underline{A}}$-morphism $f : A \longrightarrow A'$ such that $fr(x) = r'(x)f$ for all $x \in G$. Thus, if $\underline{\underline{A}}$ = A-mod, for example, then $\underline{\underline{A}}^G$ is just the category of G-representations on A-modules, i.e., it is the category A[G]-mod, where A[G] is the monoid ring of G over A (see Chapter IX).

We shall apply this construction now to the monoids $\underline{\underline{N}}$ and $\underline{\underline{Z}}$, freely generated as monoid and as group, respectively, by $1 \in \underline{\underline{N}}$. If $\underline{\underline{A}}$ is a category, a monoid homomorphism $r : \underline{\underline{N}} \longrightarrow \text{End}_{\underline{\underline{A}}}(A)$ is completely determined by $\alpha = r(1)$, which can be arbitrary. Moreover r extends to a homomorphism $\underline{\underline{Z}} \longrightarrow \text{End}_{\underline{\underline{A}}}(A)$ if and only if $\alpha \in \text{Aut}_{\underline{\underline{A}}}(A)$. If we identify r with the pair (A, α) then we see that the category $\underline{\underline{A}}^{\underline{\underline{N}}}$ is isomorphic to the category whose objects are pairs (A, α) $(A \in \underline{\underline{A}}, \alpha \in \text{End}_{\underline{\underline{A}}}(A))$ and in which a morphism (A, α) to (B, b) is an $\underline{\underline{A}}$-morphism $f : A \longrightarrow B$ such that $f\alpha = bf$. For example, $(A,\alpha) \simeq (A, \alpha')$ if and only if there is an $f \in \text{Aut}_{\underline{\underline{A}}}(A)$ such that $\alpha' = f^{-1}\alpha f$. We shall refer to $\underline{\underline{A}}^{\underline{\underline{N}}}$, as the category of endomorphisms in $\underline{\underline{A}}$. We can identify $\underline{\underline{A}}^{\underline{\underline{Z}}}$

with the full subcategory whose objects are those (A, α) for which $\alpha \in \mathrm{Aut}_{\underline{\underline{A}}}(A)$. This is called the "category of automorphisms" in $\underline{\underline{A}}$. The latter will be studied in great detail in subsequent chapters (e.g., Chapters VII and VIII), where we shall use the alternative notation

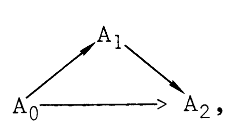
$$\Sigma\, \underline{\underline{A}} = \underline{\underline{A}}^{\underline{\underline{Z}}}$$

(1.4) SOME DIAGRAM CATEGORIES. Let D be a partially ordered set. We regard D as the set of objects of a category, also denoted D, in which $D(\alpha, b)$ has one element if $\alpha \le b$ and is otherwise empty. Composition is then forced, and it is definable because $\le$ is transitive. (We do not really need to know that $\alpha \le b$ and $b \le \alpha \Rightarrow \alpha = b$.)

As examples we have the sets $\Delta_n = \{0, 1, \ldots, n\}$ with their natural orderings. Thus Δ_0 is the <u>trivial category</u>. For any category $\underline{\underline{A}}$ we can identify $\underline{\underline{A}}$ canonically with $\underline{\underline{A}}^{\Delta_0}$. A functor $F : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ is called a <u>constant functor</u> if it factors through Δ_0. The category Δ_1 has a single nonidentity arrow, and $\underline{\underline{A}}^{\Delta_1}$ is called the <u>category of morphisms</u> in $\underline{\underline{A}}$. A functor $\Delta_1 \longrightarrow \underline{\underline{A}}$ can be identified with a morphism $\alpha : A_0 \longrightarrow A_1$. If $b : B_0 \longrightarrow B_1$ is another then a morphism $\alpha \longrightarrow b$ in $\underline{\underline{A}}^{\Delta_1}$ is a pair of morphisms $f_i : A_i \longrightarrow B_i$ $(i = 0, 1)$ such that $f_1\alpha = bf_0$. In particular it makes sense to say that "two morphisms are isomorphic".

Note that the category of endomorphisms in $\underline{\underline{A}}$ [see (1.3)] is a subcategory of the category of morphisms, but it is <u>not</u> a full subcategory. For if α and b above are endomorphisms then the morphisms $(f_0, f_1) : \alpha \longrightarrow b$ in $\underline{\underline{A}}^{\underline{\underline{N}}}$ are those for which $f_0 = f_1$.

The category $\underline{\underline{A}}^{\Delta_2}$ is the <u>category of commutative triangles</u>,



with an evident notion of morphism.

More generally, the diagrams of a fixed type in $\underline{\underline{A}}$ can be viewed as functors from the "diagram category" of the given type. As such we can speak of morphisms of diagrams.

Exercise. Let $T_n(A)$ be the ring of triangular matrices $(a_{ij})_{1 \leq i,\ j \leq n}$, $a_{ij} = 0$ if $i < j$, over a ring A. Establish an equivalence

$$(A\text{-mod})^{\Delta_n} \longrightarrow T_n(A)\text{-mod}$$

(First do the case n = 2.)

§2. REPRESENTABLE FUNCTORS

There is a general type of identity which says that by fixing a variable, we can view functions of two variables as functions of one variable whose values are functions of the remaining variable. Applied to functors, this becomes

$$\underline{\underline{C}}^{(\underline{\underline{A}} \times \underline{\underline{B}})} = (\underline{\underline{C}}^{\underline{\underline{A}}})^{\underline{\underline{B}}}$$

where $\underline{\underline{A}}$, $\underline{\underline{B}}$, and $\underline{\underline{C}}$ are categories. For any category $\underline{\underline{A}}$ we have the basic "morphism functor"

$$\underline{\underline{A}}(\quad,\quad) : \underline{\underline{A}}^0 \times \underline{\underline{A}} \longrightarrow \text{Sets}$$

By the formalism above this corresponds to a functor

$$\underline{\underline{A}}^0 \longrightarrow \text{Sets}^{\underline{\underline{A}}} \quad ; \quad \begin{array}{l} A \longmapsto \bar{A} \\ \alpha \longmapsto \bar{\alpha} \end{array}$$

called the representation functor. Explicitly,

$$\bar{A}(B) = \underline{\underline{A}}(A, B) \quad \text{and} \quad \bar{A}(b) = \underline{\underline{A}}(A, b) : c \longmapsto cb$$

If $\alpha : A \longrightarrow A'$ then $\bar{\alpha} : \bar{A} \longrightarrow \bar{A}'$ is defined by

$$\bar{\alpha}_B : \bar{A}(B) \longrightarrow \bar{A}'(B) \quad , \quad \bar{\alpha}_B(b) = b\alpha$$

(2.1) PROPOSITION (Yoneda). Let $A \in \underline{\underline{A}}$ and $F \in \text{Sets}^{\underline{\underline{A}}}$. Define

$$\phi : \text{Nat. Tran.}(\bar{A}, F) \longrightarrow F(A)$$

by $\phi(\alpha) = \alpha_A(1_A)$. Then ϕ is bijective.

Proof. If $\alpha \in F(A)$ define $\alpha' : \bar{A} \longrightarrow F$ by

$\alpha'_B(h) = (Fh)(\alpha)$ for $h : A \longrightarrow B$. Then $\phi(\alpha') = (\alpha')_A(1_A)$ $= (F1_A)(\alpha) = \alpha$. Thus ϕ is surjective. Moreover, for α as above, $\phi(\alpha)'_A(1_A) = (F1_A)(\phi(\alpha)) = \phi(\alpha) = \alpha_A(1_A)$. Therefore α and $\beta = \phi(\alpha)'$ agree on 1_A. In general, if $h : A \longrightarrow B$, then the commutivity of

$$\begin{array}{ccc} \bar{A}(A) & \xrightarrow{\alpha_A} & F(A) \\ {\scriptstyle \bar{A}(h)}\downarrow & & \downarrow{\scriptstyle F(h)} \\ \bar{A}(B) & \xrightarrow[\alpha_B]{} & F(B) \end{array}$$

shows that $\alpha_B(h) = \alpha_B\ (\bar{A}(h)(1_A)) = F(h)(\alpha_A(1_A))$ $= F(h)(\beta_A)(1_A)) = \beta_B(\bar{A}(h)(1_A)) = \beta_B(h)$. Thus ϕ is bijective, q.e.d.

(2.2) COROLLARY. *The representation functor*

$$\underline{\underline{A}}^0 \longrightarrow Sets^{\underline{\underline{A}}}$$

is faithful and full. In particular any functor isomorphism $\underline{\underline{A}}(A, \cdot) \longrightarrow \underline{\underline{A}}(B, \cdot)$ *is induced by a isomorphism* $A \longrightarrow B$.

Proof. The map

$$\bar{B}(A) = \underline{\underline{A}}^0(A, B) \xrightarrow{\text{repr. functor}} \text{Nat. Trans}(\bar{A}, \bar{B})$$

is just the map $\alpha \longmapsto \alpha'$ constructed above.

A functor $F : \underline{\underline{A}} \longrightarrow$ Sets is called *representable* if it is isomorphic to $\bar{A}$ for some $A \in \underline{\underline{A}}$. If $\alpha : \bar{A} \longrightarrow F$ is such an isomorphism then the pair (A, α) is determined up to a unique isomorphism, according to the results above. Thus an object is completely known by its morphisms into other objects. Analogous conclusions for the functors $\underline{\underline{A}}(\cdot, A)$ can be deduced by replacing $\underline{\underline{A}}$ by $\underline{\underline{A}}^0$.

We shall now define several types of objects in categories by designating the functors they are to represent. Of course this leaves open the question of their existence.

An *initial object* represents the functor $A \longmapsto \{A\}$, i.e., it has a unique morphism into any object. Dually, a *final object* admits a unique morphism from any object. An object which is both initial and final is called a *zero object*. The symbol 0 will always be used to denote a zero object. In its presence there is a unique morphism in $\underline{\underline{A}}(A, B)$ which factors as $A \longrightarrow 0 \longrightarrow B$, and we denote this morphism also by 0! Evidently $0\alpha = 0$ and $\alpha 0 = 0$ for all morphisms α.

Suppose $X_1 \xrightarrow{f_1} X' \xleftarrow{f_2} X_2$ is a pair of set maps. Then we define

$$X_1 \pi_{X'} X_2 = \{(x_1, x_2) \varepsilon X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}.$$

Given a diagram $A_1 \xrightarrow{f_1} A' \xleftarrow{f_2} A_2$ in a category $\underline{\underline{A}}$ we define the fiber product, $A_1 \pi_{A'} A_2$, by

$$\underline{\underline{A}}(B, A_1 \pi_{A'} A_2) = \underline{\underline{A}}(B, A_1) \pi_{\underline{\underline{A}}(B, A')} \underline{\underline{A}}(B, A_2) \qquad (B \varepsilon \underline{\underline{A}})$$

Explicitly, $A_1 \pi_{A'} A_2$ comes equipped with "projections" p_1, p_2 making

$$\begin{array}{ccc} A_1 \pi_{A'} A_2 & \xrightarrow{p_2} & A_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[f_1]{} & A' \end{array}$$

commutative. Moreover, given another commutative square

$$\begin{array}{ccc} B & \xrightarrow{h_2} & A_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[f_2]{} & A' \end{array}$$

there is a unique morphism $t : B \longrightarrow A_1 \pi_{A'} A_2$ such that $h_i = P_i t (i = 1, 2)$. A square of the type (1) above will be said to be <u>Cartesian</u>. It is also sometimes called a <u>pullback diagram</u>.

The dual notion associates with a diagram

$A_1 \xleftarrow{f_1} A' \xrightarrow{f_2} A_2$ a co-Cartesian (or pushout diagram)

$$\begin{array}{ccc} A_1 \amalg_{A'} A_2 & \longleftarrow & A_2 \\ \uparrow & & \uparrow \\ A_1 & \longleftarrow & A' \end{array}$$

i.e., one which is initial in the category of all such commutative squares. This defines the fiber coproduct of the given diagram.

Let $\alpha : A \longrightarrow B$ be a morphism in a category $\underline{\underline{A}}$ with a zero object. Then we define

$$\mathrm{Ker}(\alpha) = A \pi_B 0$$

and

$$\mathrm{Coker}(\alpha) = 0 \amalg_A B$$

We shall use capital letters for the objects here and small letters for the corresponding morphisms:

$$\mathrm{Ker}(\alpha) \xrightarrow{\mathrm{ker}(\alpha)} A \xrightarrow{\alpha} B \xrightarrow{\mathrm{coker}(\alpha)} \mathrm{Coker}(\alpha)$$

Given $\alpha' : A' \longrightarrow A$ such that $\alpha\alpha' = 0$ there is a unique $\alpha : A' \longrightarrow \mathrm{Ker}(\alpha)$ such that $\alpha' = \mathrm{ker}(\alpha)\alpha$. A similar property characterizes the cokernel. We further define

$$\mathrm{Im}(\alpha) = \mathrm{Ker}(\mathrm{coker}(\alpha))$$

and

$$\mathrm{Coim}(\alpha) = \mathrm{Coker}(\mathrm{ker}(\alpha))$$

It is easily checked that there is a canonical morphism

$$i : \mathrm{Coim}(a) \longrightarrow \mathrm{Im}(a)$$

such that the diagram

$$\begin{array}{ccc} \mathrm{Ker}(a) & & \mathrm{Coker}(\alpha) \\ \downarrow & & \uparrow \\ A & \xrightarrow{\alpha} & B \\ \downarrow & & \uparrow \\ \mathrm{Coim}(\alpha) & \xrightarrow{i} & \mathrm{Im}(\alpha) \end{array}$$

commutes.

Next we introduce the notion of the limit (and colimit) of a functor. Let $\underline{\underline{L}}$ and $\underline{\underline{A}}$ be categories. For each $A \in \underline{\underline{A}}$ we have the constant functor

$$c(A) : \underline{\underline{L}} \longrightarrow \underline{\underline{A}} \qquad \begin{array}{ll} L \longmapsto A & (L \in \mathrm{ob}\underline{\underline{L}}) \\ f \longmapsto 1_A & (f \in \mathrm{mor}\underline{\underline{L}}) \end{array}$$

A morphism α : A ⟶ B defines an evident natural transformation $c(\alpha)$: $c(A) \longrightarrow c(B)$, so we have a functor

$$c : \underline{\underline{A}} \longrightarrow \underline{\underline{A}}^{\underline{\underline{L}}}$$

which is evidently full and faithful. Now if $F : \underline{\underline{L}} \longrightarrow \underline{\underline{A}}$ is any functor we define its limit,

$$\underset{\leftarrow}{F} = \lim F \in \underline{\underline{A}}$$

by

$$\underline{\underline{A}}(A, \underset{\leftarrow}{F}) = \underline{\underline{A}}^{\underline{\underline{L}}}(c(A), F) \qquad (A \in \underline{\underline{A}})$$

Dually, the colimit of F,

$$\underset{\rightarrow}{F} = \operatorname{colim} F \in \underline{\underline{A}}$$

is defined by

$$\underline{\underline{A}}(\underset{\rightarrow}{F}, A) = \underline{\underline{A}}^{\underline{\underline{L}}}(F, c(A)) \qquad (A \in \underline{\underline{A}})$$

If limits always exist it is easy to see that they define a functor

$$\lim : \underline{\underline{A}}^{\underline{\underline{L}}} \longrightarrow \underline{\underline{A}}$$

and similarly for colimits.

Further remarks about limits will be made in §8. For the moment we shall discuss only the following case: Let L be a set and let $\underline{\underline{L}}$ be the category with ob $\underline{\underline{L}}$ = L and with only identities as morphisms. A functor $F : \underline{\underline{L}} \longrightarrow \underline{\underline{A}}$ is then simply a family $(F(i))_{i \in L}$ of objects of $\underline{\underline{A}}$ indexed by L. In this case the limit of F is called the product of $(F(i))_{i \in L}$, and it is denoted

$$\Pi_{i \in L} F(i)$$

The colimit is called the coproduct and is denoted

$$\amalg_{i \in L} F(i)$$

If all of the F(i) are equal to the same object A (i.e., if F = c(A)) then we write

$$A^L \quad \text{and} \quad A^{(L)}$$

for the product and coproduct, respectively. In particular, if L = {1,..., n}, we shall often write

$$A^n = A\pi\cdots\pi A \qquad \text{(n factors)}$$

and

$$A^{(n)} = A \amalg \cdots \amalg A \quad \text{(n factors)}$$

Exercise. Let $\underline{\underline{L}}$ be the category with objects $\{0, 1, 2\}$ and with only two nonidentity morphisms, $1 \longrightarrow 0$ and $2 \longrightarrow 0$. Let $F : \underline{\underline{L}} \longrightarrow \underline{\underline{A}}$ be a functor. Interpret lim F and colim F.

§3. ADDITIVE CATEGORIES

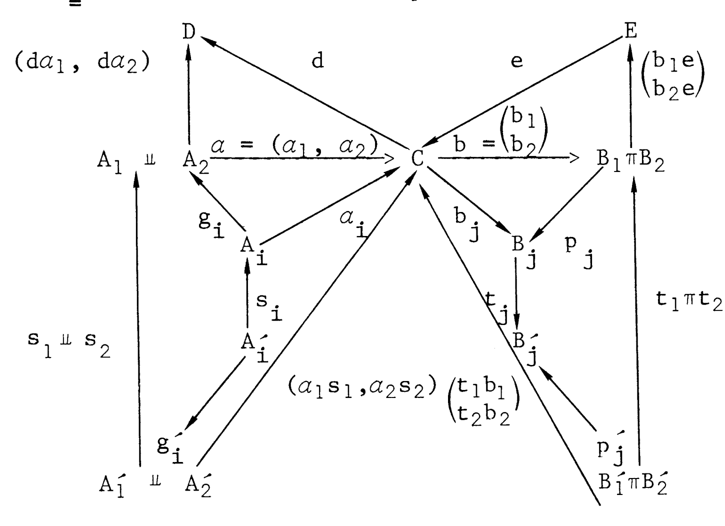
An additive category is a category $\underline{\underline{A}}$ satisfying axioms Ad Cat $i(0 \leq i \leq 3)$ below. We want the morphisms $\underline{\underline{A}}(A, B)$ to be an Abelian group. It is a remarkable fact that this structure is built into the rather primitive looking axioms that follow:

Ad Cat 0. $\underline{\underline{A}}$ has a zero object, 0.

Ad Cat 1. The product and coproduct of any two objects in $\underline{\underline{A}}$ exist.

Before stating the next axiom we shall introduce matrix notation. Reference to the following commutative diagram in $\underline{\underline{A}}$ will facilitate its explanation.

(1)



Here the q's and p's are the structure morphisms for the indicated coproducts and products, respectively.

For given morphisms α and b as above, we write $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_i = \alpha q_i$ $(i = 1, 2)$, and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, where $b_j = p_j b$ $(j = 1, 2)$. The rules for composition in this notation are then illustrated in the diagram. The formulas are:

$$d(\alpha_1, \alpha_2) = (d\alpha_1, d\alpha_2), \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e = \begin{pmatrix} b_1 e \\ b_2 e \end{pmatrix}$$

(2)

$$(\alpha_1, \alpha_2)(s_1 \amalg s_2) = (\alpha_1 s_1, \alpha_2 s_2) \quad ,$$

$$(t_1 \pi t_2)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} t_1 b_1 \\ t_2 b_2 \end{pmatrix}$$

Now $\quad \underline{\underline{A}}(A_1 \amalg A_2, B_1 \pi B_2) = \prod_{1 \leq i, \ j \leq 2} \underline{\underline{A}}(A_i, B_j)$

so we can represent a morphism α: $A_1 \amalg A_2 \longrightarrow B_1 \pi B_2$ by a <u>matrix</u>

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} , \quad \alpha_{ij} = p_i \alpha q_j .$$

Here $\alpha_{ij} : A_j \longrightarrow B_i$, (note the reverse!). Viewing α as a morphism from a coproduct and to a product respectively, we can compare this notation with that introduced above, as follows:

$$\left(\begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix}, \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix}\right) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} (\alpha_{11}, \alpha_{12}) \\ (\alpha_{21}, \alpha_{22}) \end{pmatrix}$$

In the diagram (1), we can now write the composite

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} (\alpha_1, \alpha_2) = \begin{pmatrix} b_1\alpha_1 & b_1\alpha_2 \\ b_2\alpha_1 & b_2\alpha_2 \end{pmatrix}$$

Moreover, we have the formula

$$(t_1 \pi t_2) \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} (s_1 \amalg s_2) = \begin{pmatrix} t_1\alpha_{11}s_1 & t_1\alpha_{12}s_2 \\ t_2\alpha_{21}s_1 & t_2\alpha_{22}s_2 \end{pmatrix}$$

From this it follows that

$$\phi = \phi_{A,\ B} = \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix} : \quad A \amalg B \longrightarrow A \pi B$$

is a natural transformation from the coproduct to the product.

Ad Cat 2. ϕ is an isomorphism.

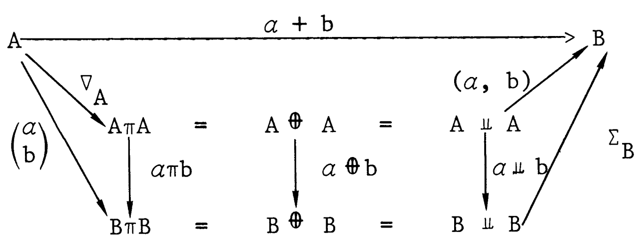
In the presence of this axiom we can use ϕ to identify $A \amalg B$ and $A\pi B$, which we then sometimes denote by $A \oplus B$ and call the direct sum of A and B. Note that we have the "diagonal morphism"

$$\Delta_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \quad A \longrightarrow A\pi A = A \oplus A$$

and the "sum morphism"

$$\Sigma_B = (1,\ 1) : \quad B \amalg B = B \oplus B \longrightarrow B$$

With these we shall introduce an addition, $a + b$, in $\underline{\underline{A}}(A,\ B)$ by the commutativity of the diagram

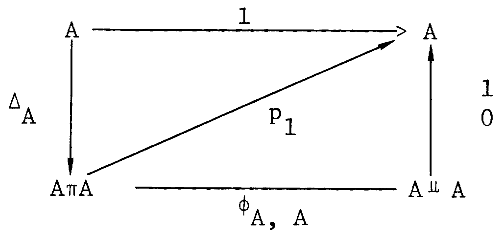


The fact that $(a\pi b)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $(1,\ 1)(a \amalg b) = (a,\ b)$ comes from (2) above. Moreover, the formulas $d(a,\ b) = (da,\ db)$ and $\begin{pmatrix} a \\ b \end{pmatrix} e = \begin{pmatrix} ae \\ be \end{pmatrix}$ in (2) show that we have

(3) $\qquad d(a + b) = da + db \quad$ and $\quad (a + b)e = ae + be$

(3.1) PROPOSITION. The addition + defined above endows each $\underline{\underline{A}}(A,\ B)$ with the structure of a commutative monoid whose neutral element is the zero morphism. Moreover composition in $\underline{\underline{A}}$ is + - bilinear.

Proof. We have just verified the last assertion. To prove that $a + 0 = a$ we first write $(a + 0) = a(1_A + 0)$, using (3). To show now that $1_A + 0 = 1_A$ we must verify that the square



is commutative. We do this by showing that each triangle commutes:

$$p_1\Delta_A = p_1(1, 1) = 1$$

$$p_1\phi_{A,\,A} = p_1\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\0\end{pmatrix}$$

Replacing $\begin{pmatrix}1\\0\end{pmatrix}$ and p_1 by $\begin{pmatrix}0\\1\end{pmatrix}$ and p_2, respectively, we see that $0 + \alpha = \alpha$ as well.

Now let α, b, c, d ε $\underline{\underline{A}}(A, B)$ and consider the composite

$$A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{\begin{pmatrix}\alpha & b\\ c & d\end{pmatrix}} B \oplus B \xrightarrow{\Sigma_B} B$$

We can identify the middle with $\begin{pmatrix}(\alpha, & b)\\(c, & d)\end{pmatrix}$ and with $\left(\begin{pmatrix}\alpha\\c\end{pmatrix}, \begin{pmatrix}b\\d\end{pmatrix}\right)$. The two resulting interpretations of the composite give us the formula

$$(\alpha + b) + (c + d) = (\alpha + c) + (b + d)$$

letting $d = 0$, we obtain, thanks to the first part of the proof,

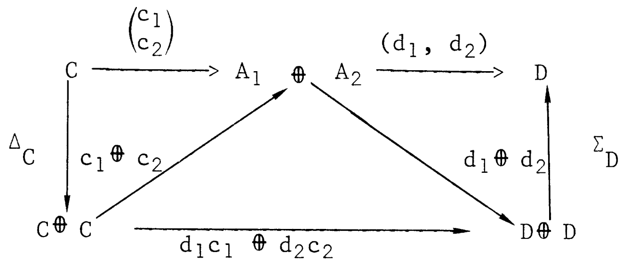
$$(\alpha + b) + c = (\alpha + c) + b$$

The case $\alpha = 0$ shows now that + is commutative. Therefore we can rewrite the last equation as:

$$c + (\alpha + b) = (c + \alpha) + b, \text{ i.e., + is associative.}$$

q.e.d.

Given $C \xrightarrow{c_i} A_i \xrightarrow{d_i} D$ $(i = 1, 2)$ we see from the commutative diagram



that

$$(d_1, d_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = d_1c_1 + d_2c_2$$

From this we deduce the usual rule for matrix multiplication

$$\begin{pmatrix} \alpha & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + bc' & \alpha b' + bd' \\ c\alpha' + dc' & cb' + dd' \end{pmatrix}$$

whenever we have

$$A_1 \oplus A_2 \xrightarrow{\begin{pmatrix} \alpha' & b' \\ c' & d' \end{pmatrix}} B_1 \oplus B_2 \xrightarrow{\begin{pmatrix} \alpha & b \\ c & d \end{pmatrix}} C_1 \oplus C_2$$

Suppose we are given $\alpha \in \underline{\underline{A}}(A, B)$. Consider

$$\alpha = \begin{pmatrix} 1_A & 0 \\ \alpha & 1_B \end{pmatrix} : A \oplus B \longrightarrow A \oplus B$$

If

$$C \xrightarrow{\begin{pmatrix} c \\ c' \end{pmatrix}} A \oplus B \xrightarrow{(d,\ d')} D$$

then $\alpha\begin{pmatrix} c \\ c' \end{pmatrix} = \begin{pmatrix} c \\ \alpha c + c' \end{pmatrix}$ and $(d, d')\alpha = (d + d'\alpha, d')$

Visibly, then, α is both a monomorphism and an epimorphism. Suppose it were an isomorphism say, with inverse $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Then the equation

$$1_{A \oplus B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & x \\ \alpha w + y & \alpha x + z \end{pmatrix}$$

shows that $w = 1$ and hence $0 = \alpha w + y = \alpha + y$. Thus an inverse for α yields a + - inverse for α. This proves that axiom Ad Cat 3 below is automatic if we know that a morphism in $\underline{\underline{A}}$ which is both a monomorphism and an epimorphism must be an isomorphism.

<u>Ad Cat 3</u>. The operation + makes each $\underline{\underline{A}}(A, B)$ an

Abelian group.

An alternative definition of additive category postulates (i) a zero object, (ii) existence of products, and (iii) an additive group structure on each $\underline{\underline{A}}(A, B)$ so that composition is bilinear. Then if $p_i : A_1 \pi A_2 \longrightarrow A_i$ are the projections we can define

$$q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : A_1 \longrightarrow A_1 \pi A_2 \quad \text{and}$$

$$q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : A_2 \longrightarrow A_1 \pi A_2$$

and we have $p_i q_i = 1_{A_i}$ and $p_i q_j = 0$ if i j. Therefore, $s = q_1 p_1 + q_2 p_2$ satisfies $p_i s = p_i = p_i \cdot 1$ $(i = 1, 2)$ and so $s = 1$. We can now show that $(A_1 \pi A_2, q_1, q_2)$ is the coproduct of A_1 and A_2. For given $b_i : A_i \longrightarrow B$ $(i = 1,2)$ define $b = (b_1, b_2) : A_1 \pi A_2 \longrightarrow B$ by $(b_1, b_2) = b_1 p_1 + b_2 p_2$. The formulas above show that $bq_i = b_i (i = 1, 2)$. If $b' : A_1 \pi A_2 \longrightarrow B$ also satisfies $b' q_i = b_i (i = 1, 2)$ then $cq_i = 0 (i = 1, 2)$, where $c = b - b'$. Hence $0 = cq_1 p_1 + cq_2 p_2 = c1 = c$. From this construction of the coproduct one can easily deduce the equivalence of this definition of additive category with the one presented above. Dually, one can obtain a definition by assuming the existence of coproducts instead of products in (ii).

The reasoning above shows that if we have a diagram

$$A_1 \underset{p_1}{\overset{q_1}{\rightleftarrows}} A \underset{q_2}{\overset{p_2}{\rightleftarrows}} A_2$$

such that $p_i q_i = 1_{A_i}$ $(i = 1, 2)$, $p_i q_j = 0$ if $i \neq j$, and $1_A = q_1 p_1 + q_2 p_2$, then $A = A_1 \oplus A_2$. More precisely, $(A, p_1, p_2) = A_1 \pi A_2$ and $(A, q_1, q_2) = A_1 \amalg A_2$.

(3.2) PROPOSITION. <u>Let</u>

$$(4)\qquad \begin{array}{ccc} A & \xrightarrow{\;b_2\;} & A_2 \\ {\scriptstyle b_1}\big\downarrow & & \big\downarrow{\scriptstyle a_2} \\ A_1 & \xrightarrow[\;a_1\;]{} & A' \end{array}$$

be a square in an additive category $\underline{\underline{A}}$, *and consider*

$$A \xrightarrow{\;b = \begin{pmatrix} b_1 \\ -b_2 \end{pmatrix}\;} A_1 \oplus A_2 \xrightarrow{\;a = (a_1,\ a_2)\;} A'$$

(i) *The square* (4) *is commutative if and only if* $ba = 0$.

(ii) *It is Cartesian if and only if* $b = \ker(a)$.

(iii) *It is co-Cartesian if and only if* $a = \operatorname{coker}(b)$.

Proof. Exercise.

A functor $T : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ between additive categories is called an *additive functor* if the maps

$$T_{A,\ B} : \underline{\underline{A}}(A,\ B) \longrightarrow \underline{\underline{B}}(TA,\ TB)$$

are all group homomorphisms. The discussion above shows that such a T must preserve products and coproducts. Conversely a functor T which preserves products or coproducts must be additive.

We shall close this section now with a proof of the "Krull-Schmidt Theorem", which asserts the uniqueness up to isomorphism of direct sum decompositims. We fix an additive category $\underline{\underline{A}}$.

(3.3) LEMMA. *Let*

$$A \oplus B \xrightarrow{\;\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\;} C \oplus D$$

If α *and* a *are both isomorphisms then* $B \simeq D$.

Proof. If $c = 0$ then the lower right coordinate of α^{-1} is an inverse for d. In general we replace α by

$$\begin{pmatrix} 1_C & 0 \\ -ca^{-1} & 1_D \end{pmatrix}\alpha = \begin{pmatrix} a & b \\ 0 & d' \end{pmatrix}$$

and reduce to the first case.

We shall say that an idempotent $e \in \mathrm{End}_{\underline{\underline{A}}}(A)$ splits if there exists a diagram $B \xrightarrow{q} A \xrightarrow{p} B$ such that $pq = 1$ and $qp = e$. We assume henceforth that all idempotents in $\underline{\underline{A}}$ split.

(3.4) LEMMA. Given a diagram $A \xrightarrow{q} B \xrightarrow{p'} A$ such that $p'q$ is an automorphism of A, there exists a $q_1 : A_1 \longrightarrow B$ such that $(B; q, q_1)$ represents B as $A \oplus A_1$.

Proof. Let $p = (p'q)^{-1}p'$. Then $pq = 1$ so $e = qp$ is idempotent. By assumption, therefore, we can find $A_1 \xrightarrow{q_1} B \xrightarrow{p_1} A$ such that $p_1q_1 = 1$ and $q_1p_1 = 1-e$. Then the data $(B; q, p; q_1, p_1)$ satisfy the required identities for $A \oplus A_1$.

A ring R is called local if a sum of two nonunits in R is a nonunit. The nonunits then constitute the unique maximal ideal of R. An object $A \in \underline{\underline{A}}$ is called indecomposable if $A \neq 0$ and if $A \simeq B \oplus C \Rightarrow B = 0$ or $C = 0$. This is equivalent to the condition that $\mathrm{End}_{\underline{\underline{A}}}(A)$ contains precisely two idempotents, 0 and 1. This is clearly the case if $\mathrm{End}_{\underline{\underline{A}}}(A)$ is a local ring $(\neq 0)$.

(3.5) LEMMA. Suppose $A \oplus B = C_1 \oplus \cdots \oplus C_n$, and assume $R = \mathrm{End}_{\underline{\underline{A}}}(A)$ is a local ring. Then there is an i such that $C_i \simeq C_i' \oplus C_i''$ where $C_i'' \simeq A$ and $B \simeq C_i' \oplus \bigoplus_{j \neq i} C_j$.

Proof. Let (q_A, p_A), (q_B, p_B), and $(q_i, p_i)(1 \leq i \leq n)$ be the morphisms associated with the two direct sum decompositions. Then in R we have $1 = p_Aq_A = p_A(\Sigma q_ip_i)q_A =$

$\Sigma p_A q_i p_i q_A$. Since R is local one of the $p_A q_i p_i q_A$ must be a unit. Relabeling, if necessary, we can assume $(p_A q_1)(p_1 q_A)$ is an antomorphism of A. According to (3. 4) there is a $q_1' : C_1' \longrightarrow C_1$ such that $(C_1; p_1 q_A, q_1')$ represents C_1 as $A \oplus C_1'$. Using this to refine the decomposition $C_1 \oplus \cdots \oplus C_n$ to $A \oplus C_1' \oplus C_2 \oplus \cdots \oplus C_n$, we obtain an isomorphism of the latter with $A \oplus B$ such that the composite

$$A \xrightarrow{q_A} A \oplus B = A \oplus C_1' \oplus C_2 \oplus \cdots \oplus C_n \xrightarrow{\text{1st proj.}} A$$ is

an isomorphism. It follows therefore from (3. 3) that $B \simeq C_1' \oplus C_2 \oplus \cdots \oplus C_n$.

(3.6) THEOREM (Krull-Schmidt). Let $\underline{\underline{A}}$ be an additive category in which all idempotents split. Let $A_i \in \underline{\underline{A}}(1 \leq i \leq n)$ be nonzero objects with local endomorphism rings, and put $A = A_1 \oplus \cdots \oplus A_n$.

(a) Any direct sum decomposition of A can be refined to one with indecomposable summands.

(b) If $A \simeq B_1 \oplus \cdots \oplus B_m$ with each B_i indecomposable then $m = n$ and there is a permutation α of $\{1, \ldots, n\}$ such that $B_i \simeq A_{\alpha(i)}$ $(1 \leq i \leq n)$.

Proof. Induction on n; the case n = 1 is clear, so suppose n > 1.

If $A \simeq C_1 \oplus \cdots \oplus C_r$ then (3. 5) implies that for some i, say, i = 1, we can write $C_1 \simeq A_1 \oplus C_1'$, so that $A_2 \oplus \cdots \oplus A_n \simeq C_1' \oplus C_2 \oplus \cdots \oplus C_r$. By induction we can refine the latter to an undecomposable decomposition. If the C_i are undecomposable to begin with, then we must have $C_1' = 0$ and the uniqueness now follows also by induction.

§4. ABELIAN CATEGORIES

An Abelian category is a category $\underline{\underline{A}}$ satisfying:

Ab Cat 0. $\underline{\underline{A}}$ is additive (see §3);

Ab Cat 1. Every morphism α in $\underline{\underline{A}}$ has a kernel and a cokernel; and

Ab Cat 2. ("First isomorphism theorem") The canonical morphism

$$\mathrm{Coim}\ (\alpha) \longrightarrow \mathrm{Im}(\alpha)$$

is an isomorphism for each morphism α in $\underline{\underline{A}}$.

The intention of these axioms is to make available, in any Abelian category, all of the elementary arguments and constructions (involving only a finite amount of data) which one performs in categories of modules. The achievement of this aim is testified to by the "Embedding Theorem," which we quote below. In view of that theorem one might protest that the notion of Abelian category is superfluous; why not speak of subcategories of categories of modules instead. This is roughly analogous to asking that we only speak of vector spaces with fixed coordinate systems, or that we speak only of groups of permutations (after all, every group is one). There are many reasons beyond lingiustic simplification that make the notion of Abelian category natural and useful. The most obvious one derives from the fact that the axioms are self-dual, so that the dual of a theorem about Abelian categories is again one. Only rarely does the dual of a category of modules have a natural representation as a category of modules. Furthermore, there is the important notion of quotient category (see Chapter VIII, §5) which would be awkward, to say the least, to formalize using only categories of modules. Of greatest importance, perhaps, is the fact that, with respect to certain infinite constructions (e.g. limits) categories of modules betray certain definite idiosyncracies.

Let $\underline{\underline{A}}$ be an Abelian category. A sequence $\cdots \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow \cdots$ is called exact at B if $\ker(b) = \mathrm{im}(a)$. A functor $T : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ between Abelian categories is called exact if it is additive, and if $TA \xrightarrow{T\alpha} TB \xrightarrow{Tb} TC$ is exact whenever $A \xrightarrow{\alpha} B \xrightarrow{b} C$ is exact. An exact sequence of the form $0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$ is called a short exact sequence. The exactness of this sequence just means that $a = \ker(b)$ and $b = \operatorname{coker}(a)$. It is easily shown that an additive functor T is exact if and only if it carries short exact sequences into short exact sequences, or, equivalently, that it preserves kernels and cokernels.

If $A \in \underline{\underline{A}}$ then the functors

$$\underline{\underline{A}}(A, .) : \underline{\underline{A}} \longrightarrow \underline{\underline{Z}}\text{-mod} \quad \text{and}$$

$$\underline{\underline{A}}(. , A) : \underline{\underline{A}}^{\circ} \longrightarrow \underline{\underline{Z}}\text{-mod}$$

are both kernel-preserving. Therefore they are exact if and only if they preserve epimorphisms. An object $P \in \underline{\underline{A}}$ is called projective if $\underline{\underline{A}}(P, .)$ is exact, i.e., if it preserves epimorphisms. Explicitly, given an epimorphism $B \xrightarrow{b} C$ in $\underline{\underline{A}}$, then it is required that every morphism $p: P \longrightarrow C$ factor through B: $p = bq$ for some $q: P \longrightarrow B$. In case $C = P$ and $p = 1_P$ this implies that every epimorphism $B \xrightarrow{b} P$ has a right inverse, and hence that $B \simeq \operatorname{Ker}(b) \oplus P$.

(4.1) EMBEDDING THEOREM (Freyd, Grothendieck, Lubkin). Let $\underline{\underline{A}}$ be an Abelian category with only a set of objects. Then there is an exact functor $E: \underline{\underline{A}} \longrightarrow \underline{\underline{Z}}$ -mod which is injective on both objects and morphisms.

The first published proof of this is Lubkin's. An elegant proof by Freyd can be found in Freyd [1] or in Mitchell [1]. Mitchell has also a useful strengthening of the theorem. He obtains a functor $E: \underline{\underline{A}} \longrightarrow R$ -mod, for a suitable ring R, which has all the properties of the E above and which is moreover full. Thus the maps $\underline{\underline{A}}(A, B) \longrightarrow \operatorname{Ham}_R(EA, EB)$ in Mitchell's theorem are isomorphisms, not just monomorphisms.

We shall adopt embedding theorem without proof. It will be used only in the verification of certain properties of finite diagrams in Abelian categories. The theorem permits us to view them as diagrams of modules and module homomorphisms. Typical properties of the functor E which are used are the following: E preserves kernels, cokernels, images, ... ; a sequence, S in $\underline{\underline{A}}$ is exact <=> ES is exact; a square S in $\underline{\underline{A}}$ is Cartesian (or co-Cartesian) <=> ES is. The set theoretical restriction in the embedding theorem is quite innocent because any finite set of data can be embedded in a full Abelian subcategory of $\underline{\underline{A}}$ to which the theorem applies.

For the rest of this section we shall work in a fixed Abelian category $\underline{\underline{A}}$.

We shall often abbreviate a monomorphism α: $A \longrightarrow B$ by writing $A \subset B$, and we then call the (undenoted) α the inclusion. We further write B/A for Coker(α). If $A_1, A_2 \subset B$ then

$$A_1 \cap A_2 = \mathrm{Ker}(B \longrightarrow (B/A_1) \oplus (B/A_2))$$

$$A_1 + A_2 = \mathrm{Im}(A_1 \oplus A_2 \longrightarrow B)$$

We then have the usual "first and second isomorphisms theorems:"

1. <u>If</u> $A \subset B \subset C$ <u>then</u> $C/B \longrightarrow (C/A)/(B/A)$ <u>is an isomorphism</u>; and
2. <u>If</u> $A, B \subset C$ <u>then</u> $A/(A \cap B) \longrightarrow (A + B)/B$ <u>is an isomorphism</u>.

A sequence $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$ is called a <u>finite filtration</u> (of length n) of the object A. If $0 = B_0 \subset B_1 \subset \cdots \subset B_m = A$ is another it is called a <u>refinement</u> of the first if there is a strictly increasing function α: $\{0,\ldots,n\} \longrightarrow \{0,\ldots,m\}$ such that $\alpha(0) = 0$, $\alpha(n) = m$, and $B_{\alpha(i)} = A_i$ $(0 \leq i \leq n)$. We call the two filtrations <u>J-H-equivalent</u> (J-H = Jordan-Holder) if there is a bijection β: $\{0,\ldots,n\} \longrightarrow \{0,\ldots,m\}$ (so n = m) such that $A_i/A_{i-1} \simeq B_{\beta(i)}/B_{\beta(i)-1}$ $(1 \leq i \leq n)$.

From the first and second isomorphism theorems one can deduce (see any algebra book) Proposition (4.2).

(4.2) PROPOSITION (Zassenhaus Lemma). <u>Any two finite</u>

filtrations of an object A have J-H-equivalent refinements.

An object A is called simple if it has precisely two subobjects (0 and A). A Jordan-Holder series of an object A is a finite filtration $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$ such that A_i/A_{i-1} is simple ($1 \leq i \leq n$). When A possesses one it is said to be of finite length. That its length, n, is well defined follows from the Jordan-Holder theorem below. The latter is a rapid consequence of the Zassenhaus Lemma.

(4.3) THEOREM (Jordan-Holder). *Let* A *be an object of finite length, and let* $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$ *be a filtration such that* $A_i/A_{i-1} \neq 0$ ($1 \leq i \leq n$). *Then this filtration can be refined to a Jordan-Holder series. Moreover any two Jordan-Holder series of* A *are J-H-equivalent.*

We shall close this section now with some basic lemmas on certain types of diagrams.

(4.4) PROPOSITION ("5-Lemma"). *Consider a commutative diagram*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \xrightarrow{a_3} & A_4 & \xrightarrow{a_4} & A_5 \\ \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_2} & & \downarrow{\scriptstyle c_3} & & \downarrow{\scriptstyle c_4} & & \downarrow{\scriptstyle c_5} \\ B_1 & \xrightarrow[b_1]{} & B_2 & \xrightarrow[b_2]{} & B_3 & \xrightarrow[b_3]{} & B_4 & \xrightarrow[b_4]{} & B_5 \end{array}$$

whose rows are exact.

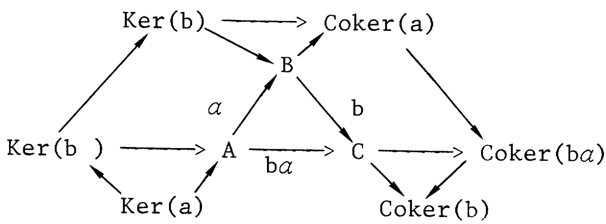
(1) *If* c_1 *is an epimorphism and if* c_2 *and* c_4 *are monomorphisms then* c_3 *is a monomorphism.*

(2) *If* c_5 *is a monomorphism and if* c_2 *and* c_4 *are epimorphisms then* c_3 *is an epimorphism.*

(3) *If* c_i $(i \neq 3)$ *are isomorphisms then so also is* c_3

Proof. Part (3) follows from (1) and (2), and (2) is the dual of one. To prove (1) it suffices, by the embedding theorem, to do so in the category $\mathbb{Z}$-mod, where it can be done by "diagram chasing". We leave the details as an exercise.

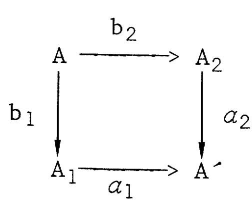
(4.5) PROPOSITION. Given morphisms $A \xrightarrow{a} B \xrightarrow{b} C$ there exist unique morphisms which make the diagram



commute, and the resulting outer perimeter sequence is exact.

Proof. The existence and uniqueness of the commutative diagram is trivial to check. For exactness we can assume it is a diagram in $\mathbb{Z}$-mod. The details are left as an exercise.

(4.6) PROPOSITION. Let



be a Cartesian square in which a_1 is an epimorphism. Then b_2 is also an epimorphism, and the induced morphism $\mathrm{Ker}(b_2) \longrightarrow \mathrm{Ker}(a_1)$ is an isomorphism.

Proof. It suffices to check this in $\mathbb{Z}$-mod, where it is a simple matter. Alternatively, apply (4.5) to the commutative triangle

$$\begin{array}{ccc} & A' \oplus A_2 & \\ \alpha_1 \oplus 1 \nearrow & & \searrow (1, -\alpha_2) \\ A_1 \oplus A_2 & \xrightarrow[(a_1,\ -a_2)]{} & A' \end{array}$$

Since α_1 is an epimorphism so also are all morphisms in the triangle. Therefore, since $A = \text{Ker}(\alpha_1, -\alpha_2)$, we have an exact sequence

$$0 \longrightarrow \text{Ker}(a_1 \oplus 1) \xrightarrow{i} A \xrightarrow{j} \text{Ker}(1, -a_2) \longrightarrow 0$$

The projection $A' \oplus A_2 \longrightarrow A_2$ induces an isomorphism $\text{Ker}(1, -\alpha_2) \longrightarrow A_2$, whose composite with j is b_2. Therefore b_2 is an epimorphism and $\ker(b_2) = i$. But manifestly the projection $A_1 \oplus A_2 \longrightarrow A_1$ induces an isomorphism $\text{Ker}(a_1 \oplus 1) \longrightarrow \text{Ker}(a_1)$. q.e.d.

(4.7) PROPOSITION ("Snake Lemma"). Given a commutative diagram

$$\begin{array}{ccccccc} (0 \longrightarrow)\ A_1' & \xrightarrow{\alpha_1} & A_1 & \xrightarrow{\alpha_1'} & A_1'' & \longrightarrow & 0 \\ \downarrow d' & & \downarrow d & & \downarrow d'' & & \\ 0 \longrightarrow\ A_2' & \xrightarrow{\alpha_2} & A_2 & \xrightarrow{\alpha_2'} & A_2'' & (\longrightarrow & 0) \end{array}$$

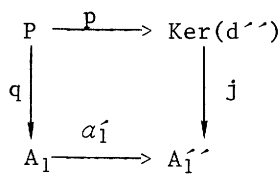
with exact rows, there is a natural morphism ∂ which makes the sequence

$$(0 \longrightarrow) \text{Ker}(d') \longrightarrow \text{Ker}(d) \longrightarrow \text{Ker}(d'') \xrightarrow{\partial} \text{Coker}(d')$$
$$\longrightarrow \text{Coker}(d) \longrightarrow \text{Coker}(d'') (\longrightarrow 0)$$

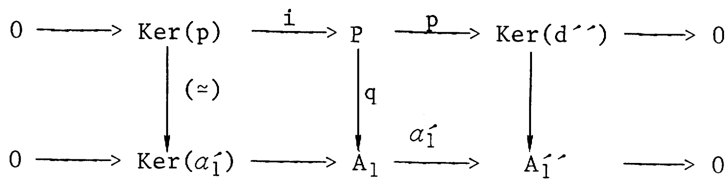
exact. (The data in parentheses are understood to occur in the conclusion if their counterparts are accepted in the hypothesis.)

Proof. We shall prove the existence of ∂. Its naturality with respect to morphisms of diagrams of the above type will be clear from the construction. The proof of exactness, which can be done in the category $\underline{\underline{Z}}$-mod, will be left as an exercise (see Bourbaki [3], §1).

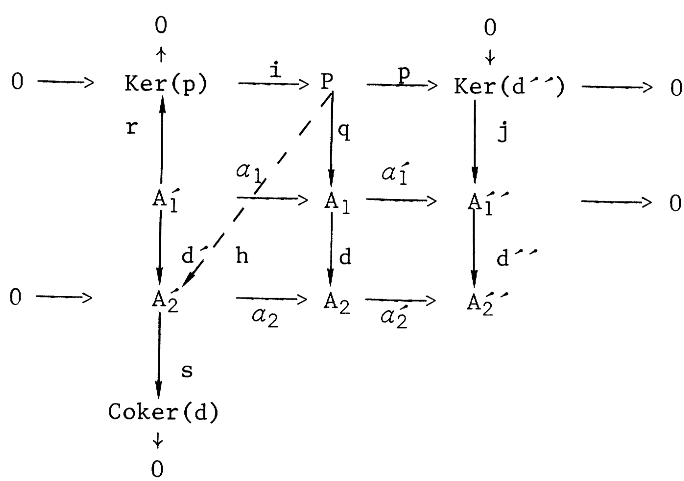
Form the fiber product



Since α_1' is an epimorphism it follows from (4.6) that we have a commutative diagram with exact rows



Since $\ker(\alpha_1') = \mathrm{im}(\alpha_1)$ we obtain an epimorphism $r\colon A_1' \longrightarrow \mathrm{Ker}(p)$ so that the following diagram is commutative:



We shall now construct h so that the diagram remains commutative. Since $\alpha_2' dq = d''\alpha_1' q = d''jp = 0p = 0$ it follows that there is a unique $h\colon P \longrightarrow A_2' = \mathrm{Ker}(\alpha_2')$ such that $dq = \alpha_2 h$.

We shall obtain ∂ as the morphism induced by sh. In order to establish that there is a ∂ which factors sh through $\mathrm{Ker}(d'') = \mathrm{Coker}(i)$ we must check that $shi = 0$. Since r is an epimorphism it suffices to show that $shir = 0$. Since $sd' = 0$ it suffices to show that $hir = d'$. Since α_2 is a monomorphism this will follow once we show that $\alpha_2 hir = \alpha_2 d'$. But $\alpha_2 hir = dqir = d\alpha_1 = \alpha_2 d'$. q.e.d.

REMARK. In a special case there is a much more direct construction of ∂. Namely, suppose there is a morphism $b\colon A_1'' \longrightarrow A_1$ splitting α_1', i.e. such that $\alpha_1' b = 1_{A_1''}$. Then $db\colon A_1'' \longrightarrow A_2$ induces $\mathrm{Ker}(d'') \longrightarrow A_2$, and it is easily seen that this induces ∂.

§5. COMPLEXES, HOMOLOGY, MAPPING CONE

We fix an Abelian category $\underline{\underline{A}}$. A graded object of $\underline{\underline{A}}$ is a sequence $C = (C_n)_{n \in \underline{\underline{Z}}}$ of objects $C_n \in \underline{\underline{A}}$. A sequence $\alpha = (\alpha_n)$ of morphisms $\alpha_n\colon C_n' \longrightarrow C_n$ is a morphism $C' \longrightarrow C$ of graded objects. We define the graded object $C(h)$ by $C(h)_n = C_{n+h}$. A morphism $C' \longrightarrow C(h)$ is sometimes called a "morphism of degree h" from C' to C. When $C_n = 0$ for $n < 0$, C is called positive. If C(h) is positive for some h then C is said to be bounded below. It is called finite if $C_n = 0$ for all but finitely many n.

A complex in $\underline{\underline{A}}$ consists of a graded object C together with a morphism $d\colon C \longrightarrow C$ of degree -1 such that $d^2 = 0$. More explicitly, $d = (d_n)$, where $d_n\colon C_n \longrightarrow C_{n-1}$ and $d_{n-1}d_n = 0$ for each n. A morphism $(C', d') \longrightarrow (C, d)$ of complexes is a morphism $\alpha\colon C' \longrightarrow C$, of graded objects, such that $\alpha d' = d\alpha$. We shall often suppress d when denoting a complex (C, d). For example, C(h) will denote the complex $(C(h), (-1)^h d)$.

Associated with a complex C are three graded objects:

$$B = B(C) : \quad B_n = \operatorname{Im}(C_{n+1} \xrightarrow{d_{n+1}} C_n) \quad (\text{"boundaries"})$$

$$Z = Z(C) : \quad Z_n = \operatorname{Ker}(C_n \xrightarrow{d_n} C_{n-1}) \quad (\text{"cycles"})$$

$$H = H(C) : \quad H_n = Z_n/B_n \quad (\text{"homology"})$$

When $H(C) = 0$, i.e., when the sequence

$$\cdots \longrightarrow C_{n+1} \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots$$

is exact, the complex C is said to be <u>acyclic</u>. A morphism $\alpha: C' \longrightarrow C$ of complexes evidently induces morphisms $Z(C') \longrightarrow Z(C)$ and $B(C') \longrightarrow B(C)$, and therefore also a homology morphism $H(a) : H(C') \longrightarrow H(C)$. Two morphisms $\alpha, b : C' \longrightarrow C$ are called <u>homotopic</u>, denoted $\alpha \simeq b$, if there is a morphism $s : C' \longrightarrow C(1)$ of graded objects such that $\alpha - b = ds + sd'$. When evaluating $H(a - b)$, the term sd' restricts to zero on $Z(C')$, and the image of ds is in $B(C)$; thus $H(\alpha - b) = 0$, i.e., $H(\alpha) = H(b)$, if $\alpha \simeq b$. A complex C is said to be <u>contractible</u> if $1_C \simeq 0$, i.e., if $1_C = ds + sd$ for some $s : C \longrightarrow C(1)$ as above. In this case we have $0 = H(0) = H(1_C) = 1_{H(C)}$, so <u>a contractible complex is acyclic</u>.

(5.1) PROPOSITION ("The Long Homology Sequence"). <u>Let $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ be an exact sequence of complexes. Then there is a morphism $\partial : H(C'') \longrightarrow H(C')$ of degree -1 which is natural with respect to morphisms of exact sequences as above, and such that the sequence</u>

$$\cdots \longrightarrow H_n(C') \longrightarrow H_n(C) \longrightarrow H_n(C'') \xrightarrow{\partial} H_n(C') \longrightarrow H_n(C) \longrightarrow \cdots$$

<u>is exact</u>.

<u>Proof</u>. Let E denote the given short exact sequence.

Then $(d', d, d'') : E \longrightarrow E(-1)$ is a morphism of short exact sequences. Taking its kernel and cokernel, respectively, we deduce from the Snake lemma that the rows of

$$\begin{array}{ccccccc} & & C'/B(C') & \longrightarrow & C/B(C) & \longrightarrow & C''/B(C'') \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z(C') & \longrightarrow & Z(C) & \longrightarrow & Z(C'') \end{array}$$

are exact. The vertical maps here are those induced by (d', d, d''). Again applying the Snake lemma, we deduce an exact sequence of kernels and cokernels,

$$H(C') \longrightarrow H(C) \longrightarrow H(C'') \xrightarrow{\partial} H(C') \longrightarrow H(C) \longrightarrow H(C'')$$

where ∂ has degree -1. q.e.d.

(5.2) COROLLARY. <u>If two of the complexes</u> C', C, C'' <u>above are acyclic then so also is the third</u>.

(5.3) COROLLARY (9-Lemma). <u>Let</u>

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_2' & \longrightarrow & C_2 & \longrightarrow & C_2'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_1' & \longrightarrow & C_1 & \longrightarrow & C_1'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_0' & \longrightarrow & C_0 & \longrightarrow & C_0'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

<u>be a commutative diagram with exact rows, and assume the composite</u> $C_2 \longrightarrow C_1 \longrightarrow C_0$ <u>is zero. Then if two of the columns are exact so also is the third</u>.

<u>Proof</u>. The hypotheses permit us to view the columns as complexes, zero except in degrees 0, 1, 2, so we can apply the last corollary.

The <u>mapping cone</u> of a morphism $a\colon C' \longrightarrow C$ of complexes is a complex, $MC(a)$, defined as follows:

$$MC(a)_n = C_n \oplus C'_{n-1}$$

$$d(\alpha)_{n+1} = \begin{pmatrix} d_{n+1} & a_n \\ 0 & -d'_n \end{pmatrix} : C_{n+1} \oplus C'_n \longrightarrow C_n \oplus C_{n-1}$$

Since $d^2 = 0$, $d'^2 = 0$, and $\alpha d' = d\alpha$ it follows that $d(\alpha)^2 = 0$. Moreover the direct sum decomposition of MC(a) yields an exact sequence of complexes,

$$(1) \qquad 0 \longrightarrow C \longrightarrow MC(\alpha) \longrightarrow C'(-1) \longrightarrow 0$$

Since $H_n(C'(-1)) = H_{n-1}(C')$ we can write the long homology sequence of (1) in the form

$$\cdots H_n(C) \longrightarrow H_n(MC(a)) \longrightarrow H_{n-1}(C') \xrightarrow{\partial} H_{n-1}(C) \longrightarrow H_{n-1}(MC(a)) \cdots$$

To compute ∂ we consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n & \longrightarrow & MC(a)_n & \longrightarrow & C'_{n-1} & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d(\alpha) & & \downarrow d' & & \\ 0 & \longrightarrow & C_{n-1} & \longrightarrow & MC(\alpha)_{n-1} & \longrightarrow & C'_{n-2} & \longrightarrow & 0 \end{array}$$

Since the rows are split ∂ is induced by the composite,

$$C'_{n-1} \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} MC(\alpha)_n = C_n \oplus C'_{n-1} \xrightarrow{d(\alpha)_n = \begin{pmatrix} d & \alpha \\ 0 & -d' \end{pmatrix}} C_{n-1}$$

(cf. the remark after the Snake lemma (4.7)). Since $\begin{pmatrix} d & \alpha \\ 0 & -d' \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ -d' \end{pmatrix}$ its restriction to $Z_{n-1}(C')$ induces $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$, and hence ∂ is induced by α: $Z_{n-1}(C') \longrightarrow C_{n-1}$. In other words, $\partial = H(\alpha)$. We have now proved:

(5.4) PROPOSITION. _Let_ α: $C' \longrightarrow C$ _be a morphism of complexes, and let_

$$0 \longrightarrow C \longrightarrow MC(\alpha) \longrightarrow C'(-1) \longrightarrow 0$$

be the exact sequence (1) above. Then its long homology sequence is isomorphic to

$$\cdots H_n(C) \longrightarrow H_n(MC(a)) \longrightarrow H_{n-1}(C') \xrightarrow{H_{n-1}(a)} H_{n-1}(C) \longrightarrow H_{n-1}(MC(a)) \ldots$$

Hence $MC(a)$ is acyclic if and only if $H(\alpha)$ is an isomorphism.

§6. RESOLUTIONS: PROJECTIVE DIMENSION

We shall work in a fixed Abelian category $\underline{\underline{A}}$. A resolution of $A \in \underline{\underline{A}}$ could be defined to be an exact sequence

$$\cdots C_n \longrightarrow \cdots \longrightarrow C_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

For technical purposes it is convenient, instead, to interpret these data as follows: view the sequence down to C_0 as a positive complex, C, identify A with the complex having only one nonzero term, A, concentrated in degree zero, and view $\varepsilon: C \longrightarrow A$ as a morphism of complexes inducing an isomorphism $H(\varepsilon): H(C) \longrightarrow H(A) = A$. We shall often use this to identify $H(C)$ with A. The length of the resolution is the least $n \geq -1$ such that $C_m = 0$ for all $m > n$. If $\underline{\underline{C}}$ is a full subcategory of $\underline{\underline{A}}$ we say that $C \xrightarrow{\varepsilon} A$ is a $\underline{\underline{C}}$-resolution of A if all $C_n \in \underline{\underline{C}}$, and we define the subcategories

$$\mathrm{Res}_\infty(\underline{\underline{C}}) \supset \mathrm{Res}(\underline{\underline{C}}) \supset \mathrm{Res}_n(\underline{\underline{C}}) \qquad (n \geq 0)$$

to be the full subcategory of objects having $\underline{\underline{C}}$-resolutions, finite $\underline{\underline{C}}$-resolutions, and $\underline{\underline{C}}$-resolutions of length $\leq n$, respectively. Thus we have $\mathrm{Res}_0(\underline{\underline{C}}) = \underline{\underline{C}}$, $\mathrm{Res}_n(\underline{\underline{C}}) \subset \mathrm{Res}_{n+1}(\underline{\underline{C}})$, and $\mathrm{Res}(\underline{\underline{C}}) = \bigcup_{\infty > n \geq 0} \mathrm{Res}_n(\underline{\underline{C}})$.

To construct resolutions and morphisms between them we shall use the following condition on a subcategory $\underline{\underline{C}}$:

(1) If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact and if $A, A'' \in \mathbf{C}$, then $A' \in \mathbf{C}$.

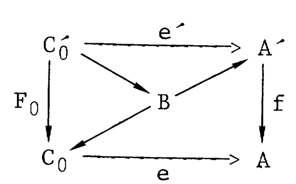
It follows easily by induction from (1) that:

(1′) If $0 \longrightarrow A_n \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0$ is exact with $A_i \in \mathbf{C}$ $(0 \le i < n)$ then $A_n \in \mathbf{C}$.

(6.1) PROPOSITION. *Let* $\mathbf{C}_0 \subset \mathbf{C}$ *be full additive subcategories of* $\mathbf{A}$ *satisfying (1) above, and suppose each object of* $\mathbf{C}$ *is a quotient of one in* $\mathbf{C}_0$. *Then* $\mathbf{C} \subset \mathrm{Res}_\infty(\mathbf{C}_0)$. *Moreover, if* $f\colon A' \longrightarrow A$ *is a morphism in* $\mathbf{C}$ *and if* $\varepsilon\colon C \longrightarrow A$ *is a* $\mathbf{C}_0$*-resolution, then there is a* $\mathbf{C}_0$*-resolution* $\varepsilon'\colon C' \longrightarrow A'$ *and a morphism* $F\colon C' \longrightarrow C$ *covering* f, *i.e. such that* $H(F) = f$. *If* C *is finite, and if* $\mathbf{C} \subset \mathrm{Res}(\mathbf{C}_0)$, *then we can choose* C' *to be finite also.*

Proof. If $A \in \mathbf{C}$ we can find an exact sequence $0 \longrightarrow B \longrightarrow C_0 \longrightarrow A \longrightarrow 0$ with $C_0 \in \mathbf{C}_0$, and then (1) implies $B \in \mathbf{C}$. Hence we can continue with B, etc., and construct a $\mathbf{C}_0$-resolution of A.

Suppose next that we are given $f\colon A' \longrightarrow A$ and a $\mathbf{C}_0$-resolution $e\colon C \longrightarrow A$ as above. Let B, $=$ $\mathrm{Ker}(C_0 \oplus A' \xrightarrow{(e,\,-f)} A)$, be the fibre product of $C_0 \xrightarrow{e} A \xleftarrow{f} A'$. Since e is an epimorphism $(e, -f)$ is also, and hence (1) implies $B \in \mathbf{C}$. Therefore we can find an epimorphism $C_0' \longrightarrow B$ with $C_0' \in \mathbf{C}_0$. We now define F_0 and e' by the commutative diagram



Since e is surjective, $B \longrightarrow A'$ is also (see (4.6)) so e' is also. Suppose now, by induction, that we have constructed a commutative diagram

$$\begin{array}{ccccccccc}
 & & C'_{n-1} & \xrightarrow{d'_{n-1}} \cdots \longrightarrow & C'_0 & \xrightarrow{e'} & A' & \longrightarrow & 0 \\
 & & {\scriptstyle F_{n-1}}\downarrow & & {\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & C_{n-1} & \xrightarrow{d_{n-1}} \cdots \longrightarrow & C_0 & \xrightarrow{e} & A & \longrightarrow & 0
\end{array}$$

with exact rows and with each $C'_i \in \underline{\underline{C}}_0$. It follows from (1´) above that $Z'_{n-1} = \operatorname{Ker}(d'_{n-1})$ and $Z_{n-1} = \operatorname{Ker}(d_{n-1})$ are in $\underline{\underline{C}}_0$. Hence we can apply the construction above to find a commutative diagram

$$\begin{array}{ccccc}
C'_n & \xrightarrow{d'} & Z'_{n-1} & \longrightarrow & 0 \\
{\scriptstyle F_n}\downarrow & & \downarrow{\scriptstyle F'} & & \\
C_n & \xrightarrow[d]{} & Z_{n-1} & \longrightarrow & 0
\end{array}$$

with exact rows, where F', and d are induced by F_{n-1}, and d_n, respectively. With d_n equal to the composite

$C'_n \xrightarrow{d'} Z'_{n-1} \subset C_{n-1}$ we have extended the resolution

C' and $F\colon\ C' \longrightarrow C$ one more step. In case C is finite and $\underline{\underline{C}} \subset \operatorname{Res}(\underline{\underline{C}}_0)$ then, when we reach an n such that $C_m = 0$ for all $m \geq n$, we can complete C' with a finite $\underline{\underline{C}}_0$-resolution of Z'_{n-1}.

<u>Exercise</u>. Show that if f is an epimorphism then the F constructed above is also an epimorphism.

(6.2) PROPOSITION. *Let $\underline{\underline{C}}_0$ be a full additive subcategory of $\underline{\underline{A}}$ satisfy* (1). *Let*

$$\begin{array}{ccc}
C' & \xrightarrow{F} & C \\
{\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle e} \\
A' & \xrightarrow{f} & A
\end{array}$$

a commutative diagram in which the verticals are $\mathbf{C}_0$*-resolutions, of lengths* d′ *and* d, *of* A′ *and* A, *respectively. If* f *is a monomorphism then* MC(F) *is a* $\mathbf{C}_0$*-resolution of* Coker(f) *of length* sup(d, d′ + 1). *If* f *is an epimorphism then* Ker(f) *has a* $\mathbf{C}_0$*-resolution of length* sup(d-1, d′).

Proof. We have the exact sequence of complexes

$$(*) \qquad 0 \longrightarrow C \longrightarrow MC(F) \longrightarrow C'(-1) \longrightarrow 0$$

(see (5.4)) which splits as a sequence of graded objects. The nonzero terms of C occur in degrees 0 to d and those of C′(-1) in degrees 1 to d′ + 1. Hence MC(F) is a positive complex in $\mathbf{C}_0$ of length sup(d, d′ + 1). Since C and C′ are resolutions they have homology only in degree zero, and there the homology sequence of (*) becomes

$$\begin{array}{ccccccccc} \cdots 0 \longrightarrow H_1(MC(F)) \longrightarrow & H_0(C') & \longrightarrow & H_0(C) & \longrightarrow H_0(MC(F)) \longrightarrow 0 \cdots \\ & \| & & \| & \\ & A' & \xrightarrow{\ f\ } & A & \end{array}$$

Thus, if f is a monomorphism, the only homology of MC(F) is $H_0(MC(F)) \simeq \mathrm{Coker}(f)$, so MC(F) is a resolution of Coker(f).

If f is an epimorphism its only homology is $H_1(MC(F)) \simeq \mathrm{Ker}(f)$. It follows that we have exact sequences

$$0 \longrightarrow Z_1(MC(F)) \longrightarrow MC(F)_1 \longrightarrow MC(F)_0 \longrightarrow 0$$

and

$$MC(F)_2 \longrightarrow Z_1(MC(F)) \longrightarrow \mathrm{Ker}(f) \longrightarrow 0$$

extracted from MC(F). The first shows that $Z_1(MC(F)) \in \mathbf{C}_0$. The second and the vanishing of the higher homology of MC(F) shows that

$$\cdots MC(F)_n \longrightarrow \cdots \longrightarrow MC(F)_2 \longrightarrow Z_1(MC(F)) \longrightarrow \mathrm{Ker}(f) \longrightarrow 0$$

defines a resolution of Ker(f) whose length is sup(d, d′+ 1) -1 = sup(d-1, d′). q.e.d.

We shall now record some of the special features of *projective resolutions*.

(6.3) PROPOSITION. (Schanuel's Lemma). Let $0 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ and $0 \longrightarrow P_1' \longrightarrow P_0' \xrightarrow{\varepsilon'} A \longrightarrow 0$ be exact sequences in $\underline{\underline{A}}$ with P_0 and P_0' projective. Then $P_0 \oplus P_1' \simeq P_0' \oplus P_1$.

Proof. The fiber product of $(\varepsilon, \varepsilon')$ yields a commutative diagram (see (4.6))

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P_1' & = & P_1' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & Q & \longrightarrow & P_0' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with exact rows and columns. Since P_0 is projective the epimorphism $Q \longrightarrow P_0$ splits, so $Q \simeq P_0 \oplus P_1'$. Similarly, since P_0' is projective, $Q \simeq P_0' \oplus P_1$. q.e.d

(6.4) COROLLARY. If $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ and $0 \longrightarrow P_n' \longrightarrow \cdots \longrightarrow P_0' \xrightarrow{\varepsilon'} A \longrightarrow 0$ are exact sequences in $\underline{\underline{A}}$ with P_i projective $(0 \leq i < n)$ then

$$P_0 \oplus P_1' \oplus P_2 \oplus \cdots \simeq P_0' \oplus P_1 \oplus P_2' \oplus \cdots$$

Proof. The case $n = 1$ is (6.3). For $n > 1$ we consider the exact sequences $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \oplus P_0' \xrightarrow{d \oplus 1} Z_0 \oplus P_0' \longrightarrow 0$ and $0 \longrightarrow P_n' \longrightarrow \cdots \longrightarrow P_2' \longrightarrow P_1' \oplus P_0 \xrightarrow{d' \oplus 1} Z_0' \oplus P_0 \longrightarrow 0$, where $Z_0 = \mathrm{Ker}(\varepsilon)$, $Z_0' = \mathrm{Ker}(\varepsilon')$, and where d and d' are induced by the original sequences. According to (6.3) we have $Z_0 \oplus P_0' \simeq Z_0' \oplus P_0$ and hence the corollary follows by induction.

Let $\underline{\underline{P}}$ be a full additive subcategory of $\underline{\underline{A}}$ which satisfies (1) and all of whose objects are projective (in $\underline{\underline{A}}$). In this case condition (1) reduces to the apparently weaker condition: $P \in \underline{\underline{P}}$ and $P \oplus Q \in \underline{\underline{P}} \Rightarrow Q \in \underline{\underline{P}}$. If $A \in \underline{\underline{A}}$ has a $\underline{\underline{P}}$-resolution then we define the $\underline{\underline{P}}$-dimension of A to be the minimal length (possibly infinite) of a $\underline{\underline{P}}$-resolution of A; it will be denoted $\underline{\underline{P}}d(A)$. Thus $\underline{\underline{P}}d(A) = -1 \Leftrightarrow A = 0$ and $\underline{\underline{P}}d(A) \leq 0 \Leftrightarrow A \in \underline{\underline{P}}$. It follows immediately from the last corollary that, if $\underline{\underline{P}}d(A) \leq n$, so there is a $\underline{\underline{P}}$-resolution $P \longrightarrow A$ of length $\leq n$, then for any $\underline{\underline{P}}$-resolution $P' \longrightarrow A$ we have $Z_m(p') \in \underline{\underline{P}}$ for all $m \geq n-1$. This implies that no matter how we start off a $\underline{\underline{P}}$-resolution of A, we can terminate it at the nth term, $\mathrm{Ker}(P'_{n-1} \longrightarrow P'_{n-2})$, and be assured (by (6.4)) that the latter is in $\underline{\underline{P}}$.

(6.5) PROPOSITION. *Let $P \xrightarrow{\varepsilon} A$ and $P' \xrightarrow{\varepsilon'} A'$ be projective resolutions and let $f\colon A' \longrightarrow A$. Then there is a morphism $F\colon P' \longrightarrow P$ covering f, and any two such F's are homotopic.*

Proof. Since P_0' is projective and ε is an epimorphism we can find F_0 to make

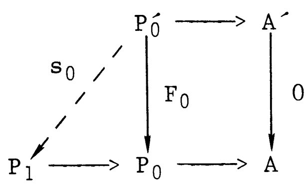
$$\begin{array}{ccc} P_0' & \xrightarrow{\varepsilon'} & A' \\ \Big\downarrow F_0 & & \Big\downarrow f \\ P_0 & \xrightarrow{\varepsilon} & A \end{array}$$

commute. This F_0 induces $f_0\colon Z_0(P') \longrightarrow Z_0(P)$, and since $P_1 \longrightarrow Z_0(P)$ is an epimorphism we can find F_1 making

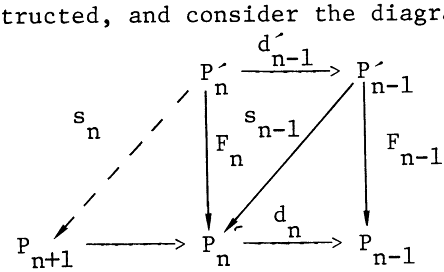
$$\begin{array}{ccc} P_1' & \longrightarrow & Z_0(P') \\ F_1 \Big\downarrow & & \Big\downarrow f_0 \\ P_1 & \longrightarrow & Z_0(P) \end{array}$$

commute. Etc...

If $G: P' \longrightarrow P$ also covers f then $F - G$ covers the morphism $0 = f - f: A' \longrightarrow A$. Therefore the last assertion follows if we show that F is homotopic to zero when $f = 0$. We need $s_n: P'_n \longrightarrow P_{n+1}$ $(n \geq 0)$ so that $F = ds + sd'$. For $n = 0$ this reads $F_0 = d_1 s_0$, since $d'_0 = 0$. Since P'_0 is projective this follows from the commutivity of



and the exactness of the bottom row. Suppose s_i $(i \leq n)$ have been constructed, and consider the diagram



The bottom row is exact and P'_n is projective, so we can solve $d_{n+1}s_n = F_n - s_{n-1}d'_{n-1}$ provided we verify that $d_n(F_n - s_{n-1}d'_{n-1}) = 0$. But $d_n s_{n-1} = F_{n-1} - s_{n-2}d'_{n-2}$, so $d_n(F_n - s_{n-1}d'_{n-1}) = d_nF_n - (F_{n-1} - s_{n-2}d'_{n-2})d'_{n-1} = d_nF_n - F_{n-1}d'_{n-1} = 0$, because $d'^2 = 0$ and $dF = Fd'$.

Remark. The proof uses only the facts that P' is projective and that the complex p is acyclic in degrees >0.

(6.6) COROLLARY. An acyclic projective complex which is bounded below is contractible.

Proof. After shifting its degrees we can view such a complex, P, as a projective resolution of 0, whereupon both 0 and 1_P cover the morphism 1_0. Hence 0 and 1_P are homotopic, i.e., P is contractible.

(6.7) PROPOSITION. *Let* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *be an exact sequence in* $\underline{\underline{A}}$. *Let* $P' \longrightarrow A'$ *and* $P'' \longrightarrow A''$ *be projective resolutions. Then there exists a differential on the graded object* $P = P' \oplus P''$ *so that the split exact sequence* $0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$ *is an exact sequence of complexes resolving* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$.

Proof. We begin by constructing $\varepsilon = (\varepsilon', h)$ so that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P'_0 & \longrightarrow & P'_0 \oplus P''_0 & \longrightarrow & P''_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon'} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon''} & & \\
0 & \longrightarrow & A' & \xrightarrow{a} & A & \xrightarrow{b} & A'' & \longrightarrow & 0
\end{array}$$

commutes, i.e., so that $bh = \varepsilon''$. This h exists because P'' is projective and b is an epimorphism.

The Snake lemma (4.7) implies ε is an epimorphisms and that $0 \longrightarrow \mathrm{Ker}(\varepsilon') \longrightarrow \mathrm{Ker}(\varepsilon) \longrightarrow \mathrm{Ker}(\varepsilon'') \longrightarrow 0$ is exact. We now repeat this construction, starting with the epimorphisms $P'_1 \longrightarrow \mathrm{Ker}(\varepsilon')$ and $P''_1 \longrightarrow \mathrm{Ker}(\varepsilon'')$, etc.

(6.8) PROPOSITION. *Let* $\underline{\underline{P}}$ *be a full additive subcategory of projective objects in* $\underline{\underline{A}}$, *satisfying (1) above, and let* $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ *be an exact sequence in* $\underline{\underline{A}}$. *If two of* A', A, A'' *have* $\underline{\underline{P}}$ *-resolutions so does the third. Suppose this is the case, and write* d', d, *and* d'' *for their respective* $\underline{\underline{P}}$ *-dimensions. Then we have*

$$d' \leq \sup(d, d'' - 1) \quad \textit{and} \quad d'' \leq \sup(d' + 1, d)$$

Moreover, $d \leq \sup(d', d'')$, *and if this inequality is strict* *we have* $d'' = d' + 1$.

Proof. Say e': $P' \longrightarrow A'$ and e: $P \longrightarrow A$ are $\underline{\underline{P}}$ -resolutions of lengths d' and d, respectively. By (6.5) we can cover f with F: $P' \longrightarrow P$ and then (6.2) says that $MC(F)$ is a $\underline{\underline{P}}$ -resolution of $A'' = \mathrm{Coker}(f)$ of length $\leq \sup(d' + 1, d)$.

If e'': $P'' \longrightarrow A''$ is a $\underline{\underline{P}}$ -resolution of length d'' we use (6.5) to cover g with G: $P \longrightarrow P''$ and then use (6.2) to obtain a $\underline{\underline{P}}$ -resolution of $A' = \mathrm{Ker}(g)$ of length $\leq \sup(d, d'' - 1)$.

On the other hand, we can use (6.7) to obtain from P' and P'' a $\underline{\underline{P}}$ -resolution $P \longrightarrow A$ of length $\leq \sup(d', d'')$. This proves all but the final assertion.

Suppose $d < \sup(d', d'')$. If $d < d'$ then we have $d' \leq d'' - 1$ and $d'' \leq \sup(d' + 1, d) = d' + 1$; hence $d'' = d' + 1$. If $d < d''$ then $d' \leq \sup(d, d'' - 1) = d'' - 1$ and $d'' \leq d' + 1$; hence again $d'' = d' + 1$. q.e.d.

(6.9) COROLLARY. *Let* $\underline{\underline{P}}$ *be as in (6.8). Then* $\mathrm{Res}_\infty(\underline{\underline{P}})$, $\mathrm{Res}(\underline{\underline{P}})$, *and* $\mathrm{Res}_n(\underline{\underline{P}})$ $(n \geq 0)$ *are all full additive subcategories of* $\underline{\underline{A}}$ *satisfying (1) above. If all but one of the terms of a finite exact sequence*

$$0 \longrightarrow A_n \longrightarrow \cdots \longrightarrow A_0 \longrightarrow 0$$

lie in $\mathrm{Res}_\infty(\underline{\underline{P}})$ *or* $\mathrm{Res}(\underline{\underline{P}})$ *then so also does the remaining term*.

§7. ADJOINT FUNCTORS

Given two functors

$$\underline{\underline{A}} \underset{S}{\overset{T}{\rightleftarrows}} \underline{\underline{B}}$$

and a natural isomorphism

$$(1) \qquad \gamma = \gamma_{A,\ B}\colon\ \mathbf{A}(A,\ SB) \longrightarrow \mathbf{B}(TA,\ B)$$

of functors $\mathbf{A}^0 \times \mathbf{B} \longrightarrow$ Sets, we say that S is an *adjoint* of T, and that T is a *coadjoint* of S. It is not difficult to see that either functor determines (via (1)) the other up to a unique isomorphism. We shall call (T, S) an *adjoint pair*.

This situation arises frequently in nature. For example, the "forgetful" functor from groups to sets has as coadjoint the free group functor. Similarly, the forgetful functor from k-algebras to k-modules (k is a commutative ring) has the tensor algebra functor as coadjoint.

(7.1) PROPOSITION. Let (T, S) be an adjoint pair of functors as above.

(1) S *preserves products, limits, final objects, kernels*,...

(2) T *preserves coproducts, colimits, initial objects, cokernels*,...

Proof. (2) is the dual of (1) and (1) follows immediately from the definitions and the natural identification $\mathbf{A}(A,\ SB) = \mathbf{B}(TA,\ B)$. We shall illustrate the latter by showing that S preserves limits (of which products are a special case, incidentally). Suppose $B = \lim F$ for some functor $F\colon\ \mathbf{L} \longrightarrow \mathbf{B}$. Then we claim that $SB = \lim SF$. We must show that they represent the same functor $\mathbf{A}^0 \longrightarrow$ Sets. By definition $\mathbf{A}(A,\ \lim SF) = \mathbf{A}^{\mathbf{L}}(c(A),\ SF)$, where $c\colon\ \mathbf{L} \longrightarrow \mathbf{A}$ is the constant functor with value A, and the adjointness identity implies $\mathbf{A}^{\mathbf{L}}(c(A),\ SF) = \mathbf{B}^{\mathbf{L}}(c(TA),\ F) = \mathbf{B}(TA,\ \lim F)$ $= \mathbf{A}(A,\ S \lim F)$.

(7.2) COROLLARY. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *above are additive categories then* S *and* T *are additive functors*.

Proof. It follows from (7.1) that both functors preserve zero objects and direct sums, and these two properties imply that a functor is additive.

Let (T, S) be an adjoint pair with isomorphism γ as in (1) above. Then for $A \varepsilon \underline{\underline{A}}$ and $B \varepsilon \underline{\underline{B}}$ we have

$$\alpha_A = \gamma_{A,\ TA}^{-1}(1_{TA}) \ : \ A \longrightarrow STA$$

and

$$\beta_B = \gamma_{SB,\ B}(1_{SB}) \ : \ TSB \longrightarrow B$$

Given morphisms a: $A' \longrightarrow A$ in $\underline{\underline{A}}$ and b: $B \longrightarrow B'$ in $\underline{\underline{B}}$, the square

$$\begin{array}{ccc}
\underline{\underline{A}}(A,\ SB) & \xrightarrow{\gamma_{A,\ B}} & \underline{\underline{B}}(TA,\ B) \\
{\scriptstyle \underline{\underline{A}}(a,\ Sb)} \downarrow & & \downarrow {\scriptstyle \underline{\underline{B}}(Ta,\ b)} \\
\underline{\underline{A}}(A',\ SB') & \xrightarrow[\gamma_{A',\ B'}]{} & \underline{\underline{B}}(TA',\ B')
\end{array}$$

commutes, because γ is natural. Thus

$$b\ \gamma_{A,B}(c)(Ta) = \gamma_{A',\ B'}((Sb)ca) \qquad (c \ \varepsilon\ \underline{\underline{A}}(A,\ SB))$$

If we apply this to $a = \alpha_A$: $A \longrightarrow STA$, $b = 1_{TA}$, and $c = 1_{STA}$, then we obtain

$$b\gamma_{STA,\ TA}(c)(Ta) = 1_{TA}\ \beta_{TA}\ T\alpha_A = \beta_{TA}\ T\alpha_A,$$

and

$$\gamma_{A,\ TA}((Sb)ca) = \gamma_{A,\ TA}(\alpha_A) = 1_{TA}$$

Thus the composite

$$TA \xrightarrow{T\alpha_A} TSTA \xrightarrow{\beta_{TA}} TA$$

is the identity on TA. Similarly it follows that the composite

$$SB \xrightarrow{\alpha_{SB}} STSB \xrightarrow{S\beta_B} SB$$

is the identity on SB for $B \in \underline{\underline{B}}$.

(7.3) PROPOSITION. _Let_ (T, S) _be an adjoint pair of functors between additive categories. If_ $A \in \underline{\underline{A}}$ _is such that_ α_A: $A \longrightarrow STA$ _is an isomorphism, then_ β_B: $TSB \longrightarrow B$ _is an isomorphism for every direct summand_, B, _of_ TA.

Proof. Since β: $TS \longrightarrow 1_{\underline{\underline{B}}}$ is a natural transformation between additive functors it will suffice to show that β_{TA} is an isomorphism. But it follows from the discussion above that $\beta_{TA}\, T\alpha_A = 1_{TA}$, and our hypothesis on A implies $T\alpha_A$ is an isomorphism; hence $\beta_{TA} = T\alpha_A^{-1}$ is one also.

(7.4) COROLLARY. _Let_ (T, S) _be an adjoint pair of functors between additive categories such that the natural transformation_ α: $1_{\underline{\underline{A}}} \longrightarrow ST$ _is an isomorphism. Suppose further that every object of_ $\underline{\underline{B}}$ _is isomorphic to a direct summand of_ TA _for some_ $A \in \underline{\underline{A}}$. _Then_ β: $TS \longrightarrow 1_{\underline{\underline{B}}}$ _is also an isomorphism, so_ S _and_ T _are inverse equivalences of categories_.

§8. DIRECT LIMITS

Let G: $\underline{\underline{C}} \longrightarrow \underline{\underline{A}}$ be a functor. Then the colimit, $\underset{\rightarrow}{G}$ = colim G, is defined by

$$\underline{\underline{A}}(\underset{\rightarrow}{G}, A) = \underline{\underline{A}}^{\underline{\underline{C}}}(G, c(A)) \qquad (A \in \underline{\underline{A}})$$

Thus, if colimits always exist then colim: $\underline{\underline{A}}^{\underline{\underline{C}}} \longrightarrow \underline{\underline{A}}$ is a functor, and it is just the coadjoint of c: $\underline{\underline{A}} \longrightarrow \underline{\underline{A}}^{\underline{\underline{C}}}$.

In this section we shall take $\underline{\underline{A}}$ to be a category of "sets with structure" and also impose certain conditions on $\underline{\underline{C}}$. The properties of colimits which we then deduce will be applied in Chapter VII.

(8.1) DEFINITION. A category $\underline{\underline{C}}$ is said to be directed if it satisfies (a) and (b) below:

(a) Given A_1, $A_2 \in \underline{\underline{C}}$, there is a $B \in \underline{\underline{C}}$ and morphisms f_i: $A_i \longrightarrow B$ $(i = 1, 2)$

(b) Given f_i: $A \longrightarrow B$ $(i = 1, 2)$ in $\underline{\underline{C}}$ there exists a g: $B \longrightarrow C$ in $\underline{\underline{C}}$ such that $gf_1 = gf_2$.

Simple induction arguments show that (a) implies (a´): Any finite collection of objects map into a common object. Moreover (b) implies (b´): Given $f_{i_1}, f_{i_2}: A_i \longrightarrow B$ $(1 \leq i \leq n)$ there is a g: $B \longrightarrow C$ such that $gf_{i_1} = gf_{i_2}$ $(1 \leq i \leq n)$.

A colimit of a functor from a directed category will be called a direct (or inductive) limit.

For the rest of this section $\underline{\underline{A}}$ will denote one of the following categories: groups, rings, modules over a ring, sets,... The conclusions apply to any such category of "sets with structure". In the proofs we shall give details only for the category of groups. Similar arguments apply to the other cases.

(8.2) PROPOSITION. Let $\underline{\underline{C}}$ be a category with only a set of isomorphism classes, and let G: $\underline{\underline{C}} \longrightarrow \underline{\underline{A}}$ be a functor. Then $\underrightarrow{G}$ exists, and it is generated by $\{Im(\gamma_A)\}$, where γ_A: $G(A) \longrightarrow \underrightarrow{G}$ $(A \in \underline{\underline{C}})$ are the canonical morphisms. Condition (8.1)(a) on $\underline{\underline{C}}$ implies that

$$\underrightarrow{G} = \bigcup Im(\gamma_A) \qquad (A \in \underline{\underline{C}})$$

Suppose $\underline{\underline{C}}$ is directed. Let $A_0 \in \underline{\underline{C}}$ and let a, b $\in$ $G(A_0)$ be such that $\gamma_{A_0}(a) = \gamma_{A_0}(b)$. Then there is a morphism f: $A_0 \longrightarrow B$ in $\underline{\underline{C}}$ such that $G(f)(a) = G(f)(b)$.

Proof. There is a full subcategory of $\underline{\underline{C}}$, which is a set and such that every object is isomorphic to one in the subcategory. The inclusion functor is then an equivalence (see (1.1)) so we can assume $\underline{\underline{C}}$ itself is a set.

Now $\underset{\rightarrow}{G}$ can be constructed from $S = \coprod GA(A \in \underline{\underline{C}})$ and the canonical morphisms $\gamma'_A : GA \longrightarrow S$. Namely, we pass to the largest quotient $p: S \longrightarrow \underset{\rightarrow}{G}$ of S such that the equations $p\gamma'_B Gf(a) = p\gamma'_A(a)$ for all f: $A \longrightarrow B$ in $\underline{\underline{C}}$ and all $a \in G(A)$, and we set $\gamma_A = p\gamma'_A$.

It is clear from the construction of $\underset{\rightarrow}{G}$ as a quotient of $G(A)$ $(A \in \underline{\underline{C}})$ that $\underset{\rightarrow}{G}$ is generated by $\{\mathrm{Im}(\gamma_A) \mid A \in \underline{\underline{C}}\}$. Condition (8.1)-(a) implies that, for any $A_1, A_2 \in \underline{\underline{C}}$, there is a $B \in \underline{\underline{C}}$ such that $\mathrm{Im}(\gamma_{A_i}) \subset \mathrm{Im}(\gamma_B)$ $(i = 1, 2)$. Hence this condition implies that G is the (set theoretic) union of the $\mathrm{Im}(\gamma_A)$.

For the last assertion we first note that the identification of a and b in $G(A_0)$, after passing to $\underset{\rightarrow}{G}$, is the consequences of data involving only a finite number of objects and morphisms in $\underline{\underline{C}}$. Consequently, there is a full subcategory $\underline{\underline{C}}_0 \subset \underline{\underline{C}}$ having only a finite number of objects such that a and b are identified already in $\underset{\rightarrow}{G}_0$, where $G_0 = G|\underline{\underline{C}}_0$. Using condition (8.1)-(a) we can enlarge $\underline{\underline{C}}_0$, if necessary, and arrange that $\underline{\underline{C}}_0$ have a final object, C, i.e., one into which each object of $\underline{\underline{C}}_0$ has a morphism. If δ_A: $G(A) \longrightarrow \underset{\rightarrow}{G}_0$ is the canonical morphism for $A \in \underline{\underline{C}}_0$, then δ_C: $G(C) \longrightarrow \underset{\rightarrow}{G}_0$ is surjective. If f_1, f_2: $A \longrightarrow C$ and if $a \in G(A)$ then since $\delta_A = \delta_C G(f_1) = \delta_C G(f_2)$ we have $\delta_C(a_1) = \delta_C(a_2)$, where $a_i = G(f_i)(a)$ $(i = 1, 2)$. Let Q be the largest quotient of G(C) in which all such identifications are made. Then any morphism $f: A \longrightarrow C$ induces the same morphism $G(A) \longrightarrow Q$, so we see that $Q = \underset{\rightarrow}{G}_0$.

Now suppose we are given a, $b \in G(A_0)$ as above so that $\delta_{A_0}(a) = \delta_{A_0}(b)$ in $\underset{\rightarrow}{G}_0$. Choose a morphism f_0: $A_0 \longrightarrow C$; then $\delta_C(\alpha) = \delta_C(\beta)$, where $\alpha = G(f_0)(a)$ and $\beta = G(f_0)(b)$.

The identification of $\delta_C(\alpha)$ and $\delta_C(\beta)$ is the consequence of a finite number of identifications, $\delta_C(\gamma_{i1}) = \delta_C(\gamma_{i2})$, where $\gamma_{ij} = G(f_{ij})(a_i)$ $(j = 1, 2)$ for some morphisms f_{i1}, f_{i2}: $A_i \longrightarrow C$ and elements $a_i \in G(A_i)$. It follows by induction from (8.1)-(b) that there is a morphism g: $C \longrightarrow C'$ in $\underline{\underline{C}}$ such that $gf_{i1} = gf_{i2}$ $(1 \le i \le n)$. Hence, $G(g)(\gamma_{i1}) = G(g)(\gamma_{i2})$ $(1 \le i \le n)$ and therefore $G(g)(\alpha) = G(g)(\beta)$ also. Putting $f = gf_0$: $A_0 \longrightarrow C'$, we have $G(f)(a) = G(f)(b)$, as required. q.e.d.

(8.3) DEFINITION. A functor F: $\underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ is said to be <u>cofinal</u> if it satisfies (a) and (b) below.

(a) Given $A' \in \underline{\underline{C}}'$ there exist $A \in \underline{\underline{C}}$ and an f: $A' \longrightarrow FA$.

(b) Given f': $FA \longrightarrow A'$ in $\underline{\underline{C}}'$ there exists an f: $A \longrightarrow B$ in $\underline{\underline{C}}$ and a g': $A' \longrightarrow FB$ in $\underline{\underline{C}}'$ such that $g'f' = Ff$.

(8.4) PROPOSITION. <u>Let</u> F: $\underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ <u>be a cofinal functor between categories having each only a set of isomorphism classes of objects. Assume that</u> $\underline{\underline{C}}'$ <u>is directed and that</u> $\underline{\underline{C}}$ <u>satisfies</u> (8.1). <u>Then if</u> G: $\underline{\underline{C}}' \longrightarrow \underline{\underline{A}}$ <u>is any functor the natural morphism</u>

$$\underset{\rightarrow}{GF} \longrightarrow \underset{\rightarrow}{G}$$

<u>is an isomorphism</u>.

<u>Proof</u>. The $\underline{\underline{A}}$-morphisms γ'_{FA}: $GFA \longrightarrow \underset{\rightarrow}{G}$ $(A \in \underline{\underline{C}})$

induce a unique α such that

$$\begin{array}{ccc} \underset{\rightarrow}{GF} & \xrightarrow{\alpha} & \underset{\rightarrow}{G} \\ {\scriptstyle \gamma_A} \nwarrow & & \nearrow {\scriptstyle \gamma'_{FA}} \\ & GFA & \end{array}$$

commutes for all $A \in \underline{\underline{C}}$. It follows that $\mathrm{Im}(\gamma_{FA}) \subset \mathrm{Im}(\alpha)$. If $A' \in \underline{\underline{C}}'$ then (8.3)(a) gives us an $A' \longrightarrow FA$ for some $A \in \underline{\underline{C}}$ and hence $\mathrm{Im}(\gamma'_{A'}) \subset \mathrm{Im}(\gamma'_{FA}) \subset \mathrm{Im}(\alpha)$. Since $\underset{\rightarrow}{G}$ is generated by the $\mathrm{Im}(\gamma'_{A'})$ it follows that α is surjective.

Suppose $\alpha(x) = \alpha(y)$. Condition (8.1)-(a) for $\underline{\underline{C}}$ implies (see (8.2)) that $\underset{\rightarrow}{GF}$ is the (directed) union of the $\mathrm{Im}(\gamma_A)$, so we can find $A \in \underline{\underline{C}}$ and a, $b \in GF(A)$ such that $x = \gamma_A(a)$ and $y = \gamma_A(b)$. Since $\underline{\underline{C}}'$ is directed and $\gamma'_{FA}(a) = \gamma'_{FA}(b)$ it follows from (8.2) above that there is a morphism $f'\colon\ FA \longrightarrow A'$ in $\underline{\underline{C}}'$ such that $Gf'(a) = Gf'(b)$. Thanks to (8.3)-(b) we can, after replacing f' by $g'f'$, if necessary, assume that $A' = FB$ and $f' = Ff$ for some $f\colon\ A \longrightarrow B$ in $\underline{\underline{C}}$. Therefore $Gf' = GFf$ and so $\gamma_A(a) = \gamma_A(b)$ already in $\underset{\rightarrow}{GF}$, i.e., $x = y$, so α is injective. q.e.d.

We shall now discuss a special type of direct limit which will be encountered in Chapter VII, §2.

Let M be an additive monoid. Then the "translation category", Tran(M), has M as its objects, and morphisms, for a, b $\in$ M,

$$\mathrm{Tran}(M)(a, b) = \{c \in M \mid a + c = b\}$$

composition of morphisms is just +.

We claim that Tran(M) <u>is a directed category</u>. Condition (8.1)-(a) is seen from the diagram $a_1 \xrightarrow{a_1} a_1 + a_2 \xleftarrow{a_1} a_2$. For condition (8.1)-(b) we are given $a \xrightarrow{c_1} b \xleftarrow{c_2} a$, i.e., $a + c_1 = b = a + c_2$. Then

$b \xrightarrow{\alpha} b + \alpha$ satisfies the requirement, $\alpha + c_1 + \alpha = \alpha + c_2 + \alpha$, of (8.1)-(b).

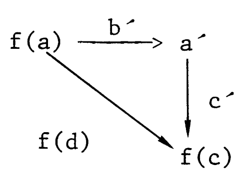
A homomorphism $f\colon\ M \longrightarrow M'$ of commutative monoids will be called *cofinal* if

(1) Given $\alpha' \in M'$, we can solve $\alpha' + b' = f(\alpha)$ for $\alpha \in M$ and $b' \in M'$.

(8.5) PROPOSITION. *The translation category* Tran(M) *of a commutative monoid* M *is directed. Moreover a cofinal homomorphism* $f\colon M \longrightarrow M'$ *of monoids induces a cofinal functor* (in the sense of (8.3)) $\mathrm{Tran}(f)\colon \mathrm{Tran}(M) \longrightarrow \mathrm{Tran}(M')$. *Therefore, if* $G\colon \mathrm{Tran}(M') \longrightarrow \underline{\underline{A}}$ *is any functor,* $\underrightarrow{G_\circ \mathrm{Tran}(f)} \longrightarrow \underrightarrow{G}$ *is an isomorphism.*

Proof. The last assertion follows from the first and Proposition (8.4), and we have already noted above that Tran(M) is directed.

To prove that Tran(f) is cofinal we must verify (8.3)-(a), which is precisely condition (1) above, and (8.3)-(b): given the top arrow of a commutative diagram



we must complete it. This amounts to solving $b' + c' = f(d)$, which we can do thanks to (1). For then $c = \alpha + d$ fills the diagram as indicated.

The following refinement of this proposition will also be used.

(8.6) PROPOSITION. *Let* $\alpha_0 = 0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ *be a sequence in a commutative monoid* M. *Write* $\alpha_{n,\,m} = \alpha_{n+1} + \cdots + \alpha_m$ if $n \le m$ $(\alpha_{n,\,n} = 0)$ and $s_n = \alpha_{0,\,n}$.

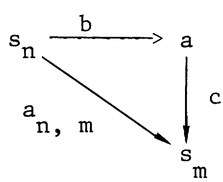
Assume that:

(2) Given $\alpha \in M$ and $n \geq 0$, there is a $b \in M$ and an $m \geq n$ such that $\alpha + b = \alpha_{n, m}$.

Let $\underline{\underline{C}} \subset \mathrm{Tran}(M)$ be the subcategory whose objects are the s_n and whose only morphisms are the $\alpha_{n, m}$. Then $\underline{\underline{C}}$ is directed and the inclusion functor is cofinal. Therefore, if $G: \mathrm{Tran}(M) \longrightarrow \underline{\underline{A}}$ is any functor we can compute $\underrightarrow{G}$ as the direct limit of the $G(s_n)$ with respect to the morphisms

$$G(s_n) \xrightarrow{G(\alpha_{n, m})} G(s_m).$$

Proof. Clearly $\underline{\underline{C}}$ is directed. For the rest it suffices, by virtue of (8.4), to establish cofinality. Condition (8.3)-(b) of cofinality requires that we complete a commutative diagram



given the top arrow. But thanks to (2) we can solve $b + c = \alpha_{n, m}$ for c and $m \geq n$, as required. Condition (8.3)-(a) follows from (2) also, in the special case $n = 0$. q.e.d.

Chapter II

CATEGORIES OF MODULES AND THEIR EQUIVALENCES

In §1 we show that an Abelian category with arbitrary coproducts, and with a "faithfully projective" object, is equivalent to a category of modules. As a preliminary to classifying equivalences, mod-A ⟶ mod-B, we show that all colimit-preserving functors are of the form $\otimes_A P$, P a bimodule. We could equally well have studied limit-preserving functors, which are of the form $\mathrm{Hom}_A(P, \cdot)$, since equivalences do both. However tensor products are more convenient for discussing composition of functors.

In §3 we analyze the structure of an equivalence mod-A ⟶ mod-B. A number of common features of A and B are deduced from its existence. In §4 we show how to construct an equivalence from a faithfully projective module. Indeed, §3 implies they are all obtained by such a construction.

The autoequivalences of mod-A, for an R-algebra A, lead to a group, $\mathrm{Pic}_R(A)$, which we study in §5. In particular, the group of "outer automorphisms" of A as an R-algebra is a subgroup of $\mathrm{Pic}_R(A)$.

§1. CHARACTERIZATION OF CATEGORIES OF MODULES

A functor between Abelian categories will be called faithfully exact if it is faithful and exact and if, further, it preserves arbitrary coproducts. It follows that such a functor preserves colimits.

Let $\mathbf{A}$ be an Abelian category and let $P \in \mathbf{A}$ represent the functor $h = \mathbf{A}(P, \cdot)$: $\mathbf{A} \longrightarrow \mathbf{Z}$-mod. Recall that P is *projective* if h is exact. P is said to be a *generator* of $\mathbf{A}$ if h is faithful. We shall call P *faithfully projective* if h is faithfully exact. Note that this requires more than that P be a projective generator, because it is not true in general that functors of the form $\mathbf{A}(P, \cdot)$ preserve coproducts. We shall see that this condition is related to the condition of finite generation for modules. If A is a ring, A is faithfully projective in mod-A.

(1.1)PROPOSITION. *Let $\mathbf{A}$ be an Abelian category with arbitrary coproducts.*

(a) $P \in \mathbf{A}$ *is a generator of $\mathbf{A}$ if and only if every object of $\mathbf{A}$ is a quotient of* $P^{(I)}$ *for some set* I.

(b) *Let* C *be a class of objects in $\mathbf{A}$ such that* (i) C *contains a generator of $\mathbf{A}$,* (ii) *arbitrary coproducts of objects in* C *are in* C, *and* (iii) *cokernels of morphisms between objects in* C *are also in* C. *Then* $C = \text{ob}\mathbf{A}$.

Proof. (a) Suppose P generates $\mathbf{A}$ and $A \in \mathbf{A}$. Then $H = \mathbf{A}(P, A)$ defines a morphism a: $P^{(H)} \longrightarrow A$, which we claim to be an epimorphism. Let b: $A \longrightarrow B$ be its cokernel. If $p \in H = h(A)$, where $h = \mathbf{A}(P, \cdot)$, then $h(b)(p) = bp$. Since $\text{Im}(p) \subset \text{Im}(a)$ and $ba = 0$, it follows that $bp = 0$, and hence $h(b) = 0$. But h is faithful, so $b = 0$, i.e., a is an epimorphism.

Conversely, if there is an epimorphism $(p_i)_{i \in I}$: $P^{(I)} \longrightarrow A$ then we will show that $\mathbf{A}(A, B) \longrightarrow \text{Hom}_{\mathbf{Z}}(hA, hB)$ is a monomorphism for all $B \in \mathbf{A}$. For if b: $A \longrightarrow B$ is such that $h(b) = 0$, then $h(b)(p_i) = bp_i = 0$ for all $i \in I$ and hence $b(p_i) = 0$. But $(p_i)_{i \in I}$ is an epimorphism, so this equation implies b = 0.

(b) If $P \in C$ is a generator of $\underline{\underline{A}}$ then it follows from part (a) that every object A in $\underline{\underline{A}}$ fits into an exact sequence $P^{(J)} \longrightarrow P^{(I)} \longrightarrow A \longrightarrow 0$. Hence conditions (i), (ii), and (iii) imply $C = \text{ob}\underline{\underline{A}}$. q.e.d.

(1.2) PROPOSITION. Let A be a ring and let $P \in$ mod-A.

(a) P is finitely generated and projective if and only if P is a direct summand of $A^{(n)}$ for some $n \geq 0$.

(b) P is a generator of mod-A if and only if A is a direct summand of $P^{(n)}$ for some $n \geq 0$.

(c) P is faithfully projective if and only if P is a finitely generated projective generator of mod-A.

Proof. (a) $\text{Hom}_A(A, \cdot)$ is isomorphic to the identity functor so A is projective, and hence likewise for $A^{(n)}$ and its direct summands. If P is finitely generated there is an epimorphism $A^{(n)} \longrightarrow P$, and the latter splits if P is projective.

(b) Every module is a quotient of $A^{(I)}$ for some I so A generates mod-A. If A is a direct summand of $P^{(n)}$, therefore, P clearly also generates. Conversely, if P generates then A is a quotient, and hence direct summand of a coproduct of copies of P. Since A is finitely generated a finite coproduct already suffices.

(c) By definition, $\text{Hom}_A(P, \cdot)$ is faithful and exact if and only if P is a projective generator. Hence it will suffice to show, for a projective module P, that P is finitely generated if and only if $h = \text{Hom}_A(P, \cdot)$ preserves coproducts. The latter condition, that $\text{Hom}_A(P, \coprod M_i) = \coprod \text{Hom}_A(P, M_i)$, just means that any $f: P \longrightarrow \coprod M_i$ has its image in the submodule generated by a finite number of the M_i's. Clearly any finitely generated module P has this property. Conversely, if P is projective, there is a split monomorphism $f: P \longrightarrow A^{(I)}$ for some I. The above condition then implies that $P \simeq f(P)$ is a direct summand of $A^{(J)}$ for some finite $J \subset I$, so P is finitely generated.

Exercise. (a) Show that a module P is finitely generated if and only if the union of a totally ordered family of proper submodules of P is a proper submodule.

(b) Show that $\mathrm{Hom}_A(P, \cdot)$ preserves coproducts if and only if the union of every (countable) chain of proper submodules is a proper submodule.

(c) Show that the conditions in (a) and (b) are not equivalent. (Examples are not easy to find.)

In the category mod-A the module A seems to play a somewhat distinguished role. This is not entirely true; any other faithfully projective module can play the same role, and fixing A in mod-A has some of the same arbitrary features as fixing a basis in a vector space. Moreover, this principle can be played backward: General theorems about faithfully projective modules need sometimes only be proved for A (cf. (5.3) below, for example).

(1.3) THEOREM (Gabriel, Mitchell). Let $\underline{\underline{A}}$ be an Abelian category with arbitrary coproducts and with a faithfully projective object P. Put $A = \underline{\underline{A}}(P, P)$. Then the functor

$$h = \underline{\underline{A}}(P, \cdot): \underline{\underline{A}} \longrightarrow \text{mod-}A$$

is an equivalence of categories, and $h(P) = A$ is the free module on one generator.

Proof. Using criterion (I, 1.1) for an equivalence we need only establish (a) and (b) below:

(a) $h_{X,Y} : \underline{\underline{A}}(X, Y) \longrightarrow \mathrm{Hom}_A(hX, hY)$ is an isomorphism for all $X, Y \in \underline{\underline{A}}$; and

(b) Every $M \in$ mod-A is isomorphic to some hX.

Fix a $Y \in \underline{\underline{A}}$ and view $h_{X,Y}$ as a natural transformation $\alpha_X : TX \longrightarrow SX$, where T and S are the indicated functors $\underline{\underline{A}}^0 \longrightarrow \underline{\underline{Z}}\text{-mod}$.

We shall prove (a) by showing that the class C of objects X for which α_X is an isomorphism (i) contains a generator, (ii) is stable under coproducts, and (iii) is "stable under cokernels." For then it follows from (1.1(b)) that $C = ob\underline{\underline{A}}$.

Since h is faithful P is a generator. Moreover $\phi_P\colon \underline{\underline{A}}(P, Y) \longrightarrow Hom_A(hP, hY) = Hom_A(A, \underline{\underline{A}}(P, Y))$ is easily seen to be the standard isomorphism, and this proves (i).

Condition (ii) follows from the fact that both S and T convert coproducts into products. For T this is clear and for S it is a consequence of our hypothesis that h preserves coproducts.

Condition (iii) means that if $X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is exact in $\underline{\underline{A}}$, then $X, Y \in C \Rightarrow Z \in C$. Now T is left exact, and, since h is exact, S is also left exact. Hence we have a commutative diagram with exact rows,

$$\begin{array}{ccccccc}
0 & \longrightarrow & TZ & \longrightarrow & TY & \longrightarrow & TX \\
 & & \downarrow{\scriptstyle \alpha_Z} & & \downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_X} \\
0 & \longrightarrow & SZ & \longrightarrow & SY & \longrightarrow & SX
\end{array}$$

and the desired conclusion follows by the 5-lemma (I, 4.4).

To prove (b), write $M = Coker(A^{(I)} \xrightarrow{f} A^{(J)})$. Then

$$f \in Hom_A((hP)^{(I)}, (hP)^{(J)}) = Hom_A(h(P^{(I)}), h(P^{(J)}))$$

so $f = h(g)$ for some $g\colon P^{(I)} \longrightarrow P^{(J)}$, thanks to part (a). By exactness, $M \simeq Coker(f) = Coker(h(g)) \simeq h(Coker\ g)$. q.e.d.

<u>Exercise</u> (Lam). Let $\underline{\underline{A}}$ be an Abelian category in which all objects are noetherian (= ascending chain condition on subobjects). Assume $\underline{\underline{A}}$ has a projective generator P, and put $A = \underline{\underline{A}}(P, P)$. Show that A is a right noetherian ring and

that $\underline{\underline{A}}(P, \cdot)$ defines an equivalence from $\underline{\underline{A}}$ to the category of finitely generated right A-modules.

§2. R-CATEGORIES: RIGHT CONTINUOUS FUNCTORS

If c lies in the center of a ring A then the endomorphisms, $x \longmapsto xc$, on A-modules constitute an endomorphism, h(c), of the identity functor on mod-A. There are no others; more precisely:

(2.1) PROPOSITION. *The "homothetie" map*

$$h: \text{ center } A \longrightarrow \text{End}(\text{Id}_{\text{mod-}A})$$

is an isomorphism of commutative rings.

Proof. Since $h(c)_A(1) = 1 \cdot c = c$ it follows that h is injective. It is clearly a ring homomorphism. Finally, suppose $t \in \text{End}(\text{Id}_{\text{mod-}A})$. Let $c = t_A(1)$. Given $x \in M \in$ mod-A define $f: A \longrightarrow M$ by $f(a) = xc$. By naturality of t,

$$\begin{array}{ccc} A & \xrightarrow{t_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow[t_M]{} & M \end{array}$$

commutes, so $t_M(x) = t_M(f(1)) = f(t_A(1)) = f(c) = xc = h(c)_M(x)$. Thus $t = h(c)$, and h is therefore surjective.

This proposition suggests that, for any category $\underline{\underline{A}}$, we define

$$\text{center } \underline{\underline{A}} = \text{End}(\text{Id}_{\underline{\underline{A}}})$$

Let R be a commutative ring and let $\underline{\underline{A}}$ be an Abelian category. Then it is easy to see that giving a ring homomorphism $R \longrightarrow$ center $\underline{\underline{A}}$ is the same as giving all the Abelian groups $\underline{\underline{A}}(X, Y)$ the structure of R-modules in such a way that composition is R-bilinear. An $\underline{\underline{A}}$ with this additional structure will be called an R-*category*. A functor T: $\underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ between R-categories is said to be an R-functor if the maps $\underline{\underline{A}}(X, Y) \longrightarrow \underline{\underline{B}}(TX, TY)$ are R-linear. When $R = \underline{\underline{Z}}$ we just recover the notions of Abelian category and additive functor.

An R-algebra is a ring A and a homomorphism $R \longrightarrow A$ whose image lies in center A. It follows therefore from Proposition (2.1) that R-algebra structures on A are equivalent to R-category structures on mod-A.

Let A and B be R-algebras and write A-mod-B for the category of left A-, right B - bimodules M, and their homomorphisms. Recall that the compatibility required of the A- and B- module structures is

$$(ax)b = a(xb) \qquad (a \in A,\ x \in M,\ b \in B)$$

If $r \in R$ then rx and xr are both defined, but not necessarily equal. Indeed, rx = xr for all $x \in M$ and $r \in R$ precisely when the bimodule structure on M makes M a left $A \otimes_R B^0$-module. Moreover, this is further equivalent to the condition that $\otimes_A M$: mod-A $\longrightarrow$ mod-B be an R-functor.

(2.2) PROPOSITION. *Let* A *and* B *be* R-*algebras, and let*

$$h:\quad (A\otimes_R B^0)\text{-mod} \longrightarrow \text{R-functors(mod-A, mod-B)}$$

be the functor defined by $h(M) = \otimes_A M$. *Then* h *is fully faithful. In particular* $M \simeq N$ *as bimodules* $\Longleftrightarrow \otimes_A M \simeq \otimes_A N$ *as functors from* mod-A *to* mod-B.

Proof. If f: $M \longrightarrow N$ is a bimodule homomorphism then $h(f) = \otimes_A f$. If h(f) = 0 then the vanishing of $A\otimes_A f$ implies f = 0, so h is faithful.

Suppose t: $hM \longrightarrow hN$ is a natural transformation, and let f: $M \longrightarrow N$ be the B-homomorphism rendering

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ A\otimes_A M & \xrightarrow[t_A]{} & A\otimes_A N \end{array}$$

commutative. Left multiplications in A are right A-linear, so t_A must preserve them, by naturality. Thus t_A, as well as the verticals, are bimodule homomorphisms, and hence likewise for f. We will prove that h is full by showing that $t = h(f)$. Let $s = t-h(f)$ and let C denote the class of $M \in$ mod-A such that $s_M = 0$. By construction C contains A. Since both hM and hN are right exact and preserve coproducts it follows now that C satisfies the hypothesis of (1.1(b)), and hence C = ob(mod-A). q.e.d.

The functors $hM = \otimes_A M$ of Proposition (2.2) (i) are cokernel preserving, and (ii) they preserve arbitrary coproducts. A functor satisfying (i) and (ii) will be called *right continuous*. The terminology is suggested by the fact that such a functor must preserve all direct limits. Among categories of modules all right continuous functors are tensor products. More precisely:

(2.3) THEOREM (Eilenberg, Watts). *Let* A, B, *and* C *be* R-*algebras. The correspondence* $M \longmapsto hM = \otimes_A M$, *from left* $A\otimes_R B^0$-*modules to right continuous* R-*functors from* mod-A *to* mod-B, *induces a bijection on isomorphism classes. If* N *is a left* $B\otimes_R C^0$-*module then* $h(M\otimes_B N) \simeq h(N)\circ h(M)$.

Remark. One is tempted to formulate this result as an equivalence of categories, as follows. Let $\underline{\underline{A}}$ and $\underline{\underline{B}}$ be the categories whose objects, in both cases, are R-algebras, and whose morphisms are

$$\underline{\underline{A}}(A, B) = \{\text{left } A\otimes_R B^0\text{-modules}\}$$

and

$$\underline{\underline{B}}(A, B) = \{\text{right continuous functors, mod-}A \longrightarrow \text{mod-}B\}$$

respectively. $\underline{\underline{B}}$ is a perfectly acceptable category, using composition of functors. For $\underline{\underline{A}}$ we would like to use $\otimes$ for composition. But then we have neither identity morphisms, nor associativity. For while $A\otimes_A M$ and M are (canonically) isomorphic, they are not equal; similarly for the associativity of $\otimes$. Thus we are compelled to pass to isomorphism classes.

Proof. If $X \in$ mod-A then $h(N)\circ h(M)(X) = h(N)(X\otimes_A M) = (X\otimes_A M)\otimes_B N \simeq X\otimes_A (M\otimes_B N)$, and the isomorphism is natural. This proves the last assertion.

The fact that h is injective on isomorphism classes is contained in (2.2). There remains only to be proved that a right continuous R-functor t: mod-$A \longrightarrow$ mod-B is of the form hM. We take $M = tA$, which is at first only a B-module. The R-algebra homomorphism

$$A \simeq \mathrm{Hom}_A(A, A) \xrightarrow{t} \mathrm{Hom}_B(M, M)$$

makes of M a left $A\otimes_R B^\circ$-module.

For $X \in$ mod-A we have maps

$$X \simeq \mathrm{Hom}_A(A, X) \xrightarrow{t} \mathrm{Hom}_B(M, tX)$$

whose composite, f_X, is A-linear with respect to the A-module structure on M just constructed. Under the canonical isomorphism

$$\mathrm{Hom}_A(X, \mathrm{Hom}_B(M, tX)) \simeq \mathrm{Hom}_B(X\otimes_A M, tX)$$

let g_X be the element on the right corresponding to f_X on the left. Since the f_X's are natural in X the g_X's are also: g: $hM \longrightarrow t$. Both hM and t are right continuous so the class C of X for which g_X is an isomorphism is stable under coproducts and cokernels. It follows therefore from

(1.1)-(b) that g is an isomorphism provided g_A is, since A generates mod-A. But g_A: $A\otimes_A M \longrightarrow tA = M$ is the standard isomorphism.

(2.4) COROLLARY AND DEFINITION. We call a left $A\otimes_R B°$-module "invertible" if it satisfies the following conditions, which are equivalent:

(a) $\otimes_A M$: mod-A ⟶ mod-B is an equivalence.

(b) There is a left $B\otimes_R A°$-module N such that $M\otimes_B N \simeq A$ and $N\otimes_A M \simeq B$ as bimodules.

(c) $M\otimes_B$: B-mod ⟶ A-mod is an equivalence.

Proof. Since an equivalence is right continuous the implications (a) ⟺ (b) follow immediately from Theorem (2.3). A left - right reflection of Theorem (2.3) shows that (b) ⟺ (c).

§3. EQUIVALENCES OF CATEGORIES OF MODULES

We fix a commutative ring R, and all of our rings, A, B,... will be R-algebras. The fact that M is a left $A\otimes_R B°$-module will sometimes be denoted by writing ${}_AM_B$, following the Cartan-Eilenberg convention.

This section contains a thorough analysis of equivalences from mod-A to mod-B. We begin by summarizing some consequences of Theorem (2.3).

(3.1) PROPOSITION. Let A and B be R-algebras and let

$$\text{mod-}A \underset{S}{\overset{T}{\rightleftarrows}} \text{mod-}B$$

be R-functors such that $ST \simeq Id_{\text{mod-}A}$ and $TS \simeq Id_{\text{mod-}B}$. Set $P = TA$ and $Q = SB$. Then we are in the situation $({}_AP_B, {}_BQ_A)$, and:

(1) $T \simeq \otimes_A P$ *and* $S \simeq \otimes_B Q$

(2) *There are bimodule isomorphisms*

$$f\colon\ P\otimes_B Q \longrightarrow A \quad \textit{and} \quad g\colon\ Q\otimes_A P \longrightarrow B$$

(3) f *and* g *may be chosen to render the diagrams*

$$\begin{array}{ccc} P\otimes_B Q\otimes_A P & \xrightarrow{f\otimes 1_P} & A\otimes_A P \\ {\scriptstyle 1_P\otimes g}\downarrow & & \downarrow{\scriptstyle \alpha} \\ P\otimes_B B & \xrightarrow[\beta]{} & P \end{array}$$

and

$$\begin{array}{ccc} Q\otimes_A P\otimes_B Q & \xrightarrow{g\otimes 1_Q} & B\otimes_B Q \\ {\scriptstyle 1_Q\otimes f}\downarrow & & \downarrow \\ Q\otimes_A A & \longrightarrow & Q \end{array}$$

commutative. (Here α and β are the obvious maps, and similarly in the other diagram.)

Proof. Everything but part (3) follows immediately from Theorem (2.3) and the fact that an equivalence is right-continuous. To prove (3), suppose we want the first diagram to commute. Since all maps are bimodule isomorphisms we at least have $\beta(1\otimes g) = u\alpha(f\otimes 1)$ for some $u \in \mathrm{Aut}_{A-B}(P)$. In particular $u \in \mathrm{Hom}_B(P, P) = \mathrm{Hom}_B(TA, TA) \simeq \mathrm{Hom}_A(A, A) = A$, so u is left multiplication by an element of A, which we shall identify with u. Being also an A-homomorphism, u must be in the center of A. Now, evidently, $u\alpha = \alpha(u \otimes 1_P)$, so if we replace f by uf we have made the first square commutative. We propose to show now that the second square automatically commutes. In order to avoid repeating this argument later we interrupt to make a definition which is suggested by the proposition above.

(3.2) DEFINITION. A set of *pre-equivalence data* (A, B, P, Q, f, g) consists of R-algebras A and B, bimodules ${}_AP_B$ and ${}_BQ_A$, and bimodule homomorphisms $f\colon\ P\otimes_B Q \longrightarrow A$ and

$g\colon Q \otimes_A P \longrightarrow B$ which are "associative" in the following sense: Writing $f(p \otimes q) = pq$ and $g(q \otimes p) = qp$ we require:

(i) $(pq)p' = p(qp')$

(ii) $(qp)q' = g(pq')$

(for all $p, p' \in P$; $q, q' \in Q$)

We shall call it a set of <u>equivalence data</u> if f and g are isomorphisms.

Now the <u>proof of Proposition</u> (3.1) is completed by:

(3.3) LEMMA. <u>Condition (ii) in the definition of pre-equivalence data follows from the other conditions, provided</u>: $d \in Q$ and $dp' = 0$ for all $p' \in P \Rightarrow d = 0$. <u>The latter condition is satisfied if</u> $\otimes_A P$ <u>is faithful</u>.

<u>Proof</u>. Given $q, q' \in Q$ and $p \in P$ we must show that $(qp)q' = g(pq')$. For any $p \in P$ we have

$$\begin{aligned} ((qp)q')p' &= (qp)(q'p') && \text{(g is left B-linear)} \\ &= q(p(q'p')) && \text{(g is right B-linear)} \\ &= q((pq')p') && \text{(condition (i))} \\ &= (q(pq'))p' && \text{(g is A-bilinear)} \end{aligned}$$

Hence, if $d = (qp)q' - q(pq')$, then $dp' = 0$ for all $p' \in P$, so $d = 0$, by hypothesis.

To prove the last assertion let $h\colon A \longrightarrow Q$ by $h(a) = da$. Then $h \otimes 1_P\colon A \otimes_A P \longrightarrow Q \otimes_A P$, followed by the isomorphism g, is zero, so $h \otimes 1_P = 0$. Therefore $h = 0$ if $\otimes_A P$ is faithful.

(3.4) THEOREM. <u>Let</u> (A, B, P, Q, f, g) <u>be a set of pre-equivalence data, and assume that</u> f <u>is surjective</u>.

1. f <u>is an isomorphism</u>.

2. P <u>and</u> Q <u>are generators as</u> A-<u>modules</u>.

3. P <u>and</u> Q <u>are finitely generated and projective as</u>

B-modules.

4. _g induces bimodule isomorphisms_

$$P \simeq \mathrm{Hom}_B(Q, B) \quad \text{and} \quad Q \simeq \mathrm{Hom}_B(P, B)$$

5. _The R-algebra homomorphisms_

$$\mathrm{End}_B(P) \longleftarrow A \longrightarrow \mathrm{End}_B(Q)^\circ$$

induced by the bimodule structures, are isomorphisms.

Proof. The hypothesis on f means that we can write

$$(*) \qquad 1 = \sum_{i \in I} p_i g_i \quad \text{in } A$$

(1) Suppose $p_j' \otimes q_j' \in \ker f$. Then using (*), we have $\Sigma p_j' \otimes q_j' = \Sigma_{j,i}(p_j' \otimes q_j')p_i q_i = \Sigma_{j,i} p_j' \otimes ((q_j' p_i)q_i) = \Sigma_{i,j}(p_j'(q_j' p_i)) \otimes q_i = \Sigma_{i,j}(p_j' q_j')(p_i \otimes q_i) = (\Sigma_j p_j' q_j')(\Sigma_i p_i \otimes q_i) = 0$, since $\Sigma_j p_j' q_j' = 0$.

(2) The linear functionals $h_i\colon P \longrightarrow A$ by $h_i(p) = pq_i$ define $h\colon P^{(I)} \longrightarrow A$, and (*) implies h is surjective, so P generates A-mod. The argument for Q is similar.

(3) Define $P \underset{h}{\overset{e}{\rightleftarrows}} B^{(I)}$ by $e(p) = (q_i p)$ and $h(b_i) = \Sigma p_i b_i$. Then $he(p) = \Sigma p_i(q_i p) = (\Sigma p_i q_i)p = p$. Thus P is finitely generated and projective, and a similar argument shows the same for Q.

(4) g induces a bimodule homomorphism $h\colon P \longrightarrow \mathrm{Hom}_B(Q, B)$ by $h(p)(q) = qp$. If $h(p) = 0$ then $p = \Sigma(p_i q_i)p = p_i(q_i p) = 0$, so h is injective. For surjectivity let $f\colon Q \longrightarrow B$ be given. Then $f(q) = f(\Sigma q(p_i q_i)) = f(\Sigma(qp_i)q_i) = \Sigma(q\, p_i)f(q_i) = \Sigma q(p_i f(q_i)) = h(p)(q)$, where $p = \Sigma p_i f(q_i)$.

Similarly $Q \simeq \mathrm{Hom}_B(P, B)$.

(5) We must show that $h\colon A \longrightarrow \mathrm{End}_B(P)$, by $h(a)(p) = ap$, is an isomorphism. If $h(a) = 0$ then $a = \Sigma a(p_i q_i) = \Sigma (ap_i)q_i = 0$, so h is injective. For surjectivity let $f\colon P \longrightarrow P$ be given. Then $f(p) = f(\Sigma(p_i q_i)p) = f(\Sigma p_i(q_i p)) = \Sigma f(p_i)(q_i p) = \Sigma(f(p_i)q_i)p = h(\alpha)(p)$, where $\alpha = \Sigma f(p_i)q_i$. Similarly $A \longrightarrow \mathrm{End}_B(Q)^\circ$ is an isomorphism.

(3.5) THEOREM. *Let* (A, B, P, Q, f, g) *be a set of equivalence data* (see definition (3.2)).

(1) P *and* Q *are both invertible bimodules* (see (2.4)).

(2) P *and* Q *are each faithfully projective both as* A-*modules and as* B-*modules*.

(3) f *and* g *induce bimodule isomorphisms of* P *and* Q *with each other's duals with respect to* A *and with respect to* B.

(4) *The* R-*algebra homomorphisms*

$$\mathrm{End}_B(P) \longleftarrow A \longrightarrow \mathrm{End}_B(Q)^\circ$$

and

$$\mathrm{End}_A(P)^\circ \longleftarrow B \longrightarrow \mathrm{End}_A(Q)$$

induced by the bimodule structure on P *and* Q, *are isomorphisms*.

(5) *The bimodule endomorphism rings of* A, B, P, *and* Q *are all isomorphic (canonically) to the centers of* A, B, mod-A, *and* mod-B (see (2.1)).

(6) *The lattice of right* A-*ideals is isomorphic, via* $\mathfrak{a} \longmapsto \mathfrak{a}P$ *with the lattice of* B-*submodules of* P, *the two sided ideals corresponding to* A-B-*submodules, or, equivalently, to fully invariant* B-*submodules. Similar conclusions apply with appropriate permutations of* (left, right), (A, B), *and* (P, Q). *In particular, by symmetry*, A and B *have isomorphic lattices of two sided ideals*.

(7) *The functor* $T = \mathrm{Hom}_A(P, \cdot) \simeq Q\otimes_A \cdot :$ A-mod $\longrightarrow$ B-mod *is an equivalence of categories. If* M ε A-mod *then* M *is finitely generated (over* A) $\Longleftrightarrow$ TM *is finitely generated (over* B). *Moreover the two sided ideals* $\mathrm{ann}_A(M)$ *in* A *and* $\mathrm{ann}_B(TM)$ *in* B *correspond under the lattice isomorphism in* (6). *In particular* M *is faithful (over* A) $\Longleftrightarrow$ TM *is faithful (over* B).

Proof. (1) follows immediately from the hypothesis and definition (2.4).

(2), (3), and (4) follow immediately from (2), (3), (4), and (5) of Theorem (3.4).

We have isomorphisms

$$\text{center } A = \mathrm{End}_{A-A}(A) \xrightarrow{\otimes_A P} \mathrm{End}_{A-B}(P),$$

$$\text{center } B = \mathrm{End}_{B-B}(B) \xrightarrow{P\otimes_B} \mathrm{End}_{A-B}(P)$$

and similar ones involving Q. Part (5) follows from these and Proposition (2.1).

We now prove (6). Since P is projective the canonical map, $\underline{a} \otimes_A P \longrightarrow \underline{a}P$, is an isomorphism for right A-ideals $\underline{a}$. Therefore the fact that $\underline{a} \rightarrowtail \underline{a}P$ is a lattice isomorphism from right A-submodules of A to B-submodules of $P = A\otimes_A P$ follows from the fact that $\otimes_A P$: mod-A $\longrightarrow$ mod-B is an equivalence. Moreover, since $A = \mathrm{End}_{\text{mod-}A}(A) = \mathrm{End}_B(P)$ the fully invariant submodules of A and P are the two sided ideals and the A-B-submodules, respectively. Clearly these correspond also under an equivalence.

The remaining assertions of (6) are clear. The isomorphism between lattices of two-sided ideals in A and B makes $\underline{a} \subset A$ and $\underline{b} \subset B$ correspond if and only if $\underline{a}P = P\underline{b}$. The conclusions above show that this does, indeed, define a bijection.

Finally, we prove (7). If M, N ε A-mod write $N^* = \mathrm{Hom}_A(N, A)$ and define h_N: $N^* \otimes_A M \longrightarrow \mathrm{Hom}_A(N, M)$ by $h_N(f \otimes x)(n) = xf(n)$. This is a natural transformation and h_A is clearly an isomorphism. Therefore, by additivity, h_N is an isomorphism if N is finitely generated and projective. By virtue of (2) and (3), therefore, $T = \mathrm{Hom}_A(P, \cdot)$ and $Q\otimes_A$ are isomorphic functors. If M ε A-mod is finitely generated then $Q \otimes_A M$ is a finitely generated B-module because Q is.

Conversely, if TM is finitely generated so is M because T is an equivalence.

Let $\underline{a} = \text{ann}_A(M)$. Then $\underline{a}\,P$ is characterized as the largest submodule of P killed by every A-homomorphism $P \longrightarrow M$. Therefore $\underline{b} = T(\underline{a}\,P)$ is the largest submodule of $B = T(P)$ killed by every B-homomorphism $B \longrightarrow TM$, i.e., $\underline{b} = \text{ann}_B(TM)$. From part (6), the ideal $\underline{c}$ in B corresponding to $\underline{a}$ is characterized by $\underline{a}\,P = P\,\underline{c}$. Therefore $T(\underline{a}\,P) = \text{Hom}(P, P\,\underline{c}) = B\underline{c} = \underline{c}$, so $\underline{c} = \underline{b}$. q.e.d.

§4. CONSTRUCTING AN EQUIVALENCE FROM A MODULE

Our treatment thus far has emphasized the symmetry inherent in equivalence data. On the other hand it follows from Theorem (3.5) that a small part of the data determines the rest.

We start from an R-algebra B and a right B-module P. From these we shall construct a set of pre-equivalence data, and then we shall determine the conditions on P for these to be equivalence data.

Set

$$A = \text{End}_B(P) \qquad \text{and} \qquad Q = \text{Hom}_B(P, B)$$

Then A is an R-algebra and P is a left $A \otimes_R B°$-module. Moreover Q is a left $B \otimes_R A°$-module with action

$$(bq)p = b(qp) \qquad (b \in B,\ q \in Q,\ p \in P)$$

and

$$(qa)p = q(ap) \qquad (q \in Q,\ a \in A,\ p \in P)$$

Next we define bimodule homomorphisms

$$f_P\colon\ P \otimes_B Q \longrightarrow A \qquad \text{and} \qquad g_P\colon\ Q \otimes_A P \longrightarrow B$$

The map g_P is just "evaluation," $g_P(q \otimes p) = qp$. We define $f_P(p \otimes q) = pq \in A = \text{End}_B(P)$ by

$$(pq)p' = p(qp') \qquad (p' \in P)$$

It now follows from Lemma (3.3) that:

(4.1) PROPOSITION. *Let* B *be an* R-*algebra and let* P *be a right* B-*module. Let* f_P *and* g_P *be the homomorphisms constructed above. Then*

$$(4.2) \quad (A = \mathrm{End}_B(P),\ B,\ P,\ Q = \mathrm{Hom}_B(P, Q),\ f_P,\ g_P)$$

is a set of pre-equivalence data (see definition (3.2)).

(4.3) EXAMPLE. Let $P = eB$ where e is idempotent. Then $B = P \oplus (1 - e)B$ so any $q\colon P \longrightarrow B$ can be extended to $\bar{q}\colon B \longrightarrow B$ by setting $\bar{q}((1 - e)B) = 0$. Thus we obtain inclusions $A = \mathrm{Hom}_B(P, P) \subset Q = \mathrm{Hom}_B(P, B) \subset B = \mathrm{Hom}_B(B, B)$. With these identifications we have

$$P = eB, \quad Q = Be, \quad \text{and} \quad A = eBe$$

and all pairings are induced by multiplication in B. In particular, $f_P\colon eB \otimes_B Be \longrightarrow A = eBe$ is surjective, and $g_P\colon Be \otimes_{eBe} eB \longrightarrow B$ has image BeB, the two-sided ideal generated by e.

(4.4) PROPOSITION. *In the notation of Proposition (5.1)*:

(a) f_P *is surjective* $\Longleftrightarrow$ P *is a finitely generated projective* B-*module, in which case* f_P *is an isomorphism.*

(b) g_P *is surjective* $\Longleftrightarrow$ P *is a generator of* mod-B, *in which case* g_P *is an isomorphism.*

(c) *(4.2) is a set of equivalence data* $\Longleftrightarrow$ P *is a* faithfully projective B-module. In this case

$$\text{mod-}A \underset{\operatorname{Hom}_B(P\,,\ \cdot)}{\overset{\otimes_A P}{\rightleftarrows}} \text{mod-}B$$

are inverse equivalences.

Proof. The implications ⟹ in (a) and (b) follow from Proposition (4.1) and Theorem (3.4). Part (c) follows from (a) and (b) (see Proposition (2.1)) and from Theorem (3.5), in view of the fact that the functors $\operatorname{Hom}_B(P, \cdot)$ and $\otimes_B \operatorname{Hom}_B(P, B)$ are isomorphic for P finitely generated and projective (cf. proof of (3.5)(7)). If P is a generator then there is an epimorphism $(q_i)_{i \in I} : P^{(I)} \longrightarrow B$, so $\operatorname{Im} g_P \supset \Sigma q_i P = B$, and g_P is surjective. The remaining implication in part (a) follows immediately from the more general:

(4.5) PROPOSITION. A right B-module P is projective ⟺ there exist families $p_i \in P$ and $q_i \in Q = \operatorname{Hom}_B(P, B)$ $(i \in I)$ such that, given $p \in P$,

(i) $q_i p = 0$ for almost all i, and

(ii) $p_i(q_i p) = p$

The families (p_i) which arise in this way are precisely the generating sets in P. Moreover the ideal $\underline{a} = \operatorname{Im} g_P = \Sigma\, qP\, (q \in Q)$ is generated, as a two-sided ideal, by $\{q_j p_i\}$. In addition $P\underline{a} = P$ and $\underline{a}^2 = \underline{a}$.

Proof. Projectivity of P is equivalent to the existence of homomorphisms $P \xrightarrow{e = (q_i)} B^{(I)} \xrightarrow{h = (p_i)} P$ such that $he = 1_P$, and the first assertion just rewrites this equation. When P is projective the h's which can occur here are precisely the epimorphisms; hence the second assertion.

To prove the third assertion suppose $p \in P$ and $q \in Q$. Then $qp = q\,\Sigma_i p_i(q_i p) = \Sigma_{i,j} q(p_j(q_j p_i))(q_i p) = \Sigma_{i,j}(qp_j)(q_j p_i)(q_i p)$. For the last assertion, (ii) implies $P\,\underline{a} = P$, and therefore $\underline{a} = QP = QPQP = \underline{a}^2$.

We close this section now by describing faithfully projective modules over commutative rings.

(4.6) LEMMA. _Let_ P _be a finitely generated module over a commutative ring_ B, _and let_ $\underline{a}$ _be a_ B-_ideal such that_ $P\,\underline{a} = P$. _Then_ $P(1 - a) = 0$ _for some_ $a \in \underline{a}$.

Proof. If $x_1, \ldots, x_n$ generate P we can solve, for each i, $x_i = \Sigma_j x_j a_{ji}$ for suitable $a_{ji} \in \underline{a}$, by hypothesis. The equations $\Sigma_j x_j(\delta_{ji} - a_{ji}) = 0 \quad (1 \leq i \leq n)$ now imply that $x_j d = 0$ $(1 \leq j \leq n)$ (Cramer's rule), where $d = \det(\delta_{ji} - a_{ji}) \equiv 1 \bmod \underline{a}$.

(4.7) PROPOSITION. _Let_ B _be a commutative ring, let_ P _be a projective_ B-_module, and let_ $\underline{a} = \operatorname{Im} g_P = \Sigma qP$ $(q \in Q = \operatorname{Hom}_B(P, B))$. _If_ $\underline{a}$ is _finitely generated, e.g., if_ B _is noetherian or if_ P _is finitely generated, then_ $\underline{a} = eB$ _for an idempotent_ e, _and_ $\operatorname{ann}_B(P) = (1 - e)B$. _Hence_ P _is a generator of_ mod-B _if and only if_ P _is faithful (i.e.,_ $\operatorname{ann}_B(P) = 0$).

Proof. Proposition (4.5) says $\underline{a}^2 = \underline{a}$. Our hypothesis makes Lemma (4.6) available (with $P = \underline{a}$) so that $\underline{a}(1 - e) = 0$ for some $e \in \underline{a}$. Clearly, then $e^2 = e$ and $\underline{a} = e\underline{a} = eB$. Moreover, by (4.5) again, $P\,\underline{a} = P$ so $P(1 - e) = 0$. Write

$e = \Sigma q_i p_i$. Then if $\alpha \in \text{ann}_B(P)$ we have $e\alpha = \Sigma (q_i p_i)\alpha = \Sigma q_i (p_i \alpha) = 0$, so $\alpha = (1 - e)\alpha \in (1 - e)B$. Thus $\text{ann}_B(P) = (1 - e)B$. Finally, P is a generator $\iff \underline{a} = B \iff e = 1 \iff \text{ann}_B(P)$ ($= (1 - e)B$) $= 0$.

(4.8) COROLLARY. <u>A module over a commutative ring is faithfully projective (in the sense of §1) if and only if it is finitely generated, projective, and faithful.</u>

<u>Examples</u>. 1. (cf. example (4.3)). Let B be the ring of matrices of the form $\begin{pmatrix} a & b \\ o & c \end{pmatrix}$ over a field k, and let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $P = eB$ is a finitely generated, projective, and faithful right B-module. However, $\text{Im}\ g_P = P \neq B$, so P is not a generator of mod-B, i.e., P is not faithfully projective. Of course B is not commutative.

2. (Kaplansky). Let B be the (commutative) ring of continuous real-valued functions on the interval [0, 1], and let P be the ideal of all functions which vanish in a neighborhood (depending on the function) of zero. It is known that P is projective. (Just construct p_i and q_i as in (4.5), using multiplication by suitable "plateau" functions for the q_i.) Moreover, it is easy to see that P is faithful. If $\underline{a} = \text{Im}\ g_P$ then $P \subset \underline{a}$, thanks to the linear functional $P \subset B$, and it is not difficult to show even that $P = \underline{a}$. Thus P is not a generator of mod-B, and therefore P is not faithfully projective. Of course P is not finitely generated.

§5. AUTOEQUIVALENCE CLASSES: THE PICARD GROUP

Let R be a commutative ring. If $\underline{\underline{A}}$ is an R-category we define

$$\text{Pic}_R(\underline{\underline{A}})$$

to be the group of isomorphism classes [T] of R-equivalences

T: $\underline{\underline{A}} \longrightarrow \underline{\underline{A}}$. The group law is induced by composition of functors.

If A is an R-algebra we define

$$\mathrm{Pic}_R(A)$$

to be the group of isomorphism classes [P] of invertible left $A \otimes_R A^\circ$-modules. The group law is: $[P]\,[Q] = [P \otimes_A Q]$. It follows from (2.4) and (3.5)-(3) that this is, indeed, a group, with $[P]^{-1} = [\mathrm{Hom}_{\text{mod-}A}(P, A)] = [\mathrm{Hom}_{A\text{-mod}}(P, A)]$. According to Theorem (2.3):

(5.1) PROPOSITION. <u>There are inverse isomorphisms</u>

$$\mathrm{Pic}_R(A) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Pic}_R(\text{mod-}A)$$

$$\alpha[P] = [\otimes_A P], \quad \underline{\text{and}} \quad \beta[T] = [TA]$$

It is intuitively clear that algebra automorphisms of A should contribute to $\mathrm{Pic}_R(\text{mod-}A)$. We shall now indicate how they appear in $\mathrm{Pic}_R(A)$.

For an R-algebra A write $\underline{\underline{\mathrm{Pic}}}_R(A)$ for the category of invertible left $A \otimes_R A^\circ$-modules, and bimodule homomorphisms. Suppose $P \in \underline{\underline{\mathrm{Pic}}}_R(A)$ and $\alpha, \beta \in \mathrm{Aut}_{R\text{-alg}}(A)$. Then we define

$${}_\alpha P_\beta$$

to be the left $A \otimes_R A^\circ$-module whose additive group is P, and whose bimodule structure is given by

$$a \cdot p = \alpha(a)p, \quad p \cdot a = p\beta(a) \quad (p \in P,\ a \in A)$$

Thus $P = {}_1P_1$, for example. Moreover, we clearly have

$${}_\alpha P_\beta \cong {}_\alpha A_1 \otimes_A P \otimes_A {}_1A_\beta$$

Suppose that $P, Q \in \underline{\underline{\mathrm{Pic}}}_R(A)$ and that $f\colon P \longrightarrow Q$ is a left A-isomorphism. Since $A = \text{Hom}_{A\text{-mod}}(P, P)°$, the left A-endomorphism $p \longmapsto f^{-1}(f(p)a)$ must be right multiplication by a unique $\alpha(a) \in A$. In other words

$$f(p\,\alpha(a)) = f(p)a \qquad (p \in \underline{P},\ a \in A)$$

Evidently $\alpha \in \text{Aut}_{R\text{-alg}}(A)$, and this equation therefore can rephrased: $f\colon {}_1P_\alpha \longrightarrow Q$ is a bimodule isomorphism. This proves part (4) of:

(5.2) PROPOSITION. Let A be an R-algebra and let $\alpha, \beta, \gamma \in \text{Aut}_{R\text{-alg}}(A)$.

(1) ${}_\alpha A_\beta \cong {}_{\gamma\alpha}A_{\gamma\beta}$ as bimodules.

(2) ${}_1A_\alpha \otimes_A {}_1A_\beta \cong {}_1A_{\alpha\beta}$ as bimodules.

(3) ${}_1A_\alpha \cong {}_1A_1$ as bimodules $\Longleftrightarrow \alpha \in \text{In Aut}(A)$, the group of inner automorphisms of A.

(4) If $P \in \underline{\underline{\text{Pic}}}_R(A)$ and if $P \cong A$ as left A-modules then $P \cong {}_1A_\alpha$ as bimodules for some $\alpha \in \text{Aut}_{R\text{-alg}}(A)$.

Proof. (1) The map $x \longmapsto \gamma(x)$ is the required isomorphism.

(2) Using (1) we have ${}_1A_\alpha \otimes_A {}_1A_\beta \cong {}_{\alpha^{-1}}A_1 \otimes_A {}_1A_\beta \cong {}_{\alpha^{-1}}A_\beta \cong {}_1A_{\alpha\beta}$.

(3) If $f\colon {}_1A_\alpha \longrightarrow {}_1A_1$ is a bimodule isomorphism then, as a left A-automorphism, $f(x) = xu$, where $u = f(1)$ is a unit in A. Moreover, $f(\alpha(a)) = f(1 \cdot a) = f(1)a$, which gives $\alpha(a)u = ua$, or $\alpha(a) = uau^{-1}$ for all $a \in A$.

Conversely, if $\alpha(a) = uau^{-1}$ for some unit $u \in A$, then

$f(x) = xu$ is a bimodule isomorphism ${}_1A_\alpha \longrightarrow {}_1A_1$.

(4) was already proved above.

The group $\mathrm{Pic}_R(A\text{-mod}) \simeq \mathrm{Pic}_R(A)$ operates on the isomorphism classes of faithfully projective left A-modules. We now describe the stability subgroups of this action.

(5.3) PROPOSITION. Let A be an R-algebra, let Q be a faithfully projective left A-module, and let $B = \mathrm{End}_A(Q)^\circ$. Then there is an exact sequence of groups,

$$(5.4)\quad 1 \longrightarrow \mathrm{InAut}(B) \longrightarrow \mathrm{Aut}_{R\text{-alg}}(B) \xrightarrow{\delta_Q} \mathrm{Pic}_R(A)$$

with

$$\mathrm{Im}\delta_Q = \{[P] \in \mathrm{Pic}_R(A) \mid P \otimes_A Q \simeq Q \text{ as left A-modules}\}$$

Proof. Suppose first that $Q = A$, so that $B = A$. Define $\delta_A(\alpha) = [{}_1A_\alpha]$. Then Proposition (5.2) tells us that δ_A is a homomorphism (part(2)), that (5.4) is exact (part (3)), and that $\mathrm{Im}(\delta_A)$ is as described above (part (4)).

In the general case set $Q^* = \mathrm{Hom}_A(Q, A)$. Then $T = \mathrm{Hom}_A(Q, \cdot) \simeq Q^* \otimes_A \cdot$: A-mod $\longrightarrow$ B-mod is an equivalence with $TQ = B$, and it induces an isomorphism h: $\mathrm{Pic}_R(A) \longrightarrow \mathrm{Pic}_R(B)$ by $h[P] = [Q^* \otimes_A P \otimes_A Q]$. We now define δ_Q as the composite $\mathrm{Aut}_{R\text{-alg}}(B) \xrightarrow{\delta_B} \mathrm{Pic}_R(B) \xrightarrow{h^{-1}} \mathrm{Pic}_R(A)$. Hence $\mathrm{Ker}(\delta_Q) = \mathrm{Ker}(\delta_B) = \mathrm{InAut}(B)$, so (5.4) is exact. If $P \in \underline{\underline{\mathrm{Pic}}}_R(A)$ then $P \otimes_A Q \simeq Q$ as left A-modules $\Longleftrightarrow$ $Q^* \otimes_A P \otimes_A Q \simeq Q^* \otimes_A Q$ as left B-modules. Since $Q^* \otimes_A Q \simeq B$ as bimodules this says that $P \otimes_A Q \simeq Q$ as left A-modules $\Longleftrightarrow$ $h[P] \in \mathrm{Im}\ \delta_B$. This establishes the alleged description of $\mathrm{Im}(\delta_Q)$, thus proving the proposition.

When $P \in \underline{\underline{\mathrm{Pic}}}_R(A)$ the elements of C = center A need not act the same on the left and right on P. If $t \in C$ then $p \longmapsto pt$, being a bimodule endomorphism of P, is left-multiplication by a unique element, $\alpha_P(t)$, which must again be in C. Thus we have, what is clearly an R-algebra homomorphism,

$$\alpha_P : \ C \longrightarrow C; \quad pt = \alpha_P(t)p \quad (p \in P,\ t \in C)$$

For example if $P = {}_1A_\alpha$ ($\alpha \in \mathrm{Aut}_{R\text{-alg}}(A)$) then $\alpha_P = \alpha|C$.

Suppose $P, Q \in \mathrm{Pic}_R(A)$. Then for $p \in P$, $q \in Q$, and $t \in C$ we have $(p \otimes q)t = p \otimes \alpha_Q(t)q = p\alpha_Q(t) \otimes q = \alpha_P(\alpha_Q(t))(p \otimes q)$. Thus

$$\alpha_{P \otimes_A Q} = \alpha_P \alpha_Q$$

Evidently $\alpha_A = 1_C$, so the invertibility of P now implies α_P is an automorphism. We have now proved:

(5.4) PROPOSITION. <u>Let</u> A <u>be an</u> R-<u>algebra with center</u> C. <u>Then there is an exact sequence</u>

$$0 \longrightarrow \mathrm{Pic}_C(A) \longrightarrow \mathrm{Pic}_R(A) \xrightarrow{h} \mathrm{Aut}_{R\text{-alg}}(C)$$

<u>where</u> $h[P] = \alpha_P$. <u>If</u> A <u>is commutative then</u>

$$0 \longrightarrow \mathrm{Pic}_A(A) \longrightarrow \mathrm{Pic}_R(A) \xrightarrow{h} \mathrm{Aut}_{R\text{-alg}}(A) \longrightarrow 1$$

<u>is exact, and h is split by</u> $\alpha \longmapsto [{}_1A_\alpha]$ (see (5.2)).

EXAMPLE. Let A be the ring of algebraic integers in a finite extension L of $\underline{\underline{Q}}$, and let G be the (Galois) group of field automorphisms of L. Evidently we can identify $G = \mathrm{Aut}_{\underline{\underline{Z}}\text{-alg}}(A)$, so that $\mathrm{Pic}_{\underline{\underline{Z}}}(A)$ is the semidirect product of G with $\mathrm{Pic}_A(A)$, which is known to be isomorphic to the ideal class group of A (see Chapter III, §7). Under this isomorphism the action of G on $\mathrm{Pic}_A(A)$ corresponds to the obvious action of field automorphisms on ideal classes.

If we take this as a description of autoequivalences of the category A-mod then we find that $\mathrm{Pic}_{\mathbb{Z}}(A\text{-mod})$ is finite (finiteness of class number: see §4 of Chapter X). In particular $\mathrm{Pic}_{\mathbb{Z}}(\mathbb{Z}\text{-mod}) = \{1\}$, i.e., any autoequivalence of the category of Abelian groups is isomorphic to the identity functor.

HISTORICAL REMARKS

Fragments of the material in this chapter have occurred, in disguised form, in many places. The questions were first clearly posed and treated systematically by Morita [1], and the basic results are sometimes called the "Morita Theorems". I have borrowed much from an unpublished exposition of S. Chase and S. Schannel, as well as from Gabriel [1].

This material leads, in a natural way, to a general form of the Wedderburn theory (see Chapter III, §1 below) and to the theory of the Brauer group of a commutative ring. This is the theme pursued in my Tata notes (Bass [4]).

Chapter III
REVIEW OF SOME RING AND MODULE THEORY

In this lengthy chapter we review a number of more or less standard topics, as may be seen from the following section titles.

§1 Semi-simplicity and Wedderburn theory.

§2 Jacobson radical and idempotents.

§3 Chain conditions, spec, and dimension.

§4 Localization, support.

§5 Integers.

§6 Homological dimension of modules.

§7 Rank, Pic, and Krull rings.

§8 Orders in semi-simple algebras.

Much of this material occurs in one or another chapter of Bourbaki. In particular, in §5 and §7 I have lifted a great deal from Bourbaki, especially from his beautiful Chapter 7 on divisors. On the other hand, a certain amount of the material here is either not standard or else not easily accessible in a form suitable for the applications to be made here.

The reader is advised to pass over this chapter and to refer to particular sections as they become relevant to the later exposition.

§1. SEMI-SIMPLICITY AND WEDDERBURN THEORY

Let A be a ring. We call $M \in$ mod-A *simple* if it has precisely two submodules (0 and M), and we call M *semi-simple* if it is a direct sum of simple modules. The ring A is called *semi-simple* if it is a semi-simple right A-module. We shall shortly see that this notion is left-right symmetric.

(1.1) PROPOSITION. *Let $M \in$ mod-A and let N and $\{S_i \mid i \in I\}$ be submodules such that each S_i is simple and such that $M = N + \Sigma S_i$. Then there is a subset $J \subset I$ such that the map*

$$f_J: N \oplus \coprod_{j \in J} S_j \longrightarrow M$$

induced by the inclusions, is an isomorphism.

Proof. By Zorn's lemma we can choose J maximal so that f_J is injective. If it is not surjective there is an $i_0 \in I - J$ such that $S_{i_0} \not\subset \mathrm{Im}(f_J)$. Since S_{i_0} is simple we must have $S_{i_0} \cap \mathrm{Im}(f_J) = 0$, and this implies $f_{J \cup \{_0\}}$ is injective, contradicting the maximality of J.

(1.2) COROLLARY. *A sum of simple modules is semi-simple. A submodule of a semi-simple module is a direct summand, and therefore is also semi-simple.*

(1.3) LEMMA ("Schur's Lemma"). *A non zero homomorphism between simple modules is an isomorphism.*

Proof. Let $f: S \longrightarrow T$ where S and T are simple and $f \neq 0$. Then $\mathrm{Im}(f) \neq 0$ so $\mathrm{Im}(f) = T$. Moreover $\mathrm{Ker}(f) \neq S$ so $\mathrm{Ker}(f) = 0$.

(1.4) PROPOSITION. *Let P be a finitely generated semi-simple module. Then there is a direct sum decomposition unique up to isomorphism, $P \simeq S_1^{n_1} \oplus \ldots \oplus S_r^{n_r}$, where the S_i*

are pairwise non-isomorphic simple modules and each $n_i > 0$. *Moreover*

$$\mathrm{End}_A(P) = \Pi\, M_{n_i}(D_i)$$

where $D_i = \mathrm{End}_A(S_i)$ *is a division ring for each* i.

Proof. P is a direct sum of simple modules, and this sum must be finite because P is finitely generated. Hence we obtain a decomposition $P = S_1^{n_1} \oplus \cdots \oplus S_r^{n_r}$ as above after collecting each group of isomorphic summands into a term of the form $S_i^{n_i}$. By Shur's lemma D_i is a division ring and $\mathrm{Hom}_A(S_i, S_j) = 0$ if $i \neq j$. Since D_i is local the uniqueness of the decomposition follows from the Krull Schmidt theorem (I, (3.6)). Moreover we have $\mathrm{End}_A(\underline{p}) = \Pi_{i,j} \mathrm{Hom}_A(S_i^{n_i}, S_j^{n_j})$ $= \Pi \mathrm{End}_A(S_i^{n_i}) = \Pi M_{n_i}(D_i)$. q.e.d.

We call a module *Artinian* if any non empty family of submodules contains minimal elements. We call the ring A *right Artinian* if A is an Artinian right A-module.

(1.5) THEOREM. *The following conditions on a ring* A *are equivalent*.

(1) A *is semi-simple*.
(2) *Every right* A-*module is semi-simple*.
(3) *Every short exact sequence of right* A-*modules splits*.
(4) A *is a finite product of full matrix rings over division rings*.
(5) A *is right Artinian and has no nonzero nilpotent two sided ideals*.

Proof. (1) ⇒ (2). Every module, being a quotient of a free module, is a sum of simple modules. Now apply (1.2).

(2) ⇒ (3) follows from (1.1).

(3) ⇒ (1). Let J be the largest semi-simple right ideal in A, i.e., by (1.2), the sum of all simple right ideals. By hypothesis $A = J \oplus J'$ for some right ideal J', which is clearly generated by one element. If $J' \neq 0$ let $J'' \subset J'$ be a maximal submodule (use Zorn). Then $J' = J'' \oplus S$ for some $S \simeq J'/J''$. Since S is simple $S \subset J$; contradiction. Therefore $J' = 0$, i.e., $J = A$.

(1) ⇒ (4) follows immediately from (1.4).

(4) ⇒ (5). It suffices to establish (4) for $M_n(D) = \text{End}_D(D^n)$, where D is a division ring. According to (II, 4.4) and (II, 3.5) $M_n(D)$ has the same lattice of two sided ideals as D, so it is simple. Moreover the lattice of right ideals is isomorphic to the lattice of D-subspaces of D^n, so it is Artinian. (Of course these facts are easy to prove directly, without appeal to Chapter II.)

The implication (5) ⇒ (1) is contained in the following more general proposition, in the special case $B = A = T$.

(1.6) PROPOSITION. *Let* T *be a two sided ideal in a ring* B. *Assume that* T *is an Artinian right* B-*module and that every nilpotent two sided* B-*ideal has zero intersection with* T. *Then* T *is a semi-simple right* B-*module generated by a central idempotent* e. *Hence* B *is the product of* $B/(1 - e)B \simeq T$ *and of* B/eB.

The proof is based on the following useful Lemma.

(1.7) LEMMA. *Let* P *be a minimal non zero right ideal in a ring* B. *Then either* $P^2 = 0$, *and* BP *is a nilpotent two sided ideal, or else* $P = eB$ *for some idempotent* e, *and* $B = P \oplus (1 - e)B$.

Proof. If $P^2 = 0$ then $(BP)^2 = BPBP = BP^2 = 0$. If

$P^2 \neq 0$ choose $x \in P$ so that $xP \neq 0$. By Schur's lemma, $p \longmapsto xp$ is an automorphism of P, so $xe = x$ for a unique $e \in P$. Since $xe^2 = xe = x$ we have $e^2 = e \neq 0$. Now $eB \subset P = eP$, and the lemma follows immediately.

Proof of (1.6). We claim every right B-submodule of T is semi-simple and is a direct summand of B. If not let $\underline{a} \subset T$ be a minimal counter-example, and let $P \subset \underline{a}$ be a minimal non-zero right ideal. If $P^2 = 0$ then BP is a nilpotent two sided ideal in T, contrary to hypothesis. Therefore (1.7) implies $P = eB$ for some idempotent e. It follows that $\underline{a} = P \oplus (1 - e)\underline{a}$. By the minimality of $\underline{a}$, $(1 - e)\underline{a}$ is semi-simple and a direct summand of B, say $B = (1 - e)\underline{a} \oplus \underline{b}$. Hence $P = eB = e\underline{b}$ is a direct summand of B, so $\underline{a} = P \oplus (1 - e)\underline{a}$ is a direct summand of B. This contradicts the fact that $\underline{a}$ was a counterexample to our claim.

We now know that T itself is a semi-simple direct summand of B, say $T = eB$ with $e^2 = e$. Let $L: B \longrightarrow A = \mathrm{End}_B(T)$ be the map defined by left multiplication (recall T is a two sided ideal). Since T is a direct summand of B, L is surjective. Hence L(T) is a two sided ideal in A. Since $1_T = L(e)$ it follows that $L(T) = A$. If $x \in T \cap \mathrm{Ker}(L)$ then $xT = 0$ so $T \times B$ is a nilpotent two sided ideal in T. By assumption then $T \times B = 0$, so also $x = e \cdot x \cdot 1 = 0$. Therefore L induces an isomorphism $T \longrightarrow A$. In particular $et = te$ for all $t \in T$. If $b \in B$ then eb and be are in T so $eb = ebe = be$. Thus e is central, and the proposition is proved.

(1.8) THEOREM(Wedderburn). Let B be a ring and suppose there is a simple generator P of mod-B (cf. (1.9) below). Then P is a faithfully projective right B-module, and:

(1) $A = \mathrm{End}_B(P)$ is a division ring. (Schur)

(2) P _is a finite dimensional left_ A-_module and_ $B = \text{End}_A(P)^\circ$. (Density theorem.)

(3) Center (B) = Center (A), _and it is a field_.

(4) B _has no two sided ideals except_ 0 _and_ B (i.e., B _is simple_) _and the lattice of left ideals of_ B _is isomorphic to the lattice of_ A-_submodules of_ P, _via_ $\mathfrak{a} \rightarrowtail P\,\mathfrak{a}$.

(5) $P \otimes_B$: B-mod ⟶ A-mod _is an equivalence of categories_.

Conversely, if A _is a division ring and if_ P _is a non-zero finite dimensional left_ A-_module, then_ P _is a faithfully projective simple right module over_ $B = \text{End}_A(P)^\circ$, _and_ $A = \text{End}_B(P)$.

Proof. If mod-B has a semi-simple generator then every module, being a quotient of a semi-simple module, is semi-simple, by (1.2). Therefore P above is projective, since all B-modules are, and it is finitely generated, being simple. This means that P is faithfully projective, so (II, 4.4) says P gives rise to a set of equivalence data, $(A, B, P, \text{Hom}_B(P, B), f_P, g_P)$. The conditions 1,..., 5 above now follow from (II, 3.5).

The converse assertion follows similarly once one verifies that P is faithfully projective, and the latter is obvious.

We close this section with a criterion for the existence of a module P as in (1.8).

(1.9) PROPOSITION. _Suppose the ring_ B _has no idempotent two sided ideals except_ 0 _and_ B _and no non-zero nilpotent two sided ideals_. (_E.g., assume_ B _is simple_.) _Suppose further that_ B _has a minimal non-zero right ideal_ P.

(*E.g., assume* B *is right Artinian.*) *Then* P *is a simple generation of* mod-B, *so Theorem* (1.8) *applies*.

Proof. Our hypothesis prevent $P^2 = 0$ so (1.7) implies $P = eB$ with $e^2 = e$. Then BeB is an idempotent two sided ideal so BeB = B, and this implies P is a generator of mod-B (see (II, 4.4)).

The results of this section are preliminary to the study of the "Brauer group" of a field (cf., e.g., Auslander-Goldman [1]). We shall not take this up here, but we do mention one fact that properly belongs to that theory: Let A and B be finite algebras over a field L which are simple and have center L. Thus they are "central simple" L-algebras. Then $A \otimes_L B$ is also a central simple L-algebra. (cf. Bourbaki [2], §10, no. 4 or Bass [4] Ch.III, Cor. 2.7.) Using this we can prove:

(1.10) PROPOSITION. *Let* L *be a field and let* A *be a semi-simple finite* L-*algebra with center* C. *Then* $\mathrm{Pic}_L(A)$ *is isomorphic to a subgroup of* $\mathrm{Aut}_{L\text{-alg}}(C)$, *which is a finite group*.

Proof. From (II, 5.4) we have an exact sequence $0 \longrightarrow \mathrm{Pic}_C(A) \longrightarrow \mathrm{Pic}_L(A) \longrightarrow \mathrm{Aut}_{L\text{-alg}}(C)$, so it suffices to show that $\mathrm{Pic}_C(A) = 0$. Write $A = \Pi A_i$ as a product of simple algebras A_i, with center C_i, and $C = \Pi C_i$. There is a homomorphism from $\mathrm{Aut}_{L\text{-alg}}(C)$ to the group of permutations of the C_i's, whose kernel is the product of the (finite) galois groups of the field extensions C_i/L. Hence $\mathrm{Aut}_{L\text{-alg}}(C)$ is finite.

Next note that $\mathrm{Pic}_C(A) = \Pi \mathrm{Pic}_{C_i}(A_i)$, clearly, so it suffices to prove that $\mathrm{Pic}_C(A) = 0$ when C is a field and A is central simple. Let $P \in \underline{\underline{\mathrm{Pic}}}_C(A)$. Viewing P as a right A-module we have $A = \mathrm{End}_A(P)$. Since A is simple this can happen only if $P \simeq A$ as right A-module. In particular

$[P: C] = [A: C]$. Now let $Q \in \underline{\underline{\text{Pic}}}_C(A)$ also. Then P and Q are left $A \otimes_C A^\circ$-modules both of dimension $[A: C]$. Since $A \otimes_C A^\circ$ is simple (see remark above) it follows that $P \simeq Q$ as $A \otimes A^\circ$-modules, i.e., as two sided A-modules. Hence $[P] = [Q]$ in $\text{Pic}_C(A)$. q.e.d.

§2. JACOBSON RADICAL AND IDEMPOTENTS

For a ring A and an $M \in$ mod-A we write

$$\text{rad}M = \bigcap \text{Ker}(h) \qquad (h: M \longrightarrow S;\ S \text{ simple})$$

Suppose $g: N \longrightarrow M$. Then $hg: N \longrightarrow S$ so $hg(\text{rad}N) = 0$ for all h as above. Thus $g(\text{rad } N) \subset \text{rad } M$, so rad is a subfunctor of the identity. In particular rad M is a fully invariant submodule of M. Applying this observation to left multiplications in A we see that rad A <u>is a two sided ideal</u>.

If $J \subset \text{rad } A$ is a two sided ideal then J is contained in every maximal right ideal of A so it follows easily that $\text{rad}(A/J) = (\text{rad } A)/J$. In particular

$$\text{rad}(A/\text{rad } A) = 0$$

If S is a simple right A-module and $x \in S$ define $f: A \longrightarrow S$ by $f(a) = xa$. Then $f(\text{rad } A) = 0$ so, since x was arbitrary, $S\cdot\text{rad } A = 0$. If $h: M \longrightarrow S$ is any homomorphism then $h(M \text{ rad } A) \subset S\cdot\text{rad } A = 0$. Thus, for any $M \in$ mod-A,

$$M\cdot\text{rad } A \subset \text{rad } M$$

(2.1) PROPOSITION. *Let* N *be a submodule of* $M \in \underline{\underline{M}}(A)$. *The following conditions are equivalent*:

(1) $N \subset \text{rad } M$.

(2) *If* H *is a submodule of* M *then* $N + H = M \Rightarrow H = M$.

Proof. (1) $\Rightarrow$ (2). Suppose $H \neq M$. Since M is finitely generated we can, by Zorn's lemma, find a maximal proper submodule L containing H. Since $N \subset \text{rad } M$ we have $N \subset L$, so $N + H \subset L$; contradiction.

Conversely, (2) clearly implies N is contained in every maximal proper submodule, hence in their intersection, which is rad M.

(2.2) PROPOSITION ("Nakayama's Lemma"). *The following conditions on a right ideal* J *in* A *are equivalent*:

(1) $J \subset \text{rad } A$.

(2) *If* $M \in \underline{\underline{M}}(A)$ *then* $MJ = M \Rightarrow M = 0$.

(2′) *If* $M \in \underline{\underline{M}}(A)$ *and if* H *is a submodule of* M, *then* $M = H + MJ \Rightarrow M = H$.

(3) $1 + J$ *consists of invertible elements* (*so* $1 + J$ *is a subgroup of* U(A)).

Proof. Since $M(\text{rad } A) \subset \text{rad } M$ for all M it follows from (2.1) that (1) $\Rightarrow$ (2′) and, in the special case M = A, that (2′) $\Rightarrow$ (1). Trivially (2′) $\Rightarrow$ (2) and conversely (2′) follows by applying (2) to M/H.

(2′) $\Rightarrow$ (3). If $x \in J$ set $u = 1 + x$. Then $A = J + uA$ so $A = uA$. Choose v so that $uv = 1$. Since $1 = uv = v + xv$ we have $v = 1 - xv \in 1 + J$ also, so v itself has a right inverse. Thus u is invertible and $v = u^{-1} \in 1 + J$.

(3) $\Rightarrow$ (1). We claim J is contained in every maximal right ideal H. If not then $J + H = A$ so $1 = x + y$ with $x \in J$, $y \in H$. Then $y = 1 - x$ is invertible, by (3), so $H = A$; contradiction.

(2.3) COROLLARY. rad A *is the intersection of the maximal left ideals in* A.

Proof. Let J be that intersection. Since rad A is a two sided ideal and $1 + \text{rad } A \subset U(A)$ we have $\text{rad } A \subset J$, by the left sided analogue of (2.2). By symmetry $J \subset \text{rad } A$.

(2.4) COROLLARY. *A nil ideal* (i.e., *one in which every element is nilpotent*) *is in* rad A.

Proof. If $x^n = 0$ then $(1 - x)^{-1} = 1 + x + \cdots + x^{n-1}$.

(2.5) COROLLARY. Let R be a commutative ring and let A be a finite R-algebra. Then $A(\text{rad } R) \subset \text{rad } A$.

Proof. Suppose $M \in \underline{\underline{M}}(A)$ and $M\cdot(\text{rad } R) = M$. Since $M \in \underline{\underline{M}}(R)$ also we conclude from (2.2)(2) that $M = 0$ and hence $A(\text{rad } R) \subset \text{rad } A$.

(2.6) PROPOSITION. Let P be a faithfully projective right A-module. Then $\text{rad } P = P\cdot(\text{rad } A)$ and $\text{rad}(\text{End}_A(P)) = \text{Hom}_A(P, \text{rad } P)$. In particular, $\text{rad } M_n(A) = M_n(\text{rad } A)$.

Proof. $M \longmapsto \text{rad } M$ and $M \longmapsto M\cdot(\text{rad } A)$ are additive functors that agree on $M = A$ and therefore on all $P \in \underline{\underline{P}}(A)$. If P is faithfully projective and $B = \text{End}_A(P)$ then $h(M) = \text{Hom}_A(P, M)$ is an equivalence from mod-A to mod-B. In particular $h(\text{rad } M) = \text{rad } h(M)$. q.e.d.

(2.7) COROLLARY. If J is a two sided ideal in rad A then $GL_n(A) \longrightarrow GL_n(A/J)$ is surjective for all $n \geq 1$.

Proof. If $u \in A$ lands in $U(A/J)$ we can solve $uv \equiv vu \equiv 1$ rad J in A. Then $uv, vu \in 1 + J \subset U(A)$ so $u \in U(A)$. Thus $U(A) \longrightarrow U(A/J)$ is surjective. Now apply this to $M_n(A) \longrightarrow M_n(A/J) = M_n(A)/M_n(J)$, using the fact that $M_n(J) \subset \text{rad } M_n(A)$ (see (2.6)).

Remark. This proof actually shows that $GL_n(A)$ is the universe image in $M_n(A)$ of $GL_n(A/J)$,

We shall call a ring A semi-local if A/rad A is semi-simple. Since A/rad A has zero radical it can have no non-zero nilpotent ideals. Hence it follows from (1.5) that A is semi-local as soon as A/rad A is right Artinian.

Moreover it is then a finite product of full matrix rings over division rings. It follows from (2.6) that $M_n(A)$ is semi-local if A is. We call A _local_ if A/rad A is a division ring. Note that this is equivalent to the definition in (I, §3). For a local ring $A = U(A) \cup \text{rad } A$.

The following proposition will be used frequently in Chapters IV and V.

(2.8) PROPOSITION. _Let_ $\underline{a}$ _be a right ideal in a semi-local ring_ A. _Let_ $b \in A$ _be such that_ $\underline{a} + bA = A$. _Then_ $\underline{a} + b$ _contains a unit of_ A.

Proof. An element of A is invertible if and only if it is invertible modulo rad A (see remark above). Hence we can, after passing to A/rad A, assume rad A = 0. Then A decomposes into a product so it suffices to solve our problem in each factor. Therefore, we can further assume $A = \text{End}_D(V)$ where V is a finite dimensional right vector space over a division ring D. In this case $\underline{a}$ is the set of all $a\colon V \longrightarrow V$ such that $aV \subset W = \underline{a}V$ (see, e.g., (II, 3.5 (6))). Since $\underline{a} + bA = A$ it follows that $W + \text{Im}(b) = V$. Choose $W_0 \subset W$ so that $V = W_0 \oplus \text{Im}(b)$. If $V = \text{Ker}(b) \oplus U$ then b induces an isomorphism from U to Im(b), so $\text{Ker}(b) \simeq W_0$. Choose a so that $aU = 0$ and a induces an isomorphism from Ker(b) to W_0. Then $aV = W_0 \subset W$ so $a \in \underline{a}$. Moreover $a + b$ is clearly an automorphism of V.

(2.9) COROLLARY. _Let_ q _be a two sided ideal in a semi-local ring_ A. _Then_ $GL_n(A) \longrightarrow GL_n(A/q)$ _is surjective for all_ $n \geq 1$.

Proof. If $u \in A$ is invertible modulo q then $q + uA = A$, so $q + u$ contains a unit of A. Thus $U(A) \longrightarrow U(A/q)$ is surjective. The corollary follows by applying this to $M_n(A)$, which is also semi-local.

We next treat the problem of lifting idempotents.

(2.10) PROPOSITION. Let J be a two sided ideal in a ring A. Suppose either that J is nil or that A is J-adically complete (i.e., A ⟶ proj. lim A/J^n is an isomorphism). Then finite sets of orthogoral idempotents can be lifted modulo J. I.e., given $a_1, \ldots, a_m \in A$ such that $a_i a_j \equiv \delta_{ij} a_i$ mod J $(1 \leq i, j \leq m)$ then there exist $e_1, \ldots, e_m \in A$ such that $e_i \equiv a_i$ mod J and such that $e_i e_j = \delta_{ij} e_i (1 \leq i, j \leq m)$.

Proof. Let $a \in A$. For any $n > 0$,

$$1 = (a + (1 - a))^{2n} = \Sigma_{0 \leq j \leq 2n} \binom{2n}{j} a^{2n-j} (1-a)^j$$

Set

$$f_n(a) = \Sigma_{0 \leq j \leq n} \binom{2n}{j} a^{2n-j} (1 - a)^j$$

$$= 1 - \Sigma_{n < j \leq 2n} \binom{2n}{j} a^{2n-j} (1 - a)^j$$

Then f_n is a polynoimal in a with integer colfficients, i.e. it lies in the ring R generated by a, and we have:

$$f_n(a) \equiv 0 \bmod a^n R;$$

$$f_n(a) \equiv 1 \bmod (1 - a)^n R$$

These imply $f_n(a)^2 \equiv f_n(a) \bmod (a(1 - a))^n R$. Since $a^n R + (1 - a)^n R = R$ clearly it follows that $(a(1 - a))^n R = a^n R \cap (1 - a)^n R$ (cf. (2.14) below). Hence we also conclude that $f_n(a) \equiv f_{n-1}(a) \bmod (a(1 - a))^{n-1} R$. At the outset we have $f_1(a) = \binom{2}{0} a^2 + \binom{2}{1} a(1 - a) = a^2 + 2a(1 - a) = 2a - a^2 = a + a(1 - a) \equiv a \bmod a(1 - a)R$. Thus $f_n(a) \equiv a \bmod a(1 - a)R$.

Now suppose $\alpha^2 - \alpha = \alpha(1 - \alpha)$ is nilpotent. Then the congruences above show that, for large n, we have $f_n(\alpha) \equiv \alpha \bmod (\alpha^2 - \alpha)R$, and $f_n(\alpha)^2 = f_n(\alpha)$. This shows we can lift an idempotent modulo a nil ideal J (for we are then given α with $\alpha^2 - \alpha \in J$.) If, on the other hand, A is J-adically complete, then we can inductively construct $e_n \in A$ such that $e_1 = \alpha$, $e_n^2 \equiv e_n \bmod J^n$, and $e_{n+1} \equiv e_n \bmod J^n$. This is because J/J^n is nilpotent, and we can use the construction above. Now the e_n converge to an $e \in A = \text{proj. lim } A/J^n$ such that $e \equiv \alpha \bmod J$ and $e^2 = e$. This proves the proposition for a single idempotent.

In general, we suppose, by induction, that $e_1, \ldots, e_{m-1}$ have been constructed as in the proposition. Then $e = e_1 + \cdots + e_{m-1}$ is idempotent and $e \equiv \alpha_1 + \cdots + \alpha_{m-1} \bmod J$. Therefore e and α_m are orthoqural idempotents mod J. Set $f = 1 - e$ and $b = f\alpha_m f$. Then evidently $b \equiv \alpha_m \bmod J$ and $eb = 0 = be$. Form the sequence $f_n(b)$ as above, so that the $f_n(b)$ converge to an idempotent e_m such that $e_m \equiv b \bmod J$. Since each $f_n(b)$ is a poloyminal in b with zero constant term and integer coefficients we have $ef_n(b) = 0 = f_n(b)e$. Therefore e and e_m are orthogonal. For $i < m$ we have $e_i e = e_i = ee_i$ so it follows that e_m is orthogonal to these e_i. q.e.d.

(2.11) PROPOSITION. Let A be a right Artinian ring, and let $J = \text{rad } A$. Every non-nilpotent right ideal in A contains a non-zero idempotent; in particular J is nilpotent. Moreover, A/J is semi-simple.

Proof. Since A/J has no non-zero nilpotent ideals its semi-simplicity follows from (1.5). If $e \in J$ is is idempotent then $eA = (eA)^2$ so $eA = 0$ by Nakayama's lemma. Therefore the first assertion implies J is nilpotent.

If the first assertion is false let the right ideal I be a minimal counterexample. Then since $I^2 \subset I$ is not nilpotent we have $I^2 = I$. Let H be minimal among the right ideals in I (e.g., I itself) such that $HI \neq 0$. Choose $x \in H$ such that $xI \neq 0$; then minimality implies $xI = H$. Choose $\alpha \in I$ so that $x\alpha = x$. Then $x\alpha^2 = x\alpha$ so $\alpha^2 - \alpha \in N = \{y \in I \mid xy = 0\}$. Since $N \subsetneq I$ the minimality of I implies N is nilpotent. Hence $\alpha^2 - \alpha$ is nilpotent. Let R be the subring generated by α. By (2.10) there is an idempotent $e \in R$ such that $e \equiv \alpha \bmod (\alpha^2 - \alpha)R$. In particular $e \equiv \alpha \bmod N$. Since $\alpha \in I$, $\alpha \notin N$ we have $e \in I$, $e \notin N$ also, and this completes the proof.

(2.12) PROPOSITION. *Let* J *be a two sided ideal contained in* rad A. *Set* $\overline{A} = A/J$ *and write* $\overline{M} = M \otimes_A \overline{A} = M/MJ$ *for* $M \in$ mod-A. *Then*

$$\overline{}: \underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}(\overline{A})$$

is a full additive functor with the following properties:

(a) *If* $f: P \longrightarrow Q$ *is a morphism in* $\underline{\underline{P}}(A)$ *such that* $\overline{f}$ *is an isomorphism then* f *is an isomorphism*.

(b) *The functor is injective on isomorphism classes of objects, and bijective if* A *is* J-*adically complete*.

Proof. Given $\overline{f}: \overline{P} \longrightarrow \overline{Q}$ there is an $f: P \longrightarrow Q$ making

$$\begin{array}{ccc} P & \xrightarrow{\quad f \quad} & Q \\ \downarrow & & \downarrow \\ \overline{P} & \xrightarrow{\quad \overline{f} \quad} & \overline{Q} \end{array}$$

commute, i.e., making the notation consistent. This is because P is projective and $Q \longrightarrow \overline{Q}$ is surjective. Thus the functor is full. If $\overline{f}$ is surjective then it follows from Nakayama's lemma, since Q is finitely generated, that f is surjective. The projectivity of Q now implies f is a split epimorphism. It follows that H = Ker(f) is finitely generated, being a direct summand of P. But $\overline{H} = 0$ because $\mathrm{Ker}(\overline{f}) = 0$ so again Nakayama implies H = 0. This proves (a) and shows that $\overline{P} \simeq \overline{Q} \Rightarrow P \simeq Q$.

There remains only to be shown that every $Q \in \underline{\underline{P}}(\overline{A})$ is isomorphic to some $\overline{P}$ ($P \in \underline{\underline{P}}(A)$) when A is J-adically complete. We can write $Q = \mathrm{Im}(\overline{e})$ where $\overline{e}$ is an idempotent in $\mathrm{End}_{\overline{A}}(\overline{A}^n) = M_n(\overline{A})$. If we can lift $\overline{e}$ to an idempotent $e \in M_n(A) = \mathrm{End}_A(A^n)$ then P = Im(e) will clearly solve our problem. Since A is J-adically complete we evidently also have $M_n(A) = \text{proj. lim } M_n(A/J^m)$, so the liftability of $\overline{e}$ follows from (2.10). q.e.d.

(2.13) COROLLARY. _If_ A _is a local ring then every_ $P \in \underline{\underline{P}}(A)$ _is free_.

Proof. Take J = rad A above. Then $\overline{A}$ is a division ring so $\overline{P}$ is free.

Let R be a commutative ring. We say two ideals $\underline{a}$ and $\underline{b}$ are _comaximal_ if $\underline{a} + \underline{b} = R$. In this case, for any M ε mod-R, the inclusion $M\underline{ab} \subset M\underline{a} \cap M\underline{b}$ is an equality. For if $x \in M\underline{a} \cap M\underline{b}$ write $1 = a + b$ ($a \in \underline{a}$, $b \in \underline{b}$) and we have $x = xa + xb \in M\underline{ab}$.

Suppose $\underline{a}$ is comaximal with $\underline{b}_i$ ($1 \le i \le n$). Write $1 = a_i + b_i$ ($a_i \in \underline{a}$, $b_i \in \underline{b}_i$, $1 \le i \le n$). In the product $1 = (a_1 + b_1)\cdots(a_n + b_n)$, all monomials lie in $\underline{a}$ except $b_1 \cdots b_n \in \underline{b}_1 \cdots \underline{b}_n$, so $\underline{a} + (\Pi \underline{b}_i) = R$.

(2.14) PROPOSITION ("Chinese Remainder Theorem," CRT). Let $\underline{a}_i$ $(1 \leq i \leq n)$ be pairwise comaximal ideals in a commutative ring R, and let $M \in$ mod-R. Then

$$\bigcap_i M\underline{a}_i = M \cdot (\Pi_i \underline{a}_i)$$

and

$$M \longrightarrow \amalg M/M\underline{a}_i$$

is surjective (with kernel $\bigcap_i M\underline{a}_i$).

Proof. The case $n = 1$ is trivial, so assume $n > 1$. Set $\underline{a}_i' = \Pi_{j \neq i}\underline{a}_j$. The remark above shows that $\underline{a}_i + \underline{a}_i' = R$ for each i. By induction and the remarks above, we have

$$\begin{aligned} M(\Pi_i \underline{a}_i) &= M\underline{a}_1\, \underline{a}_1' \\ &= M\underline{a}_1 \cap M\underline{a}_1' \\ &= M\underline{a}_1 \cap (\bigcap_{1 \leq i \leq n} M\underline{a}_i) \end{aligned}$$

Suppose we are given $x_1, ---, x_n \in M$. Write $1 = a_i + b_i$ with $a_i \in \underline{a}_i$ and $b_i \in \underline{a}_i'$. Then $b_i \equiv 1 \bmod a_i$ and $b_i \equiv 0 \bmod \underline{a}_j$ for $j \neq i$. Thus $\Sigma x_i b_i \equiv x_i \bmod M\underline{a}_i$ $(1 \leq i \leq n)$. This proves the surjectivity of $M \longrightarrow \amalg M/M\underline{a}_i$, and its kernel is evidently $\bigcap M\underline{a}_i$.

§3. CHAIN CONDITIONS, SPEC, AND DIMENSION

We call a partially ordered set X noetherian if every ascending chain in X terminates. This is easily seen to be equivalent to the "noetherian induction principle": Every non-empty subset of X has maximal elements. Dually, we call X Artinian if every descending chain terminates, and there is an equivalent "Artinian induction principle". If X^0 is the set X with ordering reversed then X is Artinian if and only if X^0 is noetherian.

Suppose now that X is a lattice. This means that each $x, y \in X$ have a supremum, $x \cup y$, and an infimum, $x \cap y$. Moreover, there are a greatest element and a least element, which we denote 1 and 0, respectively.

(3.1) PROPOSITION. _If_ X _is noetherian and Artinian then_ X _has "finite length". Specifically, every finite chain of distinct elements of_ X _can be refined to a chain_ $0 = x_0 < x_1 < \cdots < x_n = 1$ _such that no element lies properly between_ x_{i-1} and x_i $(1 \leq i \leq n)$.

Proof. It suffices to show simply that X has some finite chains as above. For then we can apply this conclusion to the lattices of elements of X lying between two successive elements of a given chain to obtain the necessary refinements.

If the conclusion fails, choose a minimal x such that it fails for the lattice of elements below x. Clearly $x \neq 0$ so we can choose y maximal among the elements strictly smaller than x. Then there is a finite chain, as required, below y, and hence also below x; contradiction.

We shall say that $x \in X$ is $\cup$-_ineducible_ if $x \neq 0$ and if $x = y \cup z \Rightarrow x = y$ or $x = z$.

(3.2) PROPOSITION("Decomposition Lemma"). _Let_ X _be an Artinian lattice and let_ x _be a non-zero element of_ X.

(a) _We can write_ $x = x_1 \cup \cdots \cup x_n$ _where each_ x_i _is_ $\cup$-_irreducible and with no inequalities among the_ x_i's.

(b) *If $\cap$ distributes over $\cup$ then every $\cup$-irreducible $y \leq x$ is $\leq$ some x_i. In particular the x_i's are then unique, and we call them the "irreducible components" of* x.

Proof. (a) If not let x be a minimal counterexample. Clearly $x \neq 0$ and x is not $\cup$-irreducible so $x = y \cup z$ with $y, z < x$. Since y and z are finite unions of $\cup$-irreducible elements (by minimality of x) so is x. After deleting redundant terms we reach the desired contradiction.

(b) If $y \leq x$ then $y = y \cap x = (y \cap x_1) \cup \cdots \cup (y \cap x_n)$. If y is $\cup$-irreducible this implies $y = y \cap x_i$, i.e., $y \leq x_i$, for some i.

(3.3) PROPOSITION. *Let* X *be a lattice and let* $x \in X$ *be such that*

(1) If $y \leq y'$, $y \cap x = y' \cap x$ and $y \cup x = y' \cup x$ then $y = y'$.

Then X *is noetherian (resp., Artinian) if and only if the lattices of elements above and below* x *are*.

Proof. If X is noetherian (resp., Artinian) then the lattices above and below x are clearly also such. Conversely, suppose (y_n) is a chain in X. For large n the chains $(y_n \cap x)$ and $(y_n \cup x)$ are stationary, by hypothesis. Hence condition (1) implies (y_n) itself becomes stationary.

Let A be a ring. We call $M \in$ mod-A *noetherian* or *Artinian* according, as its lattice of submodules is such. This lattice satisfies (1) above. For suppose X and $Y \subset Y'$ are submodules of M such that $Y \cap X = Y' \cap X$ and $Y + X = Y' + X$. Then we apply the 5-lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X/X\cap Y & \longrightarrow & M/Y & \longrightarrow & M/X+Y & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X/X\cap Y' & \longrightarrow & M/Y' & \longrightarrow & M/X+Y' & \longrightarrow & 0\end{array}$$

We call A *right noetherian* or *right Artinian* if the right A-module A is such.

(3.4) PROPOSITION. (1) *A module* M *is northerian and Artinian if and only if it has finite length. If* M *is semi-simple it is noetherian if and only if it is Artinian*.

(2) *If* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *is exact then* M *is noetherian (resp., Artinian) if and only if* M′ and M′′ *are*.

(3) A *is right noetherian (resp., Artinian) if and only if every* $M \in \underline{\underline{M}}(A)$ *is*.

(4) *A module* M *is noetherian if and only if every submodule is finitely generated*.

Proof. (1) The first assertion follows from (3.1). If M is a direct sum of simple modules then clearly M can be noetherian or Artinian only if this sum is finite, in which case M has a Jordan-Holden series, and so has finite length in the sense of (3.1), by the Jordan-Holder Theorem (I, 4.3).

(2) follows from (3.3) and the remarks above.

(3) If A is right noetherian (resp., Artinian) then (2) implies the same is true of A^n and all of its quotients for all $n > 0$. The converse is trivial.

(4) Let $\cdots H_n \subset H_{n+1} \cdots$ be a strictly ascending chain of submodules of M, and let H be their union. If H were finitely generated each generator would lie in some H_n and hence all of them would lie in H_n for large n; contradiction. Thus, if all submodules of M are finitely generated then M is noetherian. Conversely, suppose M is noetherian. If M is not finitely generated set $M_0 = (0)$ and

let M_{n+1} be generated by M_n together with some $x_{n+1} \notin M_n$. This ascending chain is impossible, so M is finitely generated. The submodules of M, being also noetherian, are likewise finitely generated.

The importance of right noetherian rings lies in the fact ((3.4)(4) above) that the category $\underline{\underline{M}}(A)$ of finitely generated right A-modules is abelian.

(3.5) PROPOSITION. A right Artinian ring is also right noetherian.

Proof. Let J = rad A. According to (2.11) A/J is semi-simple and $J^n = 0$ for some n. The modules J^{i-1}/J^i $(1 \leq i \leq n)$ are therefore semi-simple and Artinian, hence of finite length. Thus A is also of finite length, thanks to (2) and induction on n.

(3.6) THEOREM("Hilbert Basis Theorem"). Let A be a right noetherian ring and let t be an indeterminate. Then A[t] is also right noetherian.

Proof. Let J be a right ideal in A[t]. Then the set J_0 of leading coefficients of elements of J, together with 0, is clearly a right ideal in A. Choose $f_1, \text{---}, f_n \in J$ whose leading coefficients generate J_0, and let N be $\geq \deg(f_i)$ for each i. If $g \in J$ has degree $\geq N$ then we can find $g' = \Sigma f_i h_i$ with the same degree and leading coefficient as g, clearly, and then $\deg(g - g') < \deg(g)$. By an induction argument we can thus show that any $g \in J$ is of the form $g = g_0 + g_1$ where $g_1 \in \Sigma f_i A[t]$ and where $\deg(g_0) < N$. In other words $J = \Sigma_i f_i A[t] + (J \cap \Sigma_{j<N} t^j A)$. The second term is an A-submodule of $\Sigma_{j<N} t^j A \simeq A^N$, so it is a finitely generated A-module. Therefore J is a finitely generated A[t]-module. q.e.d.

We now come to some topological considerations preliminary to the introduction of the prime and maximal ideal spectra.

A topological space X is _irreducible_ if $X \neq \emptyset$ and if X is not the union of two proper closed subsets. The latter means that any two non-empty open sets intersect non-trivially.

(3.7) PROPOSITION. (a) _A subspace_ Y _of_ X _is irreducible if and only if its closure_, $\overline{Y}$, _is_.

(b) _Every irreducible subspace of_ X _is contained in a maximal one, and the latter is closed_. X _is the union of the maximal irreducible subspaces, which we call the "irreducible components" of_ X.

Proof. (a) Since Y is dense in $\overline{Y}$ every non-empty open set in $\overline{Y}$ meets Y, and two such which meet must meet in Y as well. (a) follows immediately from this.

(b) An ascending union of irreducible subspaces is irreducible, because two open sets which meet in the union meet already in one of the subspaces. Therefore, by Zorn's lemma, every irreducible subspace is contained in a maximal one. The latter is closed by part (a). The closure of a point is irreducible (by part (a)) so X is the union of the maximal irreducible subspaces. q.e.d.

We call X _noetherian_ if the lattice of open sets is noetherian, or, equivalently, if the lattice of closed sets is Artinian. It is easy to see that a noetherian space is _quasi-compact_, i.e., every open covering has a finite subcovering.

(3.8) PROPOSITION. _A noetherian space_ X _has only finitely many irreducible components. If_ Y _is a subspace of_ X _then_ Y _is also noetherian. If_ $\{Y_i\}$ _are the irreducible components of_ Y _then_ $\{\overline{Y}_i\}$ _are the irreducible components of_ $\overline{Y}$.

Proof. The first assertion follows from the decomposition lemma (3.2). If $(U_n \cap Y)_{n \geq 1}$ is an ascending chain in Y with U_n open then $(V_n = \bigcup_{i \leq n} U_i)_{n \geq 1}$ terminates and hence so does $(U_n \cap Y)$. Thus Y is noetherian. Since $Y = \cup Y_i$ we have $\overline{Y} = \cup \overline{Y}_i$, and each $\overline{Y}_i$ is irreducible, by (3.7). The decomposition lemma now implies the $\overline{Y}_i$ are the irreducible components of $\overline{Y}$ provided we verify that $\overline{Y}_i \neq \overline{Y}_j$ for $i \neq j$. Otherwise we would have $Y_i \subset \overline{Y}_j \cap Y = Y_j$, because Y_j is closed in Y, and this contradicts maximality. q.e.d.

For an irreducible closed subset Y of X we define

$$\operatorname{codim}_X(Y)$$

to be the (possibly infinite) supremum of the lengths, n, of chains $Y = Y_0 \subset Y_1 \subset \cdots \subset Y_n$ of distinct irreducible closed sets above Y in X, If Y is closed but not necessarily irreducible we define $\operatorname{codim}_X(Y)$ to be the infimum of $\operatorname{codim}_X(Y')$ where Y' ranges over all irreducible closed subsets of Y, and we may as well restrict Y' to the irreducible components of Y, clearly. In particular we have $\operatorname{codim}_X(\phi) = \infty$.

If Z is a closed subset of Y then it is easy to see that

$$\operatorname{codim}_X(Z) \geq \operatorname{codim}_Y(Z) + \operatorname{codim}_X(Y)$$

Let A be a commutative ring and write

spec(A)

for the set of prime ideals of A. If $S \subset A$ and if $F \subset \operatorname{spec}(A)$ we write

$$V(S) = \{\mathfrak{p} \mid \mathfrak{p} \supset S\}, \quad I(F) = \bigcap_{\mathfrak{p} \in F} \mathfrak{p}$$

Evidently $I(F)$ is an ideal and V and I are inclusion reversing functions such that $S \subset IV(S)$ and $F \subset VI(F)$. It follows that $V(S) \subset VIV(S) \subset V(S)$, so $VIV = V$, and similarly $IVI = I$. If $I(F) \subset I(F')$ then $F' \subset VI(F') \subset VI(F)$, and conversely $F' \subset VI(F)$ implies $I(F) \subset I(F')$. Similarly $V(S) \subset V(S') \iff S' \subset IV(S)$. We shall abbreviate the notation by writing

$$\overline{F} = V(I(F)) \qquad (F \subset \mathrm{spec}(A))$$

and

$$\sqrt{\mathfrak{a}} = I(V(\mathfrak{a})) \qquad (\mathfrak{a} \text{ an ideal in } A)$$

Thus we can restate the conclusions above:

$$I(F) \subset I(F') \iff F' \subset \overline{F}, \text{ and}$$

$$V(\mathfrak{a}) \subset V(\mathfrak{a}') \iff \mathfrak{a}' \subset \sqrt{\mathfrak{a}}$$

for ideals $\mathfrak{a}, \mathfrak{a}' \subset A$. The notation is suggested by the following proposition.

(3.9) PROPOSITION. *If* $\mathfrak{a}$ *is an ideal in* A

$$\mathfrak{a} = \{a \in A \mid a^n \in \mathfrak{a} \text{ for some } n > 0\}$$

In particular $\sqrt{(0)}$ = nil A *is the ideal of all nilpotent elements in* A. *A finitely generated ideal in* nil A *is nilpotent*.

Proof. Since, evidently, $\sqrt[A]{\mathfrak{a}}$ is the inverse image of $\sqrt[A/\mathfrak{a}]{(0)}$, it suffices to treat the case $\mathfrak{a} = (0)$. Clearly a nilpotent element belongs to every prime ideal, so it remains only to show that a non-nilpotent element, s, is excluded from some prime $\underline{p}$. Let $S = \{s^n \mid n \geq 0\}$. We shall use the localization, $S^{-1}A$, whose construction and properties are discussed in the following section. In particular, since $0 \notin S$ we have $S^{-1}A \neq 0$, so the latter has a prime ideal q, e.g., any maximal ideal will do. Then $\underline{p}$ = "$q \cap A$" is a prime of A excluding S (cf. (4.2) below).

If $a_1, \ldots, a_n \in \text{nil } A$, say $a_i^m = 0$ $(1 \leq i \leq n)$, and if $\underline{a} = \Sigma A a_i$, then it is not difficult to see that $\underline{a}^{nm} = 0$. This proves the last assertion.

If $\underline{a}_i$ is a family of ideals then clearly

$$V(\Sigma \underline{a}_i) = \cap V(\underline{a}_i)$$

Moreover, if $\underline{a}$ and $\underline{b}$ are ideals we have

$$V(\underline{ab}) = V(\underline{a} \cap \underline{b}) = V(\underline{a}) \cup V(\underline{b})$$

For these sets clearly decrease from left to right, while, conversely, if a prime $\underline{p}$ contains $\underline{a}\,\underline{b}$ it must contain $\underline{a}$ or $\underline{b}$.

The formulas above show that we can view spec(A) as a topological space with the Zariski topology, whose closed sets are those of the form V(S). The dimension of this space is called the Krull dimension of A, and it is denoted

$$\dim A = \dim \text{spec}(A)$$

If $\underline{p} \in \text{spec}(A)$ we write

$$\text{ht}(\underline{p}) = \dim(A_{\underline{p}}) \quad (= \text{codim}_{\text{spec}(A)} (V(p))).$$

(3.10) PROPOSITION. spec(A) is quasi-compact, and $\underline{p} \longmapsto V(\underline{p}) = \overline{\{\underline{p}\}}$ is an inclusion reversing bijection from spec(A) to the irreducible closed sets in spec(A). If A is noetherian then spec(A) is a noetherian space.

Proof. If $V(\underline{a}_i)$ is a family of closed sets with empty intersection then $\phi = \cap V(\underline{a}_i) = V(\Sigma\underline{a}_i)$ so $\sqrt{\Sigma\underline{a}_i} = A$. It follows that $1(= 1^n$ for all $n > 0)$ lies in $\Sigma\underline{a}_i$, by (3.9), and hence $1 \in \Sigma'\underline{a}_i = A$ where Σ' refers to some finite sum of the $\underline{a}_i$'s. Therefore $\cap' V(\underline{a}_i) = \phi$, where $\cap'$ denotes the corresponding finite intersection, and this shows that spec (A) is quasi-compact.

Since $\underline{p} = \sqrt{\underline{p}}$ for $\underline{p} \in$ spec (A), $\underline{p} \longmapsto V(\underline{p})$ is injective. Moreover, $V(\underline{p}) = \overline{\{\underline{p}\}}$, the closure of $\{\underline{p}\}$, so it is irreducible. Suppose F is an irreducible closed set. Write $F = V(\underline{a})$ with $\underline{a} = \sqrt{\underline{a}}$. We claim $\underline{a}$ is prime. For say $\underline{a} \supset \underline{b}\ \underline{c}$. Then $F \subset V(\underline{b}\ \underline{c}) = V(\underline{b}) \cup V(\underline{c})$ so $F \subset V(\underline{b})$ or $V(\underline{c})$, (say $F \subset V(\underline{b})$), because F is irreducible. Hence $\underline{b} \subset \sqrt{\underline{a}} = \underline{a}$.

If (F_n) is a decreasing chain of closed sets then $(I(F_n))$ terminates if A is noetherian and hence $(F_n) = (VI(F_n))$ terminates. q.e.d.

Let $f\colon A \longrightarrow B$ be a homomorphism of commutative rings, and let

$$^{a}f\colon \operatorname{spec}(B) \longrightarrow \operatorname{spec}(A),\ ^{a}f(q) = f^{-1}(q)$$

If $S \subset A$ then $^{a}f^{-1}(V(S)) = \{q \in \operatorname{spec}(B) \mid {}^{a}f(q) \supset S\}$ $= \{q \mid q \supset f(S)\} = V(f(S))$. Hence ^{a}f is continuous, so spec is a functor from commutative rings to topological spaces.

In case f is surjective with kernel $\underline{a}$ then ^{a}f is a homeomorphism of $\operatorname{spec}(A/\underline{a})$ onto the closed set $V(\underline{a})$.

We quote, without proof, the following result.

(3.11) THEOREM. (See Serre [2], Ch.III, Prop. 13). _If A is a commutative noetherian ring, and if_ $t_1, \ldots, t_n$ _are indeterminates, then_

$$\dim A[t_1, \ldots, t_n] = n + \dim A$$

(3.12) COROLLARY. *Suppose, above, that* $\dim A < \infty$. *Then any finitely generated commutative* A-*algebra is a noetherian ring of finite dimension*.

Proof. Any such algebra is a quotient of $A[t_1,\ldots,t_n]$ for some $n \geq 0$, so the corollary follows from (3.6) and (3.11).

The corollary applies notably when A is a field or when $A = \mathbb{Z}$. The latter case translates: *a finitely generated commutative ring is noetherian and of finite dimension*. The importance of this observation derives largely from the fact that any commutative ring is a direct union of finitely generated subrings. By this device many propositions can be reduced to the noetherian case, and we shall have occasion to use this procedure.

The maximal ideals of A constitute a subspace

$$\max (A) \subset \operatorname{spec} (A)$$

whose points are just the *closed points* of spec (A). Thus A is semi-local $\Longleftrightarrow$ max (A) is finite, and in this case max (A) is discrete. By the general remarks made earlier, max (A) is noetherian if spec (A) is.

An argument like that in proving (3.10) will show that $F \longmapsto I(F)$ is an (inclusion reversing) bijection from the irreducible closed sets in max (A) to the primes $\underline{p}$ which are intersections of maximal ideals. In particular, $\dim \max (A) \leq \dim \operatorname{spec} (A)$.

Unfortunately, there is no decent analogue of (3.11) for the maximal spectrum. Indeed it turns out that, if A is noetherian and t is an indeterminate, then

$$\dim \max(A[t]) = \dim A[t]$$

This may be arbitrarily large even though A might be local (in which case $\dim \max (A) = 0$). There is, however, the following weak result.

(3.13) PROPOSITION. *Let* A *be a commutative noetherian* *ring of dimension* d, *and assume that* $\dim(A/\mathrm{rad}\,A) < d$. *Let* T *be a free abelian group or monoid on* $n > 0$ *generators. Then* $\max(A[T])$ *is the disjoint union of a closed and an open set, each of dimension* $< d + n$.

Proof. The closed set F defined by rad A is homeomorphic to $\max((A/\mathrm{rad}\,A)[T])$, which has dimension $\dim(A/\mathrm{rad}\,A) + n < d + n$.

To show that the complement of F has dimension $<d + n$ it suffices to show that every $\underline{m} \notin F$ has height $<d + n$. For such an $\underline{m}$ we have rad $A \not\subset \underline{m}$ and hence $\underline{m} \cap A = \underline{p}$ is not maximal. To determine $\mathrm{ht}(\underline{m})$ we can pass to $A[T]_{\underline{m}}$, and the latter is a localization of $A_{\underline{p}}[T]$, which has dimension $<d + n$. q.e.d.

We close this section with a description of the connectivity of spec (A). Note first that the set of idempotents in A is a Boolean algebra, with $e_1 \cap e_2 = e_1e_2$, $e_1 \cup e_2 = e_1 + e_2 - e_1e_2$, and complementation $e \longmapsto (1 - e)$. We can exhibit the resulting partial ordering as follows: $e_1 \leq e_2 \iff e_1e_2 = e_1 \iff e_1A \subset e_2A$. In particular, an ideal can be generated by at most one idempotent. Moreover, if $e^2 = e$ then $eA = \sqrt{eA}$ so $V(eA)$ uniquely determines e.

(3.14) PROPOSITION. *The map* $e \longmapsto V((1 - e)A)$ *is an isomorphism from the Boolean algebra of idempotents in* A *to the Boolean algebra of open and closed sets in* spec (A).

Proof. Since $A = eA \oplus (1 - e)A$ it follows that spec(A) is the disjoint union of $V(eA)$ and $V((1 - e)A)$. Thus $V((1 - e)A)$ is open and closed. The remarks above show that the map is injective. If $e_1 \leq e_2$ then $(1 - e_2) \leq (1 - e_1)$, i.e., $(1 - e_2)A \subset (1 - e_1)A$, so $V((1 - e_1)A) \supset V((1 - e_2)A)$; thus the map preserves order and complementation. Since the Boolean operations are determined by the partial ordering and complementation the proposition will follow now once we show that the map is surjective.

Suppose spec (A) is the disjoint union of $V(\underline{a})$ and $V(\underline{b})$. Then spec $(A) = V(\underline{a}) \cup V(\underline{b}) = V(\underline{a} \cap \underline{b}) = V(0)$, so $\underline{c} = \underline{a} \cap \underline{b} \subset \sqrt{(0)}$ = nil A. Moreover, $V(A) = \phi = V(\underline{a}) \cap V(\underline{b}) = V(\underline{a} + \underline{b})$; so $A = \sqrt{\underline{a} + \underline{b}}$, and this implies $\underline{a} + \underline{b} = A$. Write $1 = a + b$ with $a \in \underline{a}$ and $b \in \underline{b}$. Since $A/\underline{c} = (\underline{a}/\underline{c}) \oplus (\underline{b}/\underline{c})$ it follows that a and b are orthogonal idempotents, modulo $\underline{c}$, which generate the two summands. Now since $\underline{c}$ is a nil ideal it follows from (2.10) that there are orthogonal idempotents, a' and b', such that $a' \equiv a \bmod \underline{c}$ and $b' \equiv b \bmod \underline{c}$. In particular, $a' \in a + \underline{c} \subset \underline{a}$ so $a'A \subset \underline{a} \subset a'A + \underline{c}$. Since $\underline{c}$ is nil this implies $\sqrt{a'A} = \underline{a}$ and therefore $V(\underline{a}) = V(a'A)$ with a' idempotent, as was to be shown. This concludes the proof.

§4. LOCALIZATION, SUPPORT

Most of the material in this section is quite standard, so we shall leave many elementary details to the reader.

We fix a multiplicative set S (i.e., $s, t \in S \Rightarrow st \in S$) in a commutative ring R. If $M \in$ mod-R then the "localization" of M is

$$S^{-1}M = (M \times S)/\sim$$

where $\sim$ is the equivalence relation: $(x_1, s_1) \sim (x_2, s_2)$ if $(x_1s_2 - x_2s_1)t = 0$ for some $t \in S$. We denote the class of (x, s) by x/s and we make $S^{-1}M$ an additive group by

$$(x/s) + (y/t) = (xt + ys)/st$$

In particular

(1) $x/s = 0 \iff xt = 0$ for some $t \in S$

If $x/s \in S^{-1}M$ and $a/t \in S^{-1}R$ we define $(x/s)(a/t) = (xa)/(st)$. These structures are well defined, they make $S^{-1}R$ a commutative ring, and they make $S^{-1}M$ an $S^{-1}R$-module. Moreover, there is a canonical map

$$h_M: M \longrightarrow S^{-1}M \quad (h_M(x) = xs/s \text{ for any } s \in S)$$

It is clear that h_R is a ring homomorphism (for which $h_R(S) \subset U(S^{-1}R)$) and that h_M is h_R-semi-linear; i.e., $h_M(x\alpha) = h_M(x)h_R(\alpha)$ for $x \in M$, $\alpha \in R$. Therefore, it induces a canonical $S^{-1}R$-homomorphism

$$h'_M: M \otimes_R S^{-1}R \longrightarrow S^{-1}M$$

by $$h'_M(x \otimes (\alpha/s)) = x\alpha/s$$

If $f: M \longrightarrow N$ is R-linear it induces an $S^{-1}R$-homomorphism

$$S^{-1}f: S^{-1}M \longrightarrow S^{-1}N, \; S^{-1}f(x/s) = f(x)/s$$

thus making

$$S^{-1}: \text{mod-}R \longrightarrow \text{mod-}S^{-1}R$$

an additive functor. The basic fact is:

(4.1) PROPOSITION. S^{-1} _is an exact functor, and_

$$h'_M: M \otimes_R S^{-1}R \longrightarrow S^{-1}M \quad (M \in \text{mod-}R)$$

is a natural isomorphism of functors.

Proof. If $M' \xrightarrow{g} M \xrightarrow{f} M''$ is exact then clearly $(S^{-1}f)(S^{-1}g) = 0$, so we want to show that any x/s in $\text{Ker}(S^{-1}f)$ lies in $\text{Im}(S^{-1}g)$. Since $f(x)/s = 0$ we have $f(x)t = 0$ for some $t \in S$, by (1) above. Therefore, $xt \in \text{Ker}(f) = \text{Im}(g)$; say $xt = g(y)$. Then $x/s = xt/st = g(y)/st$

$= (S^{-1}g)(y/st)$.

It is easily checked that S^{-1} preserves arbitrary coproducts. Hence $h' = (h'_M)$ is a natural transformation between right continuous functors (see (II, §2). Since h_R is evidently an isomorphism it follows that h_M is an isomorphism whenever M is of the form Coker $(R^{(I)} \longrightarrow R^{(J)})$ and this covers all $M \in$ mod-R. q.e.d.

A ring homomorphism f: $R \longrightarrow R'$ such that f(S) U(R′) factors as $f = f'h_R$ for a unique homomorphism $f'\colon\ S^{-1}R \longrightarrow R'$; $f'(a/s) = f(a)f(s)^{-1}$. In particular an R-module M on which multiplication by elements of S is bijective has a canonical $S^{-1}R$-module structure (take $R' = End_{\mathbb{Z}}(M)$). In this case $h_M\colon\ M \longrightarrow S^{-1}M$ is an isomorphism.

If $\underline{a}$ is an ideal in R we can canonically identify $S^{-1}(R/\underline{a})$ with the localization of $R/\underline{a}$ with respect to the image in $R/\underline{a}$ of S.

Let $M \in$ mod-R. Since S^{-1} is exact an R-submodule N of M leads to an $S^{-1}R$-submodule $S^{-1}N$ of $S^{-1}M$. On the other hand if H is an $S^{-1}R$-submodule of $S^{-1}M$ then we shall write, by abuse of notation,

$$H \cap M = h_M^{-1}(H)$$

Since S^{-1} is exact the function $N \longmapsto S^{-1}N$ preserves the lattice operations (sum and intersection) on submodules. Moreover, we have:

$$N \subset S^{-1}N \cap M \qquad (N \subset M)$$

(2)

$$H = S^{-1}(H \cap M) \qquad (H \subset S^{-1}M)$$

(4.2) PROPOSITION. $\mathrm{spec}(S^{-1}R) \longrightarrow \mathrm{spec}(R)$ *induces*

<u>a homeomorphism from</u> $\operatorname{spec}(S^{-1}R)$ <u>to the space of all</u> $\underline{p} \in \operatorname{spec}(R)$ <u>such that</u> $\underline{p} \cap S = \phi$, <u>the inverse map being</u> $\underline{p} \longmapsto S^{-1}\underline{p}$. <u>In case</u> $S = \{s^n \mid n \geq 0\}$ <u>for some</u> $s \in R$ <u>this identifies</u> $\operatorname{spec}(S^{-1}R)$ <u>with the (open) complement of</u> $V(s) \subset \operatorname{spec}(R)$.

<u>Proof</u>. The map $\operatorname{spec}(S^{-1}R) \longrightarrow \operatorname{spec}(R)$ is $q \rightarrowtail h_R^{-1}(q) = q \cap R$. Since $q = S^{-1}(q \cap R)$ the map is injective, and $(q \cap R) \cap S = \phi$ because $h_R(S)$ consists of units whereas q is a proper ideal.

Suppose $\underline{p} \in \operatorname{spec}(R)$ and $\underline{p} \cap S = \phi$. Consider the exact sequence $0 \longrightarrow S^{-1}\underline{p} \longrightarrow S^{-1}R \longrightarrow S^{-1}(R/\underline{p}) \longrightarrow 0$. The image of S is a multiplicative set of non-zero elements in the integral domain $R/\underline{p}$, so $S^{-1}(R/\underline{p})$ is contained in the field of fractions of $R/\underline{p}$. It follows that $S^{-1}\underline{p}$ is prime, and that $h_{R/\underline{p}}$ is injective, from which it easily follows that $\underline{p} = S^{-1}\underline{p} \cap R$. For the first assertion of the proposition, therefore, it remains only to be shown that $\underline{p} \longmapsto S^{-1}\underline{p}$ is continuous (for $\underline{p} \in \operatorname{spec}(R)$, $\underline{p} \cap S = \phi$). But the inverse image of $V(\underline{a}) \subset \operatorname{spec}(S^{-1}R)$, for an ideal $\underline{a} \subset S^{-1}R$, is $\{q \subset R \mid \underline{a} \subset q\}$, and this is easily seen to be the set of $\underline{p} \in \operatorname{spec}(R)$ such that $\underline{p} \cap S = \phi$ and $(\underline{a} \cap R) \subset \underline{p}$, a closed set.

Finally, in case $S = \{s^n\}$ we have $\underline{p} \cap S = \phi \iff s \notin \underline{p} \iff \underline{p} \notin V(s)$, for $p \in \operatorname{spec}(R)$. q.e.d.

If $\underline{p} \in \operatorname{spec}(R)$ then $S_{\underline{p}} = R - \underline{p}$ is a multiplicative set, and it is customary to write $M_{\underline{p}}$ for the localization $S_{\underline{p}}^{-1}M$. In this case $R_{\underline{p}}$ is a local ring with maximal ideal

$\underline{p}R_{\underline{p}}$. If $M \in$ mod-R we write

$$\mathrm{supp}(M) = \{\underline{p} \in \mathrm{spec}(R) \mid M_{\underline{p}} \neq 0\}$$

Occasionally we shall also consider the support of M in max(R), which is defined similarly. The localizations of M, taken altogether, do not lose essential information about M, in the following sense:

(4.3) *If* $x \in M$ *then* $x = 0 \iff x/1 = 0$ *in* $M_{\underline{m}}$ *for all* $\underline{m} \in \max(R)$. *In particular*, $M = 0 \iff M_{\underline{m}} = 0$ *for all* $\underline{m} \in \max(R)$.

This fact, together with the exactness of localization, permits one to reduce many questions to the case of local rings; e.g., the question whether or not an R-homomorphism is an isomorphism.

We shall now indicate how $\otimes$ and Hom behave under localization. If $M, N \in$ mod-R there is a commutative square

$$\begin{array}{ccc}
M \otimes_R N & \xrightarrow{h_{M \otimes_R N}} & S^{-1}(M \otimes_R N) \\
{\scriptstyle h_M \otimes_R h_N}\downarrow & & \downarrow{\scriptstyle g_M} \\
S^{-1}M \otimes_R S^{-1}N & \xrightarrow[f]{} & S^{-1}M \otimes_{S^{-1}R} S^{-1}N
\end{array}$$

where $f(x \otimes_R y) = x \otimes_{S^{-1}R} y$, and where g exists because $f(h_M \otimes_R h_N) = h_M \otimes_{S^{-1}R} h_N$ is h_R-semi-linear. The fact that localization preserves tensor products is expressed by:

(4.4) f *and* g_M *are isomorphisms*.

The case of f we leave as an exercise. As for g, it is a natural transformation between right continuous functors (see (II, §2)) and g_R is evidently an isomorphism.

Now a standard argument shows that g_M is an isomorphism for any M of the form Coker $(R^{(I)} \longrightarrow R^{(J)})$, i.e., for all M.

Before treating Hom we shall generalize the context by introducing an R-algebra A. Then if M ε mod-A we see easily that $S^{-1}A$ is an $S^{-1}R$-algebra and

$$S^{-1}: \text{ mod-A} \longrightarrow \text{mod-}S^{-1}A$$

is an exact functor isomorphic to $\otimes_R S^{-1}R$, or, equivalently, to $\otimes_A S^{-1}A$. If M, N ε mod-A we have a commutative diagram

$$(3)\qquad \begin{array}{ccc} \operatorname{Hom}_A(M, N) & \xrightarrow{h_{\operatorname{Hom}_A(M, N)}} & S^{-1}\operatorname{Hom}_A(M, N) \\ & \searrow^{S^{-1}} & \downarrow g_N \\ \operatorname{Hom}_A(S^{-1}M, S^{-1}N) & \xleftarrow[f]{} & \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \end{array}$$

where f is an inclusion, and where g_M exists because S^{-1} above is h_R-semi-linear.

(4.5) PROPOSITION. *In diagram* (3) *above* f *is an isomorphism, and the natural transformation*

$$g_M: \; S^{-1}\operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

is an isomorphism if M *is a finitely presented* A-*module* (*i.e., if* M *is of the form* Coker$(A^n \longrightarrow A^m)$ *for some* $n, m > 0$).

Proof. The assertion for f is an easy exercise which we leave to the reader. Clearly g_A is an isomorphism so g_{A^n} is for all $n > 0$. Since S^{-1} is exact, both contravariant functors of M above convert cokernels into kernels, so the 5-lemma shows that g_M is an isomorphism if there is an exact sequence $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ such that each g_{P_i} is an isomorphism. Taking $P_i = A^{n_i}$ we obtain all finitely presented M this way. q.e.d.

(4.6) PROPOSITION. (a) *Let* $M \in$ mod-A *and let* H *be an* $S^{-1}A$-*submodule of* M. *Then* $H = S^{-1}(H \cap M)$, *so the map* $H \longmapsto H \cap M$ *is an injection from the lattice of* $S^{-1}A$-*submodules of* $S^{-1}M$ *to the lattice of* A-*submodules of* M. *In particular, if* M *is a noetherian (resp., Artinian)* A-*module then* $S^{-1}M$ *is a noetherian (resp., Artinian)* $S^{-1}A$-*module*.

(b) *Assume* A *is right noetherian and let* $E = (0 \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_0 \xrightarrow{d} S^{-1}M \longrightarrow 0)$ *be an exact sequence in* $\underline{\underline{M}}(S^{-1}A)$. *Then there is an exact sequence* $E' = (0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{d_0} M \longrightarrow 0)$ *in* $\underline{\underline{M}}(A)$ *and an isomorphism* $S^{-1}E' \simeq E$ *inducing the identity on* $S^{-1}M$.

Proof. (a) The first assertion follows from (2) above, and it implies the remaining assertions.

(b) After breaking E into short exact sequences and applying an obvious induction argument, it suffices to treat the case n = 1. We are given

$$0 \longrightarrow H_1 \longrightarrow H_0 \xrightarrow{d} S^{-1}M \longrightarrow 0$$

If we can find an epimorphism $d_0\colon\ M_0 \longrightarrow M$ in $\underline{\underline{M}}(A)$, and an isomorphism $S^{-1}d_0 \simeq d$ inducing $1_{S^{-1}M}$, then the exactness of S^{-1} forces this isomorphism to induce an isomorphism

$S^{-1}M_1 \simeq H_1$, where $M_1 = \mathrm{Ker}(d_0)$. Moreover $M_1 \in \underline{\underline{M}}(A)$ because A is right noetherian, so the problem will be solved once we construct d_0.

Let X be a finite set of $S^{-1}A$-generators of H_0. After multiplying the elements of X by elements of S, if necessary, we can assume d(X) lies in the image of $h_M\colon\ M \longrightarrow S^{-1}M$. Now choose a finite set $Y \subset H_0$ so that d(Y) generates the A-module $\mathrm{Im}(h_M)$, and let $N \subset H_0$ be the A-submodule generated by X and Y. Then d induces an epimorphism $d'\colon\ N \longrightarrow \mathrm{Im}(h_M)$ in $\underline{\underline{M}}(A)$. Moreover the inclusion $i\colon\ N \longrightarrow H_0$ induces a commutative square

$$\begin{array}{ccc} N & \xrightarrow{\ i\ } & H_0 \\ {\scriptstyle h_N}\downarrow & & \downarrow{\scriptstyle h_{H_0}}\ (\simeq) \\ S^{-1}N & \xrightarrow[S^{-1}i]{} & S^{-1}H_0 \end{array}$$

in which $S^{-1}i$ is surjective by construction and injective because S^{-1} is exact.

Form the cartesian square

$$\begin{array}{ccc} M_0 & \xrightarrow{\ d_0\ } & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h_M} \\ N & \xrightarrow[d']{} & \mathrm{Im}(h_M) \end{array}$$

Since A is right noetherian $\underline{\underline{M}}(A)$ is abelian, so $M_0 \in \underline{\underline{M}}(A)$ because $M, N \in \underline{\underline{M}}(A)$. (In fact $M_0 \subset M \oplus N$.) Moreover d_0 is is an epimorphism since d' is. Finally, since $S^{-1}h_M$ is an isomorphism and S^{-1} is exact it follows that $S^{-1}f$ is also an isomorphism. Therefore

$$(S^{-1}(f), 1_{S^{-1}M}): S^{-1}(M_0 \xrightarrow{d_0} M) \longrightarrow (H_0 \xrightarrow{d} S^{-1}M)$$

is the required isomorphism.

(4.7) PROPOSITION. *Let* A *be an* R-*algebra.*

(a) *Let* $M \in \underline{\underline{M}}(A)$ *and let* $\underline{a}$ *be the annihilator of* M *as an* R-*module. Then* $\text{supp}(M) = V(\underline{a})$, *a closed set in* spec(R).

(b) *Let* $P = (0 \longrightarrow P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0 \longrightarrow 0)$ *be a finite complex in* $\underline{\underline{P}}(A)$. *Then* supp(H(P)) *is closed in* spec(R).

Proof. (a) If $s \in \underline{a}$, $s \notin \underline{p}$ then clearly $M_{\underline{p}} = 0$. Conversely, suppose $M_{\underline{p}} = 0$. If $x_1, \ldots, x_n$ generate M (as A-module) then, for each i, there is an $s_i \notin \underline{p}$ such that $x_i s_i = 0$. Therefore, $s = s_1 \cdots s_n \in \underline{a}$ and $s \notin \underline{p}$.

(b) We argue by induction on n, the case n = 0, when $H(P) = P_0$, following from (a).

If $\underline{p} \notin \text{supp}(H(P))$ we propose to find a neighborhood of $\underline{p}$ outside supp(H(P)). Let $M = \text{Coker}(d_1) \in \underline{\underline{M}}(A)$. Then $M = H_0(P)$ so $M_{\underline{p}} = 0$. Choose $s \notin \underline{p}$ such that $Ms = 0$ (using (a)). and set $S = \{s^n \mid n \geq 0\}$. We shall pass to the complex $S^{-1}P$ over the $S^{-1}R$-algebra $S^{-1}A$. According to (4.2) we can identify $\text{spec}(S^{-1}R)$ with spec(R) - V(s), an open neighborhood of $\underline{p}$. Hence it will suffice to show that $\text{supp}(H(S^{-1}P))$ is closed in $\text{spec}(S^{-1}R)$.

By construction, $S^{-1}d_1$ is surjective, so it splits (because $S^{-1}P_0 \in \underline{\underline{P}}(S^{-1}A)$). Therefore, $S^{-1}P$ is isomorphic to the direct sum of $(\cdots 0 \longrightarrow S^{-1}P_0 \xrightarrow{1_{S^{-1}P_0}} S^{-1}P_0 \longrightarrow 0 \cdots)$ and of the subcomplex $Q = (\cdots 0 \longrightarrow S^{-1}P_n \longrightarrow \cdots \longrightarrow S^{-1}P_2 \longrightarrow \mathrm{Ker}(S^{-1}d_1) \longrightarrow 0 \cdots)$ of $S^{-1}P$. Therefore, $H(S^{-1}P) = H(Q)$. Since Q has length $<n$ it follows by induction that supp $(H(Q))$ is closed. q.e.d.

§5. INTEGERS

Let R be a commutative ring and let A be an R-algebra If $X \in A$ write $R[X]$ for the R-subalgebra of A generated by X. We say $\alpha \in A$ is _integral_ over R if it satisfies the conditions below. A is _integral_ over R if each of its elements is.

(5.1) PROPOSITION. _The following conditions are equivalent_:

(1) _There is a monic polynominal_ $f(T) \in R[T]$ _such that_ $f(\alpha) = 0$.

(2) $R[\alpha]$ _is a finitely generated_ R-_module_.

(3) _There is a faithful_ $R[\alpha]$-_module_ M _finitely generated as an_ R-_module_.

Proof. (1) ⇒ (2). If $r_0 1 + r_1\alpha + \cdots + r_{n-1}\alpha^{n-1} + \alpha^n = 0$ (each $r_i \in R$) then $1, \alpha, \ldots, \alpha^{n-1}$ generate $R[\alpha]$ as an R-module.

(2) ⇒ (3). Take $M = R[\alpha]$.

(3) ⇒ (1). Say $M = \sum_{1 \le i \le n} x_i R$. We can solve

$x_i\alpha = \Sigma_j x_j r_{ij}$ with $r_{ij} \in R$ and so $\Sigma_j x_j (\alpha\delta_{ij} - r_{ij}) = 0$ $(1 \leq i \leq n)$. By Cramer's Rule we have $x_j f(\alpha) = 0$ for all j, and hence, $Mf(\alpha) = 0$, where $f(T) = \det(T\delta_{ij} - r_{ij})$. Since M is faithful, $f(\alpha) = 0$.

(5.2) COROLLARY. *A subalgebra* B *of* A *is integral over* R *if* $B \subset M \subset A$ *for some finitely generated* R-*module* M *such that* $BM \subset M$.

Proof. If $\alpha \in B$ then M is a faithful $R[\alpha]$-module.

(5.3) PROPOSITION. *Let* S *be a multiplicative set in* R. *If* A *is integral over* R *then* $S^{-1}A$ *is integral over* $S^{-1}R$.

Proof. Let $\alpha/s \in S^{-1}A$ and say $r_0 + \cdots + r_{n-1}\alpha^{n-1} + \alpha^n = 0$ with $r_i \in R$. Then $(r_0/s^n) + \cdots + (r_{n-1}/s)(\alpha/s)^{n-1} + (\alpha/s)^n = 0$.

(5.4) PROPOSITION. *Let* A *be a commutative* R-*algebra, and let* $M \in \underline{\underline{M}}(A)$

(1) *If* $\alpha_1, \ldots, \alpha_n \in A$ *are integral over* R *then* $R[\alpha_1, \ldots, \alpha_n] \in \underline{\underline{M}}(R)$.

(2) *If* $A \in \underline{\underline{M}}(R)$ *then* $M \in \underline{\underline{M}}(R)$.

Proof. (2) If $A = \Sigma\alpha_i R$ and $M = \Sigma b_j A$ then $M = \Sigma\alpha_i b_j R$.

(1) Since α_n is integral over R, and, a fortiori, over $R' = R[\alpha_1, \ldots, \alpha_{n-1}]$, it follows that $R[\alpha_1, \ldots, \alpha_n] \in \underline{\underline{M}}(R')$. By induction on n $R' \in \underline{\underline{M}}(R)$. Now apply (2).

(5.5) COROLLARY. *Let* A *be a commutative* R-*algebra and let* B *be an* A-*algebra. The set* R′ *of elements of* A *which* *are integral over* R *is an* R-*subalgebra of* A. *If* $\alpha \in A$ *is integral over* R′ *then* $\alpha \in R'$. *If* $b \in B$ *is integral over* R′ *then* b *is integral over* R.

Proof. The first assertion follows from (1) above. As for the third, suppose $c_0 + \cdots + c_{n-1}b^{n-1} + b^n = 0$ with $c_i \in R'$. Then $R' = R[c_0, \ldots, c_{n-1}] \in \underline{\underline{M}}(R)$ by (1) and $R'[b] \in \underline{\underline{M}}(R')$, so $R[b] \subset R'[b] \in \underline{\underline{M}}(R)$ by (2), and (5.2) implies b is integral over R. The second assertion follows from the third with B = A.

We call R′ above the *integral closure* of R in A. We call an integral domain *integrally closed* if it equals its integral closure in its field of fractions.

(5.6) PROPOSITION. *Let* $A \subset B$ *be commutative rings such that* B *is integral over* A, *and let* $\underline{p} \subset q$ *be primes of* A.

(a) *There is a prime* $\underline{p}'$ *of* B *such that* $\underline{p}' \cap A = \underline{p}$.

(b) *For any such* $\underline{p}'$ *there is a prime* q′ *containing* $\underline{p}'$ *such that* $q' \cap A = q$.

(c) *If* $\underline{p} = q$ *then necessarily* $\underline{p}' = q'$.

Proof. (a) Suppose first that A is local with maximal ideal $\underline{p}$. If $\underline{p}B = B$ then $\underline{p}B_0 = B_0$ for some finitely generated A-subalgebra B_0 of B, and (5.4) implies $B_0 \in \underline{\underline{M}}(A)$. However $\underline{p} \subset \text{rad } A$, so $\underline{p}B_0 = B_0$ implies $B_0 = 0$ by Nakayama's lemma, and this is impossible. Therefore, $\underline{p}B \neq B$ so there is a maximal ideal $\underline{p}'$ of B containing $\underline{p}^B$. Since $\underline{p} \subset \underline{p}' \cap A \neq A$ we have $\underline{p} = \underline{p}' \cap A$.

In the general case we pass to $A_{\underline{p}} \subset B_{\underline{p}}$ and $\underline{p}A_{\underline{p}}$. Thanks to (5.3) we can apply the conclusion above to find a prime q′′ of $B_{\underline{p}}$ such that $q'' \cap A_{\underline{p}} = \underline{p}A_{\underline{p}}$. Then $q' = q'' \cap B$ solves our problem.

(b) We pass to the integral extension $A/\underline{p} \subset B/\underline{p}'$ and apply (a) to find $q'/\underline{p}'$ lying over q/p.

(c) After passing to $A/\underline{p} \subset B/\underline{p}'$ again it suffices to show that if A and B are integral domains and if $q' \neq 0$ then $q' \cap A \neq 0$. If $b \in q'$ choose an equation $a_0 + \cdots + a_{n-1} b^{n-1} + b^n = 0$, with $a_i \in A$, of minimal degree. Then $a_0 \in bB \cap A \subset q' \cap A$, and $a_0 \neq 0$ or else we would have $a_1 + \cdots + a_{n-1} b^{n-2} + b^{n-1} = 0$.

The last condition implies chains of primes in B do not collapse at all when restricted to A. Thus we have:

(5.7) COROLLARY. spec(B) $\longrightarrow$ spec(A) *is surjective, and* dim B = dim A.

(5.8) PROPOSITION. *Let* R *be a commutative noetherian ring, let* A *be a finite* R-*algebra, and let* M $\in$ $\underline{\underline{M}}(A)$. *The following conditions are equivalent*:

(1) M *has finite length as an* A-*module*.

(2) M *has finite length as an* R-*module*.

(3) supp(M) (*in* spec(R)) *is finite and consists of maximal ideals*.

Proof. (1) $\Rightarrow$ (2). By induction on length it suffices to show that a simple A-module M has finite R-length. Since M $\in$ $\underline{\underline{M}}(R)$, clearly (see (5.4)(2)), it suffices to show that $q = \mathrm{ann}_R(M)$ is maximal, for then M is finite dimensional over R/q.

By Schur's lemma, multiplication by $a \in R$ on M is either zero or an automorphism. This shows that q is prime and that M is a vector space over the field of fractions, F, of R/q. But it is an easy exercise to see that F can be a finitely generated R/q-module only if F = R/q.

(2) $\Rightarrow$ (3). If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of R-modules then clearly supp(M) = supp(M′) $\cup$ supp(M′′). Since the implication in question has only to do with R-modules, it suffices, by induction on length, to establish (3) when M is a simple R-module, R/m (m ε max(R)). But then supp(M) = {m}.

(3) $\Rightarrow$ (1) By (4.7) we have supp(M) = V(a), where a = ann_R(M). Set R′ = R/a; then M ε M(R′) so it suffices to show that R′ is Artinian. We have spec(R′) = V($\bar{a}$), a finite set of maximal ideals. Let J = rad R′. Then (3.9) implies J = nil R′ and, since R′ is noetherian, that J is nilpotent. The Chinese Remainder Theorem (2.14), applied to the maximal ideals of R′, shows that R′/J is a finite product of fields, and hence semi-simple. For each $i \geq 1$, J^{i-1}/J^i is a noetherian (R′/J)-module, and hence of finite length. Since J is nilpotent it follows that R′ is Artinian. q.e.d.

The implication (2) $\Rightarrow$ (1) is trivial, so the proposition is proved.

A two sided ideal p in a not necessarily commutative ring A is called prime if $ab \subset p \Rightarrow a \subset p$ or $b \subset p$ for two sided ideals a and b. It suffices to have this only for a and b which contain p. Thus it is evident that a maximal two sided ideal is prime.

(5.9) PROPOSITION. Let R and A be as in (5.8) and let p be a two sided ideal in A. The following conditions are equivalent:

(1) p is maximal.

(2) p is prime and A/p is an R-module of finite length.

(3) p is the annihilator in A of a simple right A-module.

Proof. (1) $\Rightarrow$ (3). If M is a simple right ($A/\underline{p}$)-module then the inclusion $\underline{p} \subset \mathrm{ann}_A(M)$ is an equality because $\underline{p}$ is maximal.

(3) $\Rightarrow$ (2). If M is a simple right A-module then $\underline{p} = \mathrm{ann}_A(M)$ is clearly prime. By (5.8) M has finite length as an R-module, so likewise for $A/\underline{p} \subset \mathrm{End}_R(M)$.

(2) $\Rightarrow$ (1). $B = A/\underline{p}$ has finite length as an R-module, so it is an Artin ring in which the zero ideal is prime. The latter implies B has no non-zero nilpotent ideals, and that it does not decompose properly into a product of rings. Thanks to (2.11) this implies B is simple, so $\underline{p}$ is maximal. q.e.d.

We next study integrality properties of polynominals. R always denotes a commutative ring and t an indeterminate.

(5.10) LEMMA. *If* $P(t) \in R[t]$ *is monic then there is an integral extension* R' *containing* R *such that* P *is a product of linear polynomials in* $R'[t]$.

Proof. Induction on $n = \deg(P)$. We can clearly assume $n > 1$. Let $R_1 = R[t]/PR[t]$, which contains R. The residue, a, of t in R_1 is a root of P. Since P is monic we can apply the division algorithm to write $P(t) = (t - a)Q(t)$ in $R_1[t]$, where Q is monic and of degree $n - 1$. By induction we can embed R_1 in an R' which splits Q.

(5.11) COROLLARY. *Let* A *be a commutative* R-*algebra and let* R' *be the integral closure of* R *in* A. *Let* $P, Q \in A[t]$ *be monic and such that* $PQ \in R'[t]$. *Then* $P, Q \in R'[t]$.

Proof. Use (5.10) to construct an A' containing A in which P and Q factor into linear factors: $P = \Pi(t - a_i)$, $Q = \Pi(t - b_j)$. Let R'' be the integral closure of R in A'. Since $PQ \in R'[t]$, each a_i and b_j belongs to R'', being a root of PQ. Therefore $P, Q \in R''[t]$. Since $R'' \cap A = R'$ the corollary follows.

(5.12) PROPOSITION. _Let_ A _be a commutative_ R-_algebra, and let_ R′ _be the integral closure of_ R _in_ A. _Then_ R′[t] _is the integral closure of_ R[t] _in_ A[t].

Proof. Let B be the integral closure of R[t] in A[t]. Evidently, $R'[t] \subset B$. Conversely, suppose $P \in B$. Say P is a root of

$$Q(X) = F_0 + \cdots + F_{m-1} X^{m-1} + X^m \in R[t][X]$$

Choose an integer $r > \max(\deg(P), \deg(F_i)(1 \leq i \leq m))$, and set $P_1(t) = P(t) - t^r$. Then P_1 is a root of

$$Q_1(X) = Q(X + t^r) = G_0 + \cdots + G_{m-1} X^{m-1} + X^m$$

Therefore we obtain

$$(*) \qquad G_0 = -P_1(G_1 + \cdots + G_{m-1} P_1^{m-2} + P_1^{m-1})$$

The size of r guarantees that $-P_1(t)$ and $G_0(t) = Q_1(0) = Q(t^r)$ are monic. This is clear for $-P_1$, and for $Q(t^r) = F_0(t) + \cdots + F_{m-1}(t)\, t^{r(m-1)} + t^{mr}$ we need only note that $\deg(F_i(t)t^{ri}) = \deg(F_i) + ri < r(i+1) \leq rm$ for $i < m$.

Now the equation (*) implies the second factor on the right is monic also, so (5.11) implies $-P_1$, and hence also P, have coefficients in R′, since G_0 does.

(5.13) COROLLARY. _If_ R _is an integrally closed integral domain so also is_ R[t].

Proof. Let L be the field of fractions of R. Then (5.12) implies R[t] is integrally closed in L[t]. It remains only to observe, therefore, that the principal ideal domain L[t] is integrally closed. We leave this as an exercise (cf. (7.12) below.)

We close this section with some observations on the norms and traces of integral elements.

(5.14) PROPOSITION. Let A be a commutative R-algebra and let $x \in M_n(A)$. Then x is integral over R if and only if the coefficients of its characteristic polynomials, $P(t) = \det(t\cdot I - x)$, are integral over R.

Proof. Since $P(x) = 0$ (Cayley-Hamilton; cf.(XII, §1) below) x is integral over the subalgebra generated by the coefficients of P. Thus, if the latter are integral over R so also is x.

Now suppose x is integral over R. Then if $e_i (1 \leq i \leq n)$ is the standard base of A^n, the R[x]-module M generated by the e_i is in $\underline{\underline{M}}(R)$. Let u be the endomorphism of A^n defined by x, and let $N \subset \Lambda^n A^n$ be the R-module generated by all $m_1 \wedge \cdots \wedge m_n$ where $m_i \in M$ $(1 \leq i \leq n)$. Then $N \in \underline{\underline{M}}(R)$ and N is stable under $\Lambda^n u = \det(x)$. Since $e_1 \wedge \cdots \wedge e_n \in N$ it follows that N is a faithful R[det(x)]-module, so det(x) is integral over R.

Since $t\cdot I - x \in M_n(A[t])$ is integral over R[t] we see from the conclusion above that P(t) is integral over R[t]. Therefore, by (5.12), the coefficients of P are integral over R. q.e.d.

§6. HOMOLOGICAL DIMENSION OF MODULES

Let A be a ring and let $M \in$ mod-A. We write

$$hd_A(M)$$

for the minimal length (possibly infinite) of a projective resolution of M (cf. (I, §6)), and we define

$$\text{rt.gl.dim.}A = \sup hd_A(M) \quad (M \in \text{mod-}A)$$

We quote, without proof, the following useful result of Auslander (see MacLane [1], Ch. VII, Cor. 1.5):

(6.1) PROPOSITION (M. Auslander).

$$\text{rt. gl. dim. } A = \sup \text{hd}_A(M) \quad (M \in \underline{\underline{M}}(A))$$

It follows from (1.5) that rt. gl. dim. A = 0 if and only if A is semi-simple. If rt. gl. dim A ≤ 1 we call A *right hereditary*.

(6.2) PROPOSITION. (a) A *is right hereditary if and only if every right ideal is projective*.

(b) *Let* A *be right hereditary and right noetherian*. *Suppose* $M \in \underline{\underline{M}}(A)$ *and set* $T = \bigcap \text{Ker}(h)(h\colon M \longrightarrow A)$. *Then* T *is a direct summand of* M *and* M/T *is a direct sum of modules isomorphic to right ideals in* A.

Proof. (a) If $\mathfrak{a}$ is a right ideal then the exact sequence $0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$ shows (cf.(I, 6.8)) that $\text{hd}_A(A/\mathfrak{a}) \leq 1 \iff \text{hd}_A(\mathfrak{a}) \leq 0$. Thus all right ideals are projective $\iff$ every monogene,(i.e., one generator) module has $\text{hd}_A \leq 1$. The latter implies $\text{hd}_A(M) \leq 1$ for all $M \in \underline{\underline{M}}(A)$, and hence A is right hereditary by (6.1). For if M has n generators there is an exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ where M' and M'' have 1 and n - 1 generators, respectively. By (I, 6.8) $\text{hd}_A(M) \leq \sup(\text{hd}_A(M'), \text{hd}_A(M''))$ so the assertion follows by induction on n.

(b) If we show M/T is a direct sum of modules isomorphic to right ideals then it follows from (a) that M/T is projective, so $M \simeq T \oplus M/T$. Since A is right noetherian the module $M/T \in \underline{\underline{M}}(A)$ is noetherian. Among all direct summands of M/T which are direct sums of modules isomorphic to right ideals (e.g., 0) let N be a maximal one. Then $M/T = N \oplus H$ and we claim H = 0. If not there is a non-zero $h\colon H \longrightarrow A$. Since Im(h) is projective and $\neq 0$ we we have $H \simeq \mathrm{Ker}(h) \oplus \mathrm{Im}(h)$, and then $N \oplus \mathrm{Im}(h)$ contradicts the maximality of N.

We now introduce the full subcategory

$$\underline{\underline{H}}(A)$$

of modules $M \in$ mod-A which have finite $\underline{\underline{P}}(A)$-resolutions. Evidently we have

$$\underline{\underline{P}}(A) \subset \underline{\underline{H}}(A) \subset \underline{\underline{M}}(A) \subset \text{mod-}A$$

and if $M \in \underline{\underline{H}}(A)$ then $\mathrm{hd}_A(M) < \infty$. On the other hand, if $M \in \underline{\underline{M}}(A)$ and $\mathrm{hd}_A(M) < \infty$, M need not belong to $\underline{\underline{H}}(A)$. For M must be not only finitely generated, but finitely presented, and even more. In the notation of (I, §6) $\underline{\underline{H}}(A)$ is the category $\mathrm{Res}(\underline{\underline{P}}(A))$, and we have the following excerpt from (I, 6.9).

(6.3) PROPOSITION. <u>If all but one term of an exact sequence</u> $0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0$ <u>lie in</u> $\underline{\underline{H}}(A)$ <u>then so does the remaining term</u>.

A is said to be <u>right regular</u> if $\underline{\underline{H}}(A) = \underline{\underline{M}}(A)$. It follows immediately from (6.3) that A must be right noetherian. Conversely, if A is right noetherian and if $\mathrm{hd}_A(M) < \infty$ for all $M \in \underline{\underline{M}}(A)$ then A is right regular. For if $\mathrm{hd}_A(M) \leq n$ choose an exact sequence $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ with $P_i \in \underline{\underline{P}}(A)$ for $0 \leq i < n$. We may do this since A is right noetherian. Then it follows from Schanuel's lemma (cf. (I, 6.4)) that P_n is automatically projective, so $P_n \in \underline{\underline{P}}(A)$ also.

(6.4) PROPOSITION. <u>Let</u> S <u>be a multiplicative set in a commutative ring</u> R, <u>let</u> A <u>be an</u> R-<u>algebra, and let</u> $M \in$ mod-A. <u>Then</u>,

$$hd_{S^{-1}A}(S^{-1}M) \leq hd_A(M), \quad \text{rt.gl.dim.}S^{-1}A \leq \text{rt.gl.dim}A$$

and $S^{-1}A$ _is right regular if_ A _is right regular_.

Proof. If $P \longrightarrow M$ is a projective A-resolution of length n, then $S^{-1}P \longrightarrow S^{-1}M$ is a projective $S^{-1}A$-resolution (because S^{-1} is exact) and its length is $\leq n$. Since every $N \in \text{mod-}S^{-1}A$ is isomorphic to some $S^{-1}M$ the second inequality follows from the first. If A is right noetherian then so also is $S^{-1}A$, and any $N \in \underline{\underline{M}}(S^{-1}A)$ is isomorphic to $S^{-1}M$ for some $M \in \underline{\underline{M}}(A)$. Hence the last assertion follows also from the first inequality. q.e.d.

(6.5) PROPOSITION. _Let_ A _be an_ R-_algebra and let_ $M \in \text{mod-}A$. _Define_

$$U_n(M) = \{\underline{p} \in \text{spec}(R) \mid hd_{A_{\underline{p}}}(M_{\underline{p}}) \leq n\}$$

If there is an exact sequence

$$P_{n+1} \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

with $P_i \in \underline{\underline{P}}(A)$ $(0 \leq i \leq n+1)$ _then_ $U_n(M)$ _is open and_ $hd_A(M) \leq n \iff U_N(M) = \text{spec}(R)$.

Proof. Induction on n.

_n = 0_. We have an exact sequence $P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ with $P_i \in \underline{\underline{P}}(A)$ $(i = 0, 1)$. Consider the map

$$h: \ \text{Hom}_A(M, P_0) \longrightarrow \text{Hom}_A(M, M)$$

induced by ε, and let e denote the image of 1_M in Coker(h). Then clearly M is projective $\iff 1_M \in \text{Im}(h) \iff e = 0$.

Since both P_0 and M are finitely presented A-modules it follows from (4.5) that, for P ε spec(R), we can identify the corresponding map $\mathrm{Hom}_{A_{\underline{p}}}(M_{\underline{p}}, P_{0_{\underline{p}}}) \longrightarrow \mathrm{Hom}_{A_{\underline{p}}}(M_{\underline{p}}, M_{\underline{p}})$ with the localization, $h_{\underline{p}}$, of h. Thus it follows that $M_{\underline{p}}$ is $A_{\underline{p}}$-projective $\Longleftrightarrow$ $e_{\underline{p}} = e/1 \ \varepsilon$ Coker$(h)_{\underline{p}}$ (= Coker$(h_{\underline{p}})$) is zero. Now $e_{\underline{p}} = 0 \Longleftrightarrow es = 0$ for some $s \notin \underline{p} \Longleftrightarrow \underline{a} \not\subset \underline{p}$, where $\underline{a}$ = $\mathrm{ann}_R(e)$, so $U_0(M)$ is the (open) complement of $V(\underline{a})$. Moreover M is projective $\Longleftrightarrow e = 0 \Longleftrightarrow \underline{a} = R \Longleftrightarrow V(\underline{a}) = \phi \Longleftrightarrow U_0(M)$ = spec(R).

$\underline{n > 0}$. Consider the exact sequence $0 \longrightarrow K \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ where $K = \mathrm{Ker}(\varepsilon)$. Then $\mathrm{hd}_A(M) \leq n \Longleftrightarrow \mathrm{hd}_A(K) \leq n - 1$ and $U_n(M) = U_{n-1}(K)$, clearly. Since we have the exact sequence $0 \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow K \longrightarrow 0$ with $P_i \ \varepsilon\ \underline{\underline{P}}(A)$ $(1 \leq i \leq n + 1)$ it follows by induction that $U_{n-1}(K)$ is open and $\mathrm{hd}_A(K) \leq n - 1 \Longleftrightarrow U_{n-1}(K)$ = spec(R). q.e.d.

(6.6) COROLLARY. _Let_ R _be a commutative noetherian ring, let_ A _be a finite_ R-_algebra, and let_ M ε $\underline{\underline{M}}(A)$. _Then_ $\mathrm{hd}_A(M) = \sup \mathrm{hd}_{A_{\underline{m}}}(M_{\underline{m}})$ ($\underline{m}$ ε max(R)), _and if_ $\mathrm{hd}_{A_{\underline{m}}}(M_{\underline{m}}) < \infty$ _for all_ $\underline{m}$ ε max(R) _then_ $\mathrm{hd}_A(M) < \infty$. _Hence_ A _is right regular if and only if_ $A_{\underline{m}}$ _is right regular for all_ $\underline{m}$ ε max(R).

Proof. Clearly the last assertion follows from the first. Let $n = \sup \mathrm{hd}_{A_{\underline{m}}}(M_{\underline{m}})$ ($\underline{m}$ ε max(R)). By (6.4) $\mathrm{hd}_A(M) \geq n$ so we have equality if $n = \infty$. If $n < \infty$ consider $U_n(R)$. Our finiteness assumptions make the hypothesis on M in (6.5) automatic, so $U_n(R)$ is an open set whose complement contains no maximal ideals, and is therefore empty. (If $V(\underline{a}) \cap \max(R) = \phi$ then $\underline{a}$ is contained in no maximal ideal, so $\underline{a} = R$.) Thus $U_n(R) = \mathrm{spec}(R)$ and (6.5) now implies $\mathrm{hd}_A(M) \leq n$.

If $\mathrm{hd}_{A_{\underline{m}}}(M_{\underline{m}}) < \infty$ for all $\underline{m} \in \max(R)$ then (6.4) implies the same is true for all $\underline{m} \in \mathrm{spec}(R)$. Therefore, in this case the union of the $U_n(M)$ is spec(R). Since $U_n \subset U_{n+1}$ and since spec(R) is quasi-compact (even noetherian in the present case) it follows that $U_n(M) = \mathrm{spec}(R)$ for some n. Now apply (6.5) to obtain $\mathrm{hd}_A(M) \leq n$. q.e.d.

Let R be a commutative ring, let A be an R-algebra, and let S be a multiplicative set in R. We shall call $M \in$ mod-A an S-<u>torsion module</u> if $S^{-1}M = 0$, and we shall write

$$\underline{\underline{H}}_S(A) \subset \underline{\underline{M}}_S(A)$$

for the full subcategories of S-torsion modules in $\underline{\underline{H}}(A)$ and in $\underline{\underline{M}}(A)$, respectively. It is easy to see that an $M \in$ mod-A is in $\underline{\underline{M}}_S(A)$ if and only if $M \in \underline{\underline{M}}(A)$ and $Ms = 0$ for some $s \in S$. The latter means that $S \cap \mathrm{ann}_R(M) \neq \phi$. We shall say that S *is regular for* A if $\underline{\underline{H}}_S(A) = \underline{\underline{M}}_S(A)$. It is then easy to show, with the aid of (6.3), that the latter is an Abelian category in which every object is noetherian.

(6.7) PROPOSITION. *Let* R *be a commutative noetherian ring and let* A *be a finite* R-*algebra*.

(a) rt.gl.dim.A = sup $\mathrm{hd}_A(M)$, *where* M *ranges over the simple right* A-*modules. Therefore, if* R *is semi-local and if* A *is right regular then* rt.gl.dim.A $< \infty$.

(b) *Let* S *be a multiplicative set in* R. *Then* S *is regular for* A *if and only if* $A_{\underline{m}}$ *is right regular for all* $\underline{m} \in \max(R)$ *such that* $\underline{m} \cap S \neq \phi$. *In this case, if* $M \in \underline{\underline{M}}(A)$, we have

$$M \in \underline{\underline{H}}(A) \iff S^{-1}M \in \underline{\underline{H}}(S^{-1}A)$$

Proof. (a) The left side dominates the right, and equals sup $hd_A(M)$ $(M \in \underline{\underline{M}}(A))$, by Auslander's Theorem (6.1). In particular, if $n = \sup \{hd_A(M) \mid M \text{ is simple}\}$ is infinite then we have equality, so assume $n < \infty$. Given $M \in \underline{\underline{M}}(A)$, we claim $hd_A(M) \leq n$. According to (6.6) it suffices to prove this locally, so assume R is local with maximal ideal $\underline{m}$.

Let $\underline{a}$ be an ideal in R. We claim that if $M \in \underline{\underline{M}}(A)$ and if $M\underline{a} = 0$ then $hd_A(M) \leq n$. (The case $\underline{a} = 0$ will imply what we want to prove.) If not let $\underline{a}$ be a maximal counter-example (noetherian induction) and choose $M \in \underline{\underline{M}}(A)$ such that $M\underline{a} = 0$ and $hd_A(M) > n$. Then choose a maximal submodule $N \subset M$ such that $hd_A(M/N) > n$. Replacing M by M/N we can assume $hd_A(M') \leq n$ for all proper quotients M' of M. We cannot have $\underline{a} = \underline{m}$ for otherwise M would have finite length, and the homological dimension of its Jordan-Holder factors would dominate that of M (see (I, 6.8)). Thus we can choose $t \in \underline{m}$, $t \notin \underline{a}$. If $N = \mathrm{Ker}(M \xrightarrow{t} M)$ then $\mathrm{ann}_R(N) \supset \underline{a} + tR$ so $hd_A(N) \leq n$. If $N \neq 0$ then $hd_A(M/N) \leq n$ also, and hence $hd_A(M) \leq n$, contrary to assumption (using (I, 6.8) again). Therefore we have an exact sequence

$$(*) \qquad 0 \longrightarrow M \xrightarrow{t} M \longrightarrow M/Mt \longrightarrow 0$$

At this point we shall use the functor Ext, and its properties, for which the reader can consult, for example, Cartan-Eilenberg [1]. Namely (*) induces an exact sequence

$$\mathrm{Ext}^n_A(M,H) \xrightarrow{t} \mathrm{Ext}^n_A(M,H) \longrightarrow \mathrm{Ext}^{n+1}_A(M/Mt,H)$$

for all $H \in$ mod-A. Since $hd_A(M) > n$ it is known that one can choose $H \in \underline{\underline{M}}(A)$ such that $\operatorname{Ext}_A^n(M, H) \neq 0$. But the latter is a finitely generated R-module and $\operatorname{Ext}^{n+1}(M/Mt, H) = 0$. Since $t \in \operatorname{rad} R$ this contradicts Nakayama's lemma. q.e.d.

(b) Assume S is regular for A and that $\underline{m} \in \max(R)$ is such that $\underline{m} \cap S \neq \phi$. If M is a simple $A_{\underline{m}}$-module then M is a simple $A/A\underline{m}$-module so $\operatorname{hd}_A(M) < \infty$. By (6.6), $\operatorname{hd}_{A_{\underline{m}}}(M) \leq \operatorname{hd}_A(M)$. Therefore part (a) implies rt.gl.dim.$A_{\underline{m}} < \infty$; in particular $A_{\underline{m}}$ is right regular.

Conversely, assume $A_{\underline{m}}$ is right regular for every $\underline{m}$ such that $\underline{m} \cap S \neq \phi$. Let $M \in \underline{\underline{M}}(A)$ and suppose $\operatorname{hd}_{S^{-1}A}(S^{-1}M) < \infty$. We claim then that $\operatorname{hd}_A(M) < \infty$. (The opposite implication follows from (6.6)). Moreover, this assertion (in the special case $S^{-1}M = 0$) implies that S is regular for A.

It suffices, by (6.6), to show that $\operatorname{hd}_{A_{\underline{m}}}(M_{\underline{m}}) < \infty$ for all $\underline{m} \in \max(R)$. If $\underline{m} \cap S = \phi$ then $A_{\underline{m}}$ is a localization of $S^{-1}A$, and $\operatorname{hd}_{S^{-1}A}(S^{-1}M) < \infty$. If $\underline{m} \cap S \neq \phi$ then $A_{\underline{m}}$ is right regular, by hypothesis, so $\operatorname{hd}_{A_{\underline{m}}}(M_{\underline{m}}) < \infty$. q.e.d.

§7. RANK, PIC, AND KRULL RINGS

<u>All rings in this section are commutative.</u>

(7.1) THEOREM. *Let A be a commutative ring. The following conditions on $P \in$ mod-A are equivalent:*

(1) $P \in \underline{\underline{P}}(A)$.

(2) P <u>is finitely presented and</u> $P_{\underline{m}}$ <u>is a free</u> $A_{\underline{m}}$-<u>module for all</u> $\underline{m} \in \max(A)$.

(3) P <u>is finitely generated, and</u> $P_{\underline{p}}$ <u>is a free</u> $A_{\underline{p}}$-<u>module for all</u> $\underline{p} \in \mathrm{spec}(A)$. <u>If</u> $r_{\underline{p}}$ <u>is the cardinality of an</u> $A_{\underline{p}}$-<u>basis of</u> $P_{\underline{p}}$ <u>then</u> $\underline{p} \longmapsto r_{\underline{p}}$ <u>is a continuous (i.e., locally constant) function</u> $\mathrm{spec}(A) \longrightarrow \underline{\underline{Z}}$ (discrete topology).

<u>Proof</u>. (1) $\Rightarrow$ (2). Clearly P is finitely presented, and $P_{\underline{m}}$ is $A_{\underline{m}}$-free by (2.13).

(2) $\Rightarrow$ (3). Clearly P is finitely generated. If $\underline{p} \in \mathrm{spec}(A)$ embed $\underline{p}$ in $\underline{m} \in \max(A)$. Then $P_{\underline{p}}$ is a localization of $P_{\underline{m}}$ and hence is free.

Let $n = r_{\underline{p}}$. Since $P_{\underline{p}} \simeq A^n_{\underline{p}}$ we can choose a homomorphism $d\colon A^n \longrightarrow P$ such that $d_{\underline{p}}$ is an isomorphism. We then want to show that d_q is an isomorphism for all q in a neighborhood of $\underline{p}$. If we view d as the differential in a complex C (with two non-zero terms) then supp(H(C)) is closed, by (4.7). If $q \notin \mathrm{supp}(H(C))$ then C_q is acyclic, i.e., d_q is an isomorphism.

(3) $\Rightarrow$ (2). Given $\underline{p} \in \mathrm{spec}(A)$ we can construct $d\colon A^n \longrightarrow P$ such that $d_{\underline{p}}$ is an isomorphism, as above. We claim, as above, that d is an isomorphism in a neighborhood of $\underline{p}$. Since P is finitely generated, Coker (d)s = 0 for some $s \notin \underline{p}$. Moreover, $r_q = n$ for all q in some neighborhood of $\underline{p}$, by hypothesis. If the complement of this neighborhood is $V(\underline{a})$ we can choose $t \in \underline{a}$, $t \notin \underline{p}$. If U is the complement

of $V(st)$ then for all $q \in U$, d_q is surjective (because $s \notin q$) and $r_q = n$ (because $t \notin q$ and hence $\underline{a} \not\subset q$). But an epimorphism $A_q^n \longrightarrow A_q^n$ is an isomorphism, so d_q is an isomorphism for all $q \in U$.

With this conclusion we see that (2) will follow once we prove: Suppose, for each $\underline{p} \in \text{spec}(A)$, there is an $s \notin \underline{p}$ such that, if $S = \{s^n \mid n \geq 0\}$, $S^{-1}P$ is a finitely presented $S^{-1}A$-module. Then P is a finitely presented A-module. (In the case above we use the element st constructed there, in which case $S^{-1}P \simeq (S^{-1}A)^n$.)

To prove this we first use the quasi-compactness of spec(R)(see (3.10)) to find $s_1, \ldots, s_n$ such that $S_i^{-1}P$ is finitely presented for each i, where $S_i = \{s_i^n\}$, and such that the complements of the $V(s_i)$ cover spec(R). Let

$$0 \longrightarrow K \longrightarrow A^m \longrightarrow P \longrightarrow 0$$

be an exact sequence. Then $S_i^{-1}K$ is finitely generated so there is a finite set $X_i \subset K$ whose image in $S_i^{-1}K$ generates the latter as $S_i^{-1}A$-module. Then the submodule $M \subset K$ generated by $\bigcup X_i$ is such that $M_{\underline{p}} = K_{\underline{p}}$ for all $\underline{p}$, and hence $M = K$.

(2) $\Rightarrow$ (1). $U_0(P) = \{\underline{p} \mid hd_{A_{\underline{p}}} P_{\underline{p}}) \leq 0\} = \text{spec}(A)$, by hypothesis, so $P \in \underline{\underline{P}}(A)$ by (6.5), which applies because P is finitely presented. q.e.d.

If $P \in \underline{\underline{P}}(A)$ we shall write

$$[P:\ A]:\ \text{spec}\ (A) \longrightarrow \underline{\underline{Z}}$$

for the continuous function described in part (3) above, and call this the _rank_ of P. We shall also write

$$P^* = \mathrm{Hom}_A(P, A)$$

For any module M we have h_P: $P^*\otimes_A M \longrightarrow \mathrm{Hom}_A(P, M)$ defined by $h_P(f \otimes m)(x) = mf(x)$. It is a natural transformation and h_A is clearly an isomorphism so, by additivity, h_P is an isomorphism for all $P \in \underline{\underline{P}}(A)$.

(7.2) PROPOSITION. _Let_ $P, Q \in \underline{\underline{P}}(A)$. _Then_

$$[P^*: A] = [P: A]$$

$$[P \oplus Q: A] = [P: A] + [Q: A]$$

and

$$[P \otimes_A Q: A] = [\mathrm{Hom}_A(P, Q): A] = [P: A][Q: A]$$

Moreover, P _is faithful (and hence faithfully projective_ (cf. (II, §1))) _if and only if_ $[P: A]$ _is everywhere positive_.

Proof. The formulas are obvious. The set of points where $[P: A]$ is non-zero is $\mathrm{supp}(P) = V(\mathrm{ann}(P))$. q.e.d.

(7.3) PROPOSITION. _Let_ $f: A \longrightarrow B$ _be a homomorphism of commutative rings, inducing_ ${}^a f: \mathrm{spec}(B) \longrightarrow \mathrm{spec}(A)$. _Let_ $P \in \underline{\underline{P}}(A)$ _and_ $M \in$ mod-A. _Then the natural homomorphisms_

$$(P \otimes_A M)\otimes_A B \longrightarrow (P\otimes_A B) \otimes_B (M \otimes_A B) \quad \text{and}$$

$$\mathrm{Hom}_A(P, M)\otimes_A B \longrightarrow \mathrm{Hom}_B(P\otimes_A B, M\otimes_A B)$$

are isomorphisms. Moreover

$$[P \otimes_A B: \ B] = [\underline{p}: \ A] \circ {}^{a}f$$

Proof. The tensor isomorphism is well known (and valid without restriction on P) and the Hom isomorphism follows, by additivity, from the special case P = A, when it is clear.

If $q \in \mathrm{spec}(B)$ and $\underline{p} = {}^{a}f(q) = f^{-1}(q)$ then B_q is a localization of the $A_{\underline{p}}$-algebra $B_{\underline{p}}$. Since $P_{\underline{p}}$ is $A_{\underline{p}}$-free, $(P_{\underline{p}} \otimes_{A_{\underline{p}}} B_{\underline{p}}) = (P \otimes_A B)_{\underline{p}}$, and its localization $(P \otimes_A B)_q$, are free of the same ranks as $B_{\underline{p}}$- and B_q-modules, respectively. I.e.,

$$[(P \otimes_A B)_q: \ B_q] = [P_{\underline{p}}: \ A_{\underline{p}}] \qquad \text{q.e.d.}$$

(7.4) PROPOSITION. Suppose P, Q $\in$ mod-A are such that $P \otimes_A Q \simeq A^n$ for some $n > 0$. Then P, Q $\in \underline{\underline{P}}(A)$ and they are both faithfully projective.

Proof. If $\{x_i \otimes y_i \mid 1 \leq i \leq m\}$ generates $P \otimes_A Q$ (which is finitely generated) then define $h: \ A^m \longrightarrow P$ by sending the basis elements onto the x_i's. Then $h \otimes_A Q: \ A^m \otimes_A Q \longrightarrow P \otimes_A Q \simeq A^n$ is surjective, and hence splits. Likewise, then, $h \otimes_A Q \otimes_A P$ is a split epimorphism. But the latter is isomorphic to a direct sum of n copies of h, so h is a split epimorphism. This shows that $P \in \underline{\underline{P}}(A)$, and $Q \in \underline{\underline{P}}(A)$ by symmetry. Moreover they are faithful because $P \otimes_A Q$ is.

We shall next study the category

$$\underline{\underline{\mathrm{Pic}}}\,(A) = \underline{\underline{\mathrm{Pic}}}_A(A)$$

introduced in (II, §5). The emphasis there was on two sided

A-modules, but the fact that we now view A as an algebra over itself means that the elements of A operate on objects of $\underline{\underline{\text{Pic}}}(A)$ in the same way on the right and the left. Hence we may view its objects simply as right A-modules. As such, the condition for $P \in$ mod-A to be a member of $\underline{\underline{\text{Pic}}}(A)$ is that P should be invertible, in the sense that there is a $Q \in$ mod-A such that $P \otimes_A Q \simeq A$. In this case the theory of Chapter II shows that we must have $Q \simeq \text{Hom}_A(P, A) = P^*$. Moreover, the isomorphism classes, $[P]$, of these invertible modules form a group,

$$\text{Pic}(A)$$

with multiplication $[P][Q] = [P \otimes_A Q]$.

(7.5) PROPOSITION. The following conditions on $P \in$ mod-A are equivalent.

(1) P is invertible, i.e., $P \in \underline{\underline{\text{Pic}}}(A)$.

(2) $P \in \underline{\underline{P}}(A)$ and $[P: A] = 1$

(2′) $P \in \underline{\underline{P}}(A)$ and $\text{End}_A(P) = A$.

(3) $P \in \underline{\underline{M}}(A)$ and $P_{\underline{m}} \simeq A_{\underline{m}}$ for all $\underline{m} \in \max(A)$.

Proof. (1) $\Rightarrow$ (2). Since $P^* \otimes_A P \simeq A$ we have $P \in \underline{\underline{P}}(A)$, by (7.4), and $[P^*: A][P: A] = [P: A]^2 = 1$. Since $[P: A]$ takes non-negative integer values we have $[P: A] = 1$.

(2) $\Rightarrow$ (2′). If $[P: A] = 1$ the inclusion $A \subset \text{End}_A(P)$ is locally an equality, and hence an equality.

(2′) $\Rightarrow$ (3). We know $P_{\underline{m}} \simeq A_{\underline{m}}^n$ for some $n \geq 0$, and $A_{\underline{m}} = \text{End}_{A_{\underline{m}}}(P_{\underline{m}})$ implies $n = 1$.

(3) $\Rightarrow$ (1). We have $P_{\underline{p}} \simeq A_{\underline{p}}$ for all $\underline{p} \in \text{spec}(A)$, as may be seen by localizing first at some $\underline{m} \in \max(A)$ containing $\underline{p}$. Now (7.1) implies $P \in \underline{\underline{P}}(A)$. Let $h: P^* \otimes_A P \longrightarrow A$

by $h(f \otimes x) = f(x)$. Since P is finitely presented we can identify $(P^*)_{\underline{p}} = (P_{\underline{p}})^*$, and so $h_{\underline{p}}$ is an isomorphism for all $\underline{p} \in \text{spec}(A)$. Hence h is an isomorphism, so $P \in \underline{\underline{\text{Pic}}}(A)$.

(7.6) COROLLARY. A homomorphism $A \longrightarrow B$ of commutative rings induces a functor $\otimes_A B$: $\underline{\underline{\text{Pic}}}(A) \longrightarrow \underline{\underline{\text{Pic}}}(B)$ converting $\otimes_A$ to $\otimes_B$, and hence also a homomorphism Pic(A) $\longrightarrow$ Pic(B). (The latter makes Pic a functor).

Proof. This follows from (7.3) and criterion (2) above.

Now let S be a multiplicative set of non-divisors of zero in A. If M is an A-submodule of $S^{-1}A$ then there is an induced monomorphism $S^{-1}M \longrightarrow S^{-1}A$ which is an isomorphism precisely when M generates $S^{-1}A$ as an $S^{-1}A$-module, i.e., when $(S^{-1}A)M = S^{-1}A$. In this case we have $1 = (a/s)\, x$ for some $x \in M$, and hence $s = ax \in M \cap S$. Conversely, if $M \cap S \neq \emptyset$ then clearly $(S^{-1}A)M = S^{-1}A$. If M satisfies these equivalent conditions we call M a non-degenerate A-submodule of $S^{-1}A$.

If M and N are two such, say $s \in M \cap S$ and $t \in N \cap S$, then st belongs to M + N, to $M \cap N$, and to $M \cdot N$, the submodule generated by all xy ($x \in M$, $y \in N$).

Define

$$N: \; M = \{b \in S^{-1}A \mid bM \subset N\}$$

If $b \in N: M$ then $h_b(x) = bx$ defines an element $h_b \in \text{Hom}_A(M, N)$, and hence a map $(N: M) \longrightarrow \text{Hom}_A(M, N)$ which is clearly a homomorphism. If $h_b = 0$ then $b = 0$ because b kills a non-divisor of zero in $S \cap N$. Moreover, given $h \in \text{Hom}_A(M, N)$ then we have $S^{-1}h \in \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \simeq \text{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}A) = S^{-1}A$. Thus $S^{-1}h(x) = bx$ for some $b \in S^{-1}A$.

Since $S^{-1}h(M) \subset N$ we have $b \in N: M$ and therefore $h = h_b$. We record this:

(7.7) PROPOSITION. *If* M *and* N *are non-degenerate submodules of* $S^{-1}A$ *then the natural homomorphism* $N: M \longrightarrow \mathrm{Hom}_A(M, N)$ *is bijective. In particular the inclusion* $M \subset A:(A: M)$ *is isomorphic to the natural homomorphism* $M \longrightarrow M^{**}$. *Moreover,* M^* *is reflexive.*

To obtain the last assertion we can identify $M^* = A: M$. If $M \subset N$ then $N^* \subset M^*$. Therefore, since $M \subset M^{**}$, $M^* \subset (M^*)^{**} = (M^{**})^* \subset M^*$.

An A-submodule $M \subset S^{-1}A$ is called an *invertible submodule* of $S^{-1}A$ if $M \cdot N = A$ for some $N \subset S^{-1}A$. Evidently M and N must then be non-degenerate. If we choose $s \in S \cap M \cap N$ then $Ms \subset MN = A$ and so

$$As \subset M \subset As^{-1}$$

When S is the set of all non-divisors of zero in A we call $S^{-1}A$ the *full ring of fractions* of A. We call $\mathfrak{a} \subset A$ an *invertible ideal* if it is invertible in the full ring of fractions.

(7.8) THEOREM. *Let* M *be a non-degenerate* A-*submodule of* $S^{-1}A$. *The following conditions are equivalent*:

(1) M *is an invertible submodule* of $S^{-1}A$

(2) $M \in \underline{\underline{P}}(A)$

(3) $M \in \underline{\underline{Pic}}(A)$

(4) $M \in \underline{\underline{M}}(A)$ *and* $M_{\mathfrak{m}}$ *is generated by one element for each* $\mathfrak{m} \in \max(A)$.

Proof. (1) $\Rightarrow$ (2). If $MN = A$ write $1 = \Sigma m_i n_i$, with $m_i \in M$ and $n_i \in N$. Define h_i: $M \longrightarrow A$ by $h_i(m) = n_i m$. Then for all $m \in M$, $m = \Sigma m_i n_i m = \Sigma m_i h_i(m)$ so $M \in \underline{\underline{P}}(A)$ by (II, 4.5).

(2) $\Rightarrow$ (3). Since $S^{-1}M = S^{-1}A$ we have $T^{-1}M \simeq T^{-1}A$ where $T = \{t^n \mid n \geq 0\}$ for some $t \in S$. Given $\underline{p} \in \text{spec}(A)$ there is a prime $q \subset \underline{p}$ such that $t \notin q$. For otherwise $t/1$ would be in nil $A_{\underline{p}}$, by (3.9), and we would have $t^n s = 0$ for some $n > 0$ and $s \notin \underline{p}$, contradicting the fact that t is not a divisor of zero. Now since $A_q = (A_{\underline{p}})_{q_{\underline{p}}}$ we have $[M_{\underline{p}} : A_{\underline{p}}] = [M_q : A_q]$. Since $T \cap q = \emptyset$, A_q is a localization of $T^{-1}A$, so $[M_q : A_q] = 1$.

The implications (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial. We conclude the proof by showing (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1).

(4) $\Rightarrow$ (3). Since M is non-degenerate it is a faithful A-module. Since M is finitely generated $M_{\underline{m}}$ is also a faithful $A_{\underline{m}}$-module. For if X is a finite set of generators of M and if $a/s \in A_{\underline{m}}$ annihilates $M_{\underline{m}}$ then Xa is annihilated by some $t \notin \underline{m}$. This follows because Xa is finite and it becomes zero in $M_{\underline{m}}$. Thus $at \in \text{ann}_A(M) = 0$ and therefore $a/s = at/st = 0$.

By assumption, $M_{\underline{m}}$ has one generator. Being also faithful it is $\simeq A_{\underline{m}}$.

(2) $\Rightarrow$ (1). By (7.7) we can identify M with $A: M$. Thus if $M \in \underline{\underline{P}}(A)$ it follows from (II, 4.5) that there are $m_i \in M$, $n_i \in A: M$ $(i \in I)$, for some finite set I, such that $m = \Sigma m_i n_i m$ for all $m \in M$. Since M is faithful we have $\Sigma m_i n_i = 1$ and so $M(A: M) = A$. q.e.d.

(7.9) PROPOSITION. *Let* M *and* N *be* A-*submodules of* $S^{-1}A$ *with* M *invertible. Then* $N = (N: M)M$, $N: M = N(A: M)$, *and the natural homomorphism*

$$M \otimes_A N \longrightarrow MN$$

is an isomorphism.

Proof. Let $i: N \longrightarrow S^{-1}A$ be the inclusion. Since M is projective, $M \otimes_A N \longrightarrow M \otimes_A S^{-1}A$ is a monomorphism. We can identify $M \otimes_A S^{-1}A$ with $S^{-1}A$, and then the image of $M \otimes_A N$ is MN.

Let $M' = A: M$. Then $M'N \subset N: M$, and $M(N: M) \subset N$, clearly. Therefore, since $MM' = A$, we have $N: M = M'M(N: M) \subset M'N$, and $N = MM'N \subset M(N: M)$, thus completing the proof.

We shall denote by

$$\text{Pic}(A, S)$$

the set of invertible A-submodules of $S^{-1}A$. It is a group under multiplication. Moreover, if $M \in \text{Pic}(A, S)$ then $M \in \underline{\underline{\text{Pic}}}(A)$, by (7.8), and the map

$$\text{Pic}(A, S) \longrightarrow \text{Pic}(A)$$

$M \longmapsto [M]$, is, according to (7.9) above, a homomorphism. If $b \in U(S^{-1}A)$ then Ab is invertible with $A: Ab = Ab^{-1}$. Thus we obtain a homomorphism $U(S^{-1}B) \longrightarrow \text{Pic}(A, S)$.

(7.10) PROPOSITION. *Let* S *be a multiplicative set of non-divisors of zero in* A, *as above. Then the sequence*

$$0 \longrightarrow U(A) \longrightarrow U(S^{-1}A) \longrightarrow \text{Pic}(A,S) \longrightarrow \text{Pic}(A) \longrightarrow \text{Pic}(S^{-1}A)$$

is exact.

Proof. Since $A \longrightarrow S^{-1}A$ is injective so also is

$U(A) \longrightarrow U(S^{-1}A)$. If $b \in U(S^{-1}A)$ then $bA = A \iff b \in U(A)$, clearly. If $M \in \mathrm{Ker}(\mathrm{Pic}(A, S) \longrightarrow \mathrm{Pic}(A))$ then we can choose $b' \in A$: $M \simeq \mathrm{Hom}_A(M, A)$ and $b \in M$: $A \simeq \mathrm{Hom}_A(A, M)$ inducing inverse isomorphisms

$$M \underset{b}{\overset{b'}{\rightleftarrows}} A$$

It follows that $M = bA$ and $bb' = 1$ so $b \in U(S^{-1}A)$.

If $M \in \mathrm{Pic}(A, S)$ then $S^{-1}M = S^{-1}A$ so $[M] \in \mathrm{Ker}(\mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(S^{-1}A))$. Conversely, if P lies in this kernel choose an h: $P \longrightarrow A$ such that $S^{-1}h$ is an isomorphism. Since S consists of non-divisors of zero and $P \in \underline{\underline{P}}(A)$, the map $P \longrightarrow S^{-1}P$ is injective, and hence h is also injective. Thus $P \simeq hP \subset S^{-1}A$. According to (7.8), $hP \in \mathrm{Pic}(A, S)$, and this completes the proof.

Now assume A is an integral domain and that $S' = A - \{0\}$. Thus $L = S^{-1}A$ is the field of fractions of A. Since $\mathrm{Pic}(L) = 0$ clearly it follows from (7.8) and (7.10) that an ideal $\underline{a} \subset A$ is invertible as an A-module if and only if it is an invertible as a submodule of L, in the sense discussed above. We shall then say simply that $\underline{a}$ is an invertible ideal. A is called a <u>Dedekind ring</u> if every non-zero ideal in A is invertible, and a <u>discrete valuation ring</u> (DVR) if it is a local Dedekind ring. For example a principal ideal domain is a Dedekind ring.

(7.11) PROPOSITION. <u>The following conditions on a local ring</u> A <u>with maximal ideal</u> $\underline{p} \neq 0$ <u>are equivalent</u>:

(1) A <u>is a</u> DVR.

(2) A <u>is noetherian and</u> $\underline{p} \in \underline{\underline{P}}(A)$.

(3) A <u>is an integral domain</u>, $\underline{p} = pA$ <u>is principal, and</u> $U(L) = U(A) \times \{p^n \mid n \in \underline{\underline{Z}}\}$, <u>where</u> L <u>is the field</u>

<u>of fractions of</u> A.

<u>Proof</u>. (1) $\Rightarrow$ (2). Every invertible module lies in $\underline{\underline{P}}(A)$ so A is noetherian and $\underline{p} \in \underline{\underline{P}}(A)$.

(2) $\Rightarrow$ (3). Since A is local $\underline{p} \simeq A^n$ for some $n > 0$, by (2.13) and our assumption that $\underline{p} \neq 0$. Since two elements, a and b of A cannot be linearly independent (for $ab = ba$!) we must have $n = 1$. Thus $\underline{p} = pA \simeq A$ and p is not a divisor of zero. The last property implies that $p\underline{a} = \underline{a}$ where $\underline{a} = \bigcap_n \underline{p}^n A$. Since A is noetherian and $p \in \text{rad } A$ Nakayama's lemma implies $\underline{a} = 0$. If $a \neq 0$ in A let $n \geq 0$ be the largest integer such that $a \in p^n A$. We have just seen that n exists. Write $a = up^n$; then $u \in U(A)$ for otherwise $u \in pA$ and $a \in p^{n+1}A$. Finally, if $b = vp^m$ with $v \in U(A)$ then $ab = uvp^{n+m} \neq 0$, so A is an integral domain. The decomposition $U(L) = U(A) \times \{p^n\}$ now follows easily from the remarks above.

(3) $\Rightarrow$ (1). Clearly (3) implies every non-zero ideal is principal and hence invertible. q.e.d.

Let A and $\underline{p}$ be as in (7.11). If $\underline{a} \neq 0$ is an A-module in L such that $d\underline{a} \subset A$ for some $d \neq 0$ in A, then we can write $dA = \underline{p}^n$ and $d\underline{a} = \underline{p}^m$ for some $n, m \geq 0$, and then $\underline{a} = \underline{p}^{m-n} = p^{n-m}A$. Thus every such $\underline{a}$ is a power, positive or negative, of $\underline{p}$. We shall write

$$v_{\underline{p}}(\underline{p}^n) = n \quad \text{and} \quad v_{\underline{p}}(x) = v_{\underline{p}}(xA) \quad \text{for} \quad x \in U(L)$$

Thus $v_{\underline{p}}: U(L) \longrightarrow \underline{\underline{Z}}$ is a homomorphism. Also, if we define $v_{\underline{p}}(0) = \infty$, with the usual conventions, then

$$v_{\underline{p}}(a + b) \geq \min(v_{\underline{p}}(a), v_{\underline{p}}(b))$$

(i.e., α, b ε $\underline{p}^n \Rightarrow \alpha + b \in \underline{p}^n$), with equality when $v_{\underline{p}}(\alpha)$ and $v_{\underline{p}}(b)$ differ. Also $A = \{\alpha \mid v_{\underline{p}}(\alpha) \geq 0\}$ and $\underline{p} = \{\alpha \mid v_{\underline{p}}(\alpha) > 0\}$.

Note that since the only non-zero ideals in A are of the form $\underline{p}^n$, spec(A) = $\{(0), \underline{p}\}$ and dimA = 1. Moreover, if $\alpha = up^{-n}$ with $n > 0$ then $A[\alpha] = L$, and hence α is not integral over A. This shows that: A DVR is integrally closed and of dimension ≤ 1. (We put ≤ 1 to allow for fields.)

(7.12) THEOREM. Let A be an integral domain. The following conditions are equivalent:

(1) A is a Dedekind ring.

(2) A is hereditary (see (6.2)).

(3) A is noetherian and $A_{\underline{p}}$ is a DVR for all $\underline{p} \in$ max(A).

(4) A is noetherian, integrally closed, and dim A $\leq$ 1.

Proof. The equivalence of (1) and (2) follows from (6.2), by virtue of (7.8).

(1) and (2) $\Rightarrow$ (3). By (7.8) an invertible ideal is finitely generated, so A is noetherian. By (6.4) $A_{\underline{p}}$ is hereditary for all $\underline{p}$.

(3) $\Rightarrow$ (4). For each $\underline{p} \in$ max(A), $A_{\underline{p}}$ is a DVR and hence integrally closed of dimension ≤ 1, by the remark before the statement of the theorem. In particular ht($\underline{p}$) = dim($A_{\underline{p}}$) ≤ 1 for all $\underline{p} \in$ max(A) so dim A ≤ 1. We show that A is integrally closed, moreover, by noting that the inclusion

$$A \subset B = \bigcap_{\underline{p} \in \max(A)} A_{\underline{p}}$$

is an equality. This follows because $A_{\underline{p}} \subset B_{\underline{p}} \subset A_{\underline{p}}$ for all $\underline{p} \in \max(A)$.

The implication (4) $\Rightarrow$ (1) will be proved in a more general form in §8, (8.6).

Let A be an integral domain with field of fractions L, and write $Ht_1(A)$ for the set of prime ideals of height one in A. A is called a <u>Krull ring</u> if it satisfies the following conditions:

(i) $A_{\underline{p}}$ is a DVR for all $\underline{p} \in Ht_1(A)$.

(ii) $A = \cap A_{\underline{p}}$ $(\underline{p} \in Ht_1(A))$. (The intersection is taken in L).

(iii) An $a \neq 0$ in A is contained in only finitely many $\underline{p} \in Ht_1(A)$.

Conditions (i) and (ii) imply A is integrally closed, because each $A_{\underline{p}}$ is. Condition (iii) is valid in any noetherian integral domain. For then the $\underline{p}$ of height one containing a correspond to certain irreducible components of spec(A/aA), and there are only finitely many of these since A/aA is noetherian. We mention, without proof, the following example:

(7.13) PROPOSITION. <u>A noetherian integrally closed integral domain is a Krull ring.</u>

Condition (iii) was pointed out above and condition (i) follows from (7.12). Thus only condition (ii) is left unproved. However, since $A = \cap A_{\underline{m}}$ $(\underline{m} \in \max(A))$ for any integral domain we see that (ii) is automatic if $\dim A \leq 1$, i.e., if every maximal ideal has height ≤ 1. Thus, we have proved that <u>a Dedekind ring is a Krull ring of dimension</u> ≤ 1. The converse is proved below (7.14).

For a Krull ring A we define the <u>divisor group</u> D(A) to be the free Abelian group with basis $Ht_1(A)$. We view D(A) as a partially ordered group whose positive elements are those with positive coordinates w.r.t. the basis $Ht_1(A)$. An A-module $\underline{a} \subset L$ is called a fractional ideal if $d\underline{a} \subset A$ for some $d \neq 0$ in A, and we write Frac(A) for the set of non-zero fractional ideals. It is easy to check that if $\underline{a}, \underline{b} \in$ Frac(A) then $\underline{a} + \underline{b}$, $\underline{a} \cap \underline{b}$, $\underline{a}\underline{b}$, and $\underline{a}: \underline{b} = \{x \in L \mid x\underline{b} \subset \underline{a}\}$ are also in Frac(A). There is a natural map

$$\text{div}: \text{Frac}(A) \longrightarrow D(A)$$

$$\text{div}(\underline{a}) = \Sigma v_{\underline{p}}(\underline{a})\underline{p} \qquad (\underline{p} \in Ht_1(A))$$

Here $v_{\underline{p}}$ is the valuation associated with the DVR $A_{\underline{p}}$: $\underline{a}_{\underline{p}} = (\underline{p}A_{\underline{p}})^{v_{\underline{p}}(\underline{a})}$. It follows easily from condition (iii), and the fact that $d\underline{a}$ A for some $d \neq 0$, that $v_{\underline{p}}(\underline{a})$ is defined and equals zero for almost all $\underline{p}$. In case $x \in U(L)$ we shall abbreviate: div(x) = div(xA).

The following formulas are obvious:

$$\text{div}(\underline{a}\ \underline{b}) = \text{div}(\underline{a}) + \text{div}(\underline{b})$$

$$\text{div}(\underline{a} + \underline{b}) = \inf(\text{div}(\underline{a}), \text{div}(\underline{b}))$$

$$\text{div}(\underline{a} \cap \underline{b}) = \sup(\text{div}(\underline{a}), \text{div}(\underline{b}))$$

If $\underline{a} \in$ Frac(A) write $\tilde{\underline{a}} = \cap \underline{a}_{\underline{p}}$ (all $\underline{p}$'s here are understood to vary over $Ht_1(A)$) and call $\underline{a}$ divisorial if $\underline{a} = \tilde{\underline{a}}$. Since $\underline{a} \subset \tilde{\underline{a}}$ we have $\underline{a}_{\underline{p}} \subset \tilde{\underline{a}}_{\underline{p}} \subset \underline{a}_{\underline{p}}$ for all $\underline{p}$, so $\tilde{\underline{a}}_{\underline{p}} = \underline{a}_{\underline{p}}$ for all $\underline{p}$, and hence $\tilde{\underline{a}}$ is divisorial. Moreover, $\text{div}(\tilde{\underline{a}}) = \text{div}(\underline{a})$ and, since $\text{div}(\underline{a})$ determines $\tilde{\underline{a}}$, we have $\text{div}(\underline{a}) = \text{div}(\underline{b}) \iff \tilde{\underline{a}} = \tilde{\underline{b}}$.

If $\underline{a}, \underline{b} \in$ Frac(A) then $(\underline{a}: \underline{b})_{\underline{p}} \subset (\underline{a}_{\underline{p}}: \underline{b}_{\underline{p}})$ for all $\underline{p}$

(always in $Ht_1(A)$) so $(\underline{a}: \underline{b})^{\sim} \subset \cap (\underline{a}_{\underline{p}}: \underline{b}_{\underline{p}})$. Now $d\tilde{\underline{b}} \subset \tilde{\underline{a}} \iff$ $d\underline{b}_{\underline{p}} \subset \underline{a}_{\underline{p}}$ for all $\underline{p} \iff d\underline{b} \subset \underline{a}_{\underline{p}}$ for all $\underline{p} \iff d\underline{b} \subset \tilde{\underline{a}}$, i.e., $(\tilde{\underline{a}}: \tilde{\underline{b}}) = \cap (\underline{a}_{\underline{p}}: \underline{b}_{\underline{p}}) = (\tilde{\underline{a}}: \underline{b})$. But we also have $(\tilde{\underline{a}}: \underline{b}) \subset (\tilde{\underline{a}}: \underline{b})^{\sim} \subset \cap (\tilde{\underline{a}}_{\underline{p}}: \underline{b}_{\underline{p}}) = (\underline{a}_{\underline{p}}: \underline{b}_{\underline{p}})$. Putting these relationships together we conclude that

$$(\underline{a}: \underline{b})^{\sim} = \cap (\underline{a}_{\underline{p}}: \underline{b}_{\underline{p}}) = (\tilde{\underline{a}}: \underline{b}) = (\tilde{\underline{a}}: \tilde{\underline{b}})$$

In particular

$$\operatorname{div}(\underline{a}: \underline{b}) = \operatorname{div}(\underline{a}) - \operatorname{div}(\underline{b})$$

and $(\underline{a}: \underline{b})$ is divisorial if $\underline{a}$ is. This formula shows that $A: \underline{a}$ is divisorial and $\operatorname{div}(A: \underline{a}) = -\operatorname{div}(\underline{a})$, so it follows that

$$(*) \qquad \tilde{\underline{a}} = A: (A: \underline{a})$$

(7.14) PROPOSITION. If A is a Krull ring of dimension ≤ 1 then A is a Dedekind ring.

Proof. We can assume A is not a field, so that $Ht_1(A) = \max(A)$. Therefore every $\underline{a} \in \operatorname{Frac}(A)$ is divisorial, and $\operatorname{div}: \operatorname{Frac}(A) \longrightarrow D(A)$ is an injective homomorphism of monoids. Since Im(div) is a group so also is Frac(A), i.e., all $\underline{a} \in \operatorname{Frac}(A)$ are invertible. q.e.d.

It follows from (*) above that the group Cart(A) ("Cartier divisors") of invertible ideals consists of divisorial ideals, so we have a monomorphism $\operatorname{Cart}(A) \longrightarrow D(A)$. If $S = A - \{0\}$ then $\operatorname{Cart}(A) = \operatorname{Pic}(A, S)$ in the notation of (7.10). There is a commutative diagram with exact rows

$$
(1)\qquad
\begin{array}{ccccccccc}
U(A) & \longrightarrow & U(L) & \xrightarrow{\text{div}} & D(A) & \longrightarrow & C(A) & \longrightarrow & 0 \\
\| & & \| & & \uparrow & & \uparrow & & \\
U(A) & \longrightarrow & U(L) & \longrightarrow & \mathrm{Cart}(A) & \longrightarrow & \mathrm{Pic}(A) & \longrightarrow & 0
\end{array}
$$

whose bottom row is the sequence of (7.10) (using the fact that Pic(L) = 0). Since Cart(A) ⟶ D(A) is injective Pic(A) ⟶ C(A) is also.

Now let S be any multiplicative set in A - {0}. Then it is easily seen that $S^{-1}A$ is a Krull ring whose primes of height one correspond to those of A not meeting S. With this identification we can write

$$D(A) = D(S^{-1}A) \oplus D(A, S)$$

where D(A, S) is the subgroup generated by $\{\underline{p} \in Ht_1(A) \mid \underline{p} \cap S \neq \emptyset\}$. We then easily deduce a commutative diagram

$$
(2)\qquad
\begin{array}{ccccccccc}
U(A) & \to & U(S^{-1}A) & \xrightarrow{\text{div}} & D(A,S) & \to & C(A) & \to & C(S^{-1}A)\to 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
U(A) & \to & U(S^{-1}A) & \longrightarrow & \mathrm{Pic}(A,S) & \to & \mathrm{Pic}(A) & \to & \mathrm{Pic}(S^{-1}A)
\end{array}
$$

whose rows are exact and whose verticals are monomorphisms. The bottom row comes from (7.10).

(7.15) PROPOSITION. *If* S *above is generated by elements which generate prime ideals then* C(A) ⟶ $C(S^{-1}A)$ *is an isomorphism, and hence* Pic(A) ⟶ $\mathrm{Pic}(S^{-1}A)$ *is a monomorphism* (in diagram (2) above).

Proof. Let $(p_i)_{i \in I}$ be generators of S such that p_iA is prime. If $\underline{p}$ is prime and $\underline{p} \cap S \neq \emptyset$ then $\underline{p}$ contains a product of the p_i's, and hence some $p_i \in \underline{p}$. Since p_iA is prime it follows that $p_iA = \underline{p}$ if $ht(\underline{p}) = 1$, so $\underline{p} \in$ Im $(U(S^{-1}A) \xrightarrow{\text{div}} D(A, S))$. This is true for all such $\underline{p}$ so div is surjective. The proposition now follows from the properties of diagram (2).

We call a ring A <u>factorial</u> is it is a Krull ring for which C(A) = 0.

(7.16) PROPOSITION. *Let* A *be factorial and let* $\underline{a} \in$ Frac(A) *be divisorial. Then* $\underline{a}$ *is principal and* $\underline{a} = \Pi \underline{p}^{v_{\underline{p}}(\underline{a})}$ $(\underline{p} \in Ht_1(A))$.

Proof. If $\underline{p} \in Ht_1(A)$ there is an $a \neq 0$ such that $\underline{p} =$ div(a). Since aA and (clearly) also $\underline{p}$ are divisorial it follows that $\underline{p} = a$A. Now given $\underline{a}$ as above set $\underline{b} = \Pi \underline{p}^{v_{\underline{p}}(a)}$. Then $\underline{b}$ is principal, as we have just seen, and hence divisorial. Since div($\underline{a}$) = div($\underline{b}$) and $\underline{a}$ is divisorial, by assumption, we have $\underline{a} = \underline{b}$. q.e.d.

Let A be any commutative ring and let T be a multiplicative set in A. We say T is <u>factorial for</u> A is $A_{\underline{m}}$ is factorial for all $\underline{m} \in \max(A)$ such that $\underline{m} \cap T \neq \phi$.

(7.17) PROPOSITION. *Let* A *be a commutative noetherian ring and let* T *be a multiplicative set of non-divisors of zero which is factorial for* A. *Then* Pic(A) ⟶ Pic(T^{-1}A) *is surjective, and* Pic(A, T) *is a free Abelian group with* $M = \{\underline{p} \in Ht_1(A) \mid \underline{p} \cap T \neq \phi\}$ *as a basis*.

Proof. We shall assume for the proof that, if S is the set of all non-divisors of zero, that Pic(S^{-1}A) = 0. It is known (see, e.g., Bourbaki [4], §5, no. 7, Remarque 2) that S^{-1}A is semi-local, and we shall prove in (IX, 3.5) that Pic(B) = 0 if B is semi-local. Thus the assumption is justifiable.

We will first show that if P ∈ $\underline{\underline{M}}$(A) is reflexive (i.e., $P \longrightarrow P^{**}$ is an isomorphism) and if $T^{-1}P \in \underline{\underline{\text{Pic}}}(T^{-1}A)$ then $P \in \underline{\underline{\text{Pic}}}(A)$.

We must show that $P_{\underline{m}} \simeq A_{\underline{m}}$ for all $\underline{m} \in \max(A)$. If $\underline{m} \cap T = \emptyset$ then $P_{\underline{m}}$ is a localization of $T^{-1}P$ so this follows because $T^{-1}P \in \underline{\underline{\text{Pic}}}(T^{-1}A)$. If $\underline{m} \cap T \neq \emptyset$ then $A_{\underline{m}}$ is factorial, by hypothesis. Since $T \subset S$ and since $\text{Pic}(S^{-1}A) = 0$ it follows that $S^{-1}P = S^{-1}(T^{-1}P) \simeq S^{-1}A$. Let $S_{\underline{m}}$ denote the image of S in $A_{\underline{m}}$. Since S consists of non-divisors of zero it is easily checked that $0 \notin S_{\underline{m}}$. Further $S_{\underline{m}}^{-1}P_{\underline{m}} \simeq (S^{-1}P)_{\underline{m}} \simeq (S^{-1}A)_{\underline{m}} \simeq S_{\underline{m}}^{-1}A_{\underline{m}}$. Moreover, since we are dealing with finitely generated modules over noetherian rings Hom commutes with localization. Thus $P_{\underline{m}}$ is reflexive, so $P_{\underline{m}} \subset S_{\underline{m}}^{-1}P_{\underline{m}} \simeq S_{\underline{m}}^{-1}A_{m}$ and $P_{\underline{m}}$ is isomorphic to a reflexive, hence divisorial, fractional ideal of $A_{\underline{m}}$. Now (7.16) implies $P_{\underline{m}} \simeq A_{\underline{m}}$.

To show $\text{Pic}(A) \longrightarrow \text{Pic}(T^{-1}A)$ is surjective suppose $Q \in \underline{\underline{\text{Pic}}}(T^{-1}A)$. Since $\text{Pic}(S^{-1}A) = 0$ it follows that $Q \simeq \underline{a}$ for some ideal $\underline{a} \subset T^{-1}A$ such that $S^{-1}\underline{a} = S^{-1}A$. Set $\underline{a}_0 = \underline{a} \cap A$. Then $\underline{a}_0 \cap S \neq \emptyset$ so $\underline{a}_0$ is a non-degenerate A-submodule of $S^{-1}A$ in the sense of (7.7). Moreover, (7.7) implies $\underline{b} = A: (A: \underline{a}_0) \simeq \underline{a}_0^{**}$ is reflexive. We have $T^{-1}\underline{b} \simeq (T^{-1}\underline{a}_0)^{**} = \underline{a}^{**} \simeq \underline{a}$ since $\underline{a}$ is invertible. Now the last paragraph shows that $\underline{b} \in \underline{\underline{\text{Pic}}}(A)$, and $[Q] \in \text{Pic}(T^{-1}A)$ is the image of $[\underline{b}] \in \text{Pic}(A)$.

Next suppose $\underline{p} \in M$, i.e., $\text{ht}(\underline{p}) = 1$ and $\underline{p} \cap T \neq \emptyset$. If $\underline{m} \cap T = \emptyset$ then $\underline{p}_{\underline{m}} = A_{\underline{m}}$ and if $\underline{m} \cap T \neq \emptyset$ then $\underline{p}_{\underline{m}}$ is either

$A_{\underline{m}}$ or a prime of height one in a factorial ring. Thus $\underline{p}$ is invertible, and hence $\underline{p} \in \text{Pic}(A, T)$. It follows now from (7.11) that $A_{\underline{p}}$ is a DVR. Thus we can define a "divisor homomorphism" div: $\text{Pic}(A, T) \longrightarrow \underline{\underline{Z}}^{(M)}$ by $\text{div}(\underline{a}) = \Sigma v_{\underline{p}}(\underline{a})\underline{p}$ ($\underline{p} \in M$). This is a homomorphism of partially ordered groups, where $\underline{a} \in \text{Pic}(A, T)$ is ≥ 0 if $\underline{a} \subset A$. We have seen that it is surjective ($\text{div}(\underline{p}) = \underline{p}$ for $\underline{p} \in M$). The ordering makes it sufficient, for injectivity, to show that if $\underline{a} \in \text{Pic}(A, T)$ and $\underline{a} \subset A$ then $\text{div}(\underline{a}) = 0 \Rightarrow \underline{a} = A$, i.e., that $\underline{a}_{\underline{m}} = A_{\underline{m}}$ for all $\underline{m} \in \max(A)$. This is true if $\underline{m} \cap T = \phi$ because $\underline{a} \in$ Pic (A, T). Otherwise $A_{\underline{m}}$ is factorial, and $\underline{a}_{\underline{m}}$ is invertible in $A_{\underline{m}}$. If $\underline{p} \in \text{Ht}_1(A)$ and $\underline{a} \subset \underline{p}$ then $\underline{p} \cap T \neq \phi$ so $\underline{p} \in M$. Thus $\underline{a}_{\underline{m}}$ belongs to no primes of height one in $A_{\underline{m}}$ so (7.16) implies $\underline{a}_{\underline{m}} = A_{\underline{m}}$. q.e.d.

(7.18) COROLLARY. Let $R \subset A$ be commutative noetherian rings, and let $T \subset R$ be a multiplicative set of non-divisors of zero (in A) which is factorial for R and for A. Let M and M′ denote the sets of prime ideals of height one in R, respectively, in A, which meet T. Assume that if $\underline{p} \in M$ then $\underline{p}' = \underline{p}A$ is a prime ideal and $\underline{p}' \cap R = p$. Then $\underline{p}' \in M'$ and the resulting map $M \longrightarrow M'$ is bijective. It induces an isomorphism $\text{Pic}(R, T) \longrightarrow \text{Pic}(A, T)$, $\underline{a} \longmapsto \underline{a}A$.

Proof. If $\underline{p} \in M$ then $\underline{p}$ is invertible, by (7.17), and hence $\underline{p}' = \underline{p}A$ is an invertible A-ideal. (For if $1 = \Sigma a_i b_i$ with $a_i \in \underline{p}$ and $b_i\underline{p} \subset R$ for each i, then also $b_i\underline{p}' \subset A$ for each i.) Hence $A_{\underline{p}'}$ is a local ring with invertible maximal ideal, so (7.11) implies $A_{\underline{p}'}$ is a DVR. In particular

$ht(\underline{p}') = 1$, and this shows that $\underline{p}' \in M'$. Conversely, if $\underline{p}' \in M'$ then $\underline{p} = \underline{p}' \cap R$ is a prime that meets T, and it therefore contains some $\underline{p}_0 \in M$. Since $\underline{p}_0 \subset \underline{p}$ we have $\underline{p}_0 A \subset \underline{p}A \subset \underline{p}'$. Since $ht(\underline{p}') = 1$ this implies that $\underline{p}_0 A = \underline{p}A = \underline{p}'$, because $\underline{p}_0 A$ is prime. Therefore $\underline{p} = \underline{p}' \cap A = \underline{p}_0 A \cap A = \underline{p}_0 \in M$. This establishes that $\underline{p} \longmapsto \underline{p}A$ is a bijection $M \longrightarrow M'$.

If $\underline{a} \in Pic(R, T)$ then $\underline{a}A \in Pic(A, T)$ because $\underline{a}A$ still meets T, and $\underline{a}A$ is invertible over A (cf. beginning of the proof above). The resulting map $Pic(R, T) \longrightarrow Pic(A, T)$ is a homomorphism. According to (7.17) these are free Abelian groups with bases M and M′, respectively. Hence the first part of the proof shows that the homomorphism is an isomorphism.

We shall close this section now by quoting, without proof, the following basic results.

(7.19) PROPOSITION. *Let* A *be a Krull ring and let* t *be an indeterminate. Then* A[t] *is a Krull ring, and* $A \longrightarrow A[t]$ *induces isomorphisms* $C(A) \longrightarrow C(A[t])$ *and* $Pic(A) \longrightarrow Pic(A[t])$. (See Bourbaki [7], §1, nos. 9-10).

(7.20) THEOREM. *Let* A *be an integral domain with field of fractions* L, *and let* A′ *be the integral closure of* A *in a finite field extension* L′ *of* L.

(a) *If* A *is a finitely generated algebra over a field then* A′ *is a finite* A-*algebra, i.e.*, $A' \in \underline{\underline{M}}(A)$. (See Bourbaki [5], §3, no. 2.)

(b) *Suppose* A *is a Krull ring. Then* A′ *is a Krull ring. Moreover, if* L′ *is separable over* L *then* A′ *is contained in a finitely generated* A-*module in* L′. (See Bourbaki [7], §1, no. 8 and [5], §1, no. 7. Cf., also (8.5) below.)

(7.21) THEOREM. *Let* A *be a commutative noetherian* ring and let S be a multiplicative set which is regular for A (see §6). Then S is factorial for A.

By (6.7) we know that if $\underline{m} \in \max(A)$ and if $\underline{m} \cap S \neq \phi$ then $A_{\underline{m}}$ is regular. The theorem asserts that $A_{\underline{m}}$ must then also be factorial. Thus we want to know that a regular local ring is factorial. This is a well known theorem of Auslander -Buchshaum [1].

§8. ORDERS IN SEMI-SIMPLE ALGEBRAS

We fix an integral domain R with field of fractions L. Our purpose is to study certain R-algebras A contained in semi-simple L-algebras. This material will be applied in Chapter XI to examples like group rings, $A = R\pi$, where π is a finite group.

Let $V \in \underline{\underline{M}}(L)$. An R-lattice in V is an R-submodule $M \subset V$ satisfying the following conditions, which are equivalent:

(i) $ML = V$ and M is contained in a finitely generated R-module $N \subset V$.

(ii) There are free R-modules F, F′ of rank [V: L] such that $F \subset M \subset F' \subset V$.

Since L is a localization of R ($L = R_{(0)}$) an inclusion $M \subset V$ induces a monomorphism $M \otimes_R L \longrightarrow V$ with image ML. If $F \subset V$ is R-free of rank [V: L] then dimension count shows that FL = V. Thus (ii) $\Rightarrow$ (i) is clear. Conversely, if ML = V let F be the R-module generated by an L-basis for V in M. Suppose $x \in V$. Since $(V/F) \otimes_R L = 0$ we have $x\alpha \in F$ for some $\alpha \neq 0$. Taking products we can find one α which does this for a finite set of x's. Therefore, if $N \in \underline{\underline{M}}(R)$ we have $N\alpha \subset F$ for some $\alpha \neq 0$ in R. It follows that $F \subset M \subset \alpha^{-1}F$, thus proving (ii).

Similar arguments will show that if M is an R-lattice in V and if N is an R-submodule of V, then N is an R-lattice if and only if $M \subset N \subset \alpha^{-1}M$ for some $\alpha \neq 0$ in R. More generally, N is an R-lattice if it can be sandwiched between two R-lattices. We leave these remarks as well as the following proposition as exercises. All L-modules are assumed finite dimensional.

(8.1) PROPOSITION. (Bourbaki [7], §4, no. 1, Prop. 3)

(1) If M_1 and M_2 are R-lattices in V then so also are $M_1 + M_2$ and $M_1 \cap M_2$.

(2) If $W \subset V$ are L-modules and if M is an R-lattice in V then $M \cap W$ is an R-lattice in W.

(3) If $f: V_1 \times \cdots \times V_n \longrightarrow V$ is a multilinear map of L-modules, and if M_i is an R-lattice in V_i $(1 \leq i \leq n)$ then the R-module generated by $f(M_1 \times \cdots \times M_n)$ is an R-lattice in the L-module generated by $f(V_1 \times \cdots \times V_n)$.

(4) Let $M \subset V$ and $N \subset W$ be R-lattices. Then $N: M$ is an R-lattice in $\mathrm{Hom}_L(V, W)$, where $N: M = \{h \mid h(M) \subset N\}$ is caronically isomorphic to $\mathrm{Hom}_R(M,N)$.

(5) If S is a multiplicative set in A - {0}, and if M is an R-lattice in V, then $S^{-1}M$ is an $S^{-1}R$-lattice in V.

The next proposition is a basic tool for constructing and enlarging lattices.

(8.2) PROPOSITION. Assume R is a Krull ring, and let M be an R-lattice in the L-module V. Suppose we are given, for each $\underline{p} \in Ht_1(R)$, an $R_{\underline{p}}$-lattice $N_{\underline{p}}$ in V. Then a necessary and sufficient condition for the existence of an R-lattice N' in V such that $N'_{\underline{p}} = N_{\underline{p}}$ for all $\underline{p} \in Ht_1(R)$ is that $N_{\underline{p}} = M_{\underline{p}}$ for all but finitely many $\underline{p} \in Ht_1(R)$. In this case

$\tilde{N} = \cap N_{\underline{p}}$ $(\underline{p} \in Ht_1(R))$ <u>is the largest such</u> R-<u>lattice</u>.

<u>Proof</u>. We first show that if N' is any lattice (in V) then $M_{\underline{p}} = N'_{\underline{p}}$ for almost all $\underline{p}$. For we can choose $a \neq 0$ in R such that $aM \subset N' \subset a^{-1}M$, and $a \in U(R_{\underline{p}})$ for almost all $\underline{p}$.

Next suppose $N_{\underline{p}} = M_{\underline{p}}$ for almost all $\underline{p}$, and set $\tilde{N} = \cap N_{\underline{p}}$ (all $\underline{p}$'s here range over $Ht_1(R)$). If N' is a lattice such that $N'_{\underline{p}} = N_{\underline{p}}$ for all $\underline{p}$ then clearly $N' \subset \tilde{N}$. Thus the proposition will follow once we show that $\tilde{N}$ is an R-lattice. According to the first part of the proof our hypothesis on the $N_{\underline{p}}$'s is independent of the lattice M with which we compare them. Thus there is no loss in assuming that M is R-free. Since $R = \cap R_{\underline{p}}$ it then follows that $M = \cap M_{\underline{p}}$.

Let $I = \{\underline{p} \mid M_{\underline{p}} \neq N_{\underline{p}}\}$, a finite set. For each $\underline{p} \in I$ there is an $a_{\underline{p}} \neq 0$ in $R_{\underline{p}}$ such that

$$a_{\underline{p}} M_{\underline{p}} \subset N_{\underline{p}} \subset a_{\underline{p}}^{-1} M_{\underline{p}}$$

and we can certainly take $a_{\underline{p}} \in R$, after changing it by a unit in $R_{\underline{p}}$. Set $a = \prod_{\underline{p} \in I} a_{\underline{p}} \in R$. Then $aM_{\underline{p}} \subset N_{\underline{p}} \subset a^{-1}M_{\underline{p}}$ for all $\underline{p}$. We have arranged this for the $\underline{p} \in I$, and $M_{\underline{p}} = N_{\underline{p}}$ if $\underline{p} \notin I$. Taking intersections we have $aM \subset \tilde{N} \subset a^{-1}M$, thanks to the fact that $M = \cap M_{\underline{p}}$. Thus $\tilde{N}$ is a lattice. q.e.d.

We shall call an R-lattice M in V <u>divisorial</u> if $M = \cap M_{\underline{p}}$.

(8.3) COROLLARY. *If* M *is an* R-*lattice in* V *then the divisorial* R-*lattices in* M *satisfy the ascending chain condition.*

Proof. Let $D^1 \subset D^2 \subset \cdots$ be a chain of divisorial R-lattices contained in M, and let $I = \{\underline{p} \in Ht_1(R) \mid D^1_{\underline{p}} \neq M_{\underline{p}}\}$. If $\underline{p} \in I$ then, since $M_{\underline{p}}$ is a noetherian $R_{\underline{p}}$-module, the chain $D^1_{\underline{p}} \subset D^2_{\underline{p}} \subset \cdots$ stablizes. Since I is finite there is an n such that $D^n_{\underline{p}} = D^m_{\underline{p}}$ for all $m \geq n$ and all $\underline{p} \in I$. For $\underline{p} \notin I$ we have $D^1_{\underline{p}} = M_{\underline{p}}$ so this property persists. Therefore, for $m \geq n$, $D^m = \cap D^m_{\underline{p}} = \cap D^n_{\underline{p}} = D^n$, because the D's are divisorial. q.e.d.

(8.4) COROLLARY. *Let* M *and* N *be* R-*lattices in* V *and* W, *respectively, and let* S *be a multiplicative set in* R. *Assume* N *is divisorial. Then* $S^{-1}(N\colon M) = (S^{-1}N)\colon (S^{-1}M)$. *Moreover*

$$(N\colon M)^{\sim} = (N\colon M) = (N\colon \tilde{M})$$

Proof. Suppose $h\colon V \longrightarrow W$ and $hS^{-1}M \subset S^{-1}N$. Let $I = \{\underline{p} \in Ht_1(R) \mid hM_{\underline{p}} \not\subset N_{\underline{p}}\}$. This is a finite set and if $\underline{p} \in I$ then $\underline{p} \cap S \neq \phi$. Choose $a_{\underline{p}} \in \underline{p} \cap S$ such that $a_{\underline{p}} hM_{\underline{p}} \subset N_{\underline{p}}$ for each $\underline{p} \in I$ and set $s = \prod_{\underline{p} \in I} a_{\underline{p}}$. Then $shM_{\underline{p}} \subset N_{\underline{p}}$ for all $\underline{p} \in I$, and therefore for all $\underline{p} \in Ht_1(R)$. It follows that $shM \subset \cap N_{\underline{p}} = \tilde{N} = N$, so $h = sh/s \in S^{-1}(N\colon M)$. The opposite inclusion $S^{-1}(N\colon M) \subset S^{-1}N\colon S^{-1}M$ is obvious.

Using the first part of the proof and the fact that

$N = \tilde{N}$ we have $\dot{N}: \tilde{M} \subset N: M \subset (N: M)^{\sim} = \cap(N: M)_{\mathfrak{p}} = \cap(N_{\mathfrak{p}}: M_{\mathfrak{p}}) = \{h \mid hM_{\mathfrak{p}} \subset N_{\mathfrak{p}}$ for all $\mathfrak{p}\} = (\tilde{N}: \tilde{M}) = (N: \tilde{M})$.

q.e.d.

For the rest of this section we assume R is a Krull ring. If Λ is a finite L-algebra we call an R-algebra $A \subset \Lambda$ an R-order in Λ if $AL = \Lambda$ and if each element of A is integral over R.

Let M be an R-lattice in Λ. Then $M \cdot M$ is also an R-lattice (see (8.1)(3)) so $aM \cdot M \subset M$ for some $a \neq 0$ in R. Setting $N = a^{-1}M$ we have $N \cdot N \subset N$ so $R \cdot 1 + N$ is an R-algebra in Λ which is also an R-lattice. In particular it is an R-order in Λ, so R-orders exist. Our first aim is to show that, if Λ is semi-simple, that any R-order is contained in a maximal one, and that the latter are sometimes R-lattices.

Suppose first that Λ is simple with center L. By the theory of central simple algebras (Bourbaki [2]) there is a field extension L' of L and an isomorphism $\alpha: \Lambda \otimes_L L' \longrightarrow M_n(L')$ (where $[\Lambda: L] = n^2$). We then define the reduced trace and reduced norm by

$$\mathrm{Trd}_{\Lambda/L}(x) = \mathrm{Tr}(\alpha(x \otimes 1))$$

$$\mathrm{Nrd}_{\Lambda/L}(x) = \det(\alpha(x \otimes 1))$$

Since α is determined up to an inner automorphism of $M_n(L')$ the definitions are insensitive to the choice of α. It is then easy to see that they are unchanged if we enlarge L', and therefore it is independent of L', as we see by embedding two field extensions in a common one. Finally, it is known that L' can be chosen to be a galois extension, say with group G. Then one checks that $\mathrm{Trd}_{\Lambda/L}(x)$ and $\mathrm{Nrd}_{\Lambda/L}(x)$, for $x \in \Lambda$, are fixed by G, and hence lie in L. In conclusion, we have an L-linear map

$$\mathrm{Trd}_{\Lambda/L}: \Lambda \longrightarrow L$$

and a multiplicative map

$$\mathrm{Nrd}_{\Lambda/L}: \quad \Lambda \longrightarrow L$$

Moreover:

$$x \in U(\Lambda) \iff \mathrm{Nrd}_{\Lambda/L}(x) \neq 0$$

and

$$\Lambda \times \Lambda \longrightarrow L, \qquad (x, y) \longmapsto \mathrm{Trd}_{\Lambda/L}(xy)$$

is a non-degenerate bilinear form. The first assertion depends on the observation that, if $x \in U(\Lambda)$, then $x^{-1} \in L[x]$. The second follows from the fact that the trace form on $M_n(L')$ is non-degenerate, and that a degenerate form cannot become non-degenerate under extension of the base field.

Suppose $x \in \Lambda$ is integral over R. Then, in the notation above, $y = \alpha(x \otimes 1) \in M_n(L')$ is integral over R. It follows therefore from (5.14) that $P(t) = \det(t \cdot I - y)$ has coefficients which are integral over R. In particular, since R is integrally closed:

(1) <u>If</u> $x \in \Lambda$ <u>is integral over</u> R <u>then</u> $\mathrm{Trd}_{\Lambda/L}(x)$ <u>and</u> $\mathrm{Nrd}_{\Lambda/L}(x)$ <u>lie in</u> R.

Now let Λ be any semi-simple L-algebra. Write $\Lambda = \Pi\Lambda_i$ where Λ_i is simple, with center C_i. Then we define $\mathrm{Trd}_{\Lambda_i/L} = \mathrm{Tr}_{C_i/L} \circ \mathrm{Trd}_{\Lambda_i/C_i}$, and we define $\mathrm{Trd}_{\Lambda/L}((x_i)) = \Sigma\mathrm{Trd}_{\Lambda_i/L}(x_i)$. We similarly define $\mathrm{Nrd}_{\Lambda/L}((x_i)) = \Pi\mathrm{Nrd}_{\Lambda_i/L}(x_i)$ where $\mathrm{Nrd}_{\Lambda_i/L} = N_{C_i/L} \circ \mathrm{Nrd}_{\Lambda_i/C_i}$. It is easy to see that property (1) above remains valid in this more general setting.

If each C_i is a separable field extension of L then Λ is called a <u>separable</u> L-<u>algebra</u>. This is equivalent to the condition that $\Lambda \otimes_L L'$ is semi-simple for all field extensions L' of L. In this case $Tr_{C_i/L}: C_i \longrightarrow L$ is not zero for each i so it follows easily that $(x, y) \longmapsto Trd_{\Lambda/L}(xy)$ is a non-degenerate bilinear form.

(8.5) THEOREM. <u>As above, let</u> R <u>be a Krull ring with field of fractions</u> L <u>and let</u> Λ <u>be a semi-simple finite</u> L-<u>algebra. Then every</u> R-<u>order</u> A <u>in</u> Λ <u>is contained in a maximal</u> R-<u>order. If</u> Λ <u>is a separable</u> L-<u>algebra, or if</u> R <u>is a finitely generated algebra over a field, then</u> A <u>is an</u> R-<u>lattice in</u> Λ. <u>Moreover</u>, A <u>is then maximal if and only if</u> A <u>is a divisorial</u> R-<u>lattice and</u> $A_{\underline{p}}$ <u>is a maximal</u> $R_{\underline{p}}$-<u>order in</u> Λ <u>for each</u> $\underline{p} \in Ht_1(R)$.

<u>Proof</u>. <u>Case 1</u>. Λ is separable over R.

Choose a basis $e_1, \ldots, e_n \in A$ for Λ. Since $Trd_{\Lambda/L}(xy)$ is a non-degenerate form we can find $e_1', \ldots, e_n' \in \Lambda$ such that $Trd_{\Lambda/L}(e_i e_j') = \delta_{ij}$. If B is an R-order containing A write $b = \Sigma e_j' \alpha_j$ with $\alpha_j \in L$. Since $e_i b \in B$ is integral over R for each i, we have $Trd_{\Lambda/L}(e_i b) = \Sigma_j Tr(e_i e_j')\alpha_j = \alpha_i \in R$, by (1) above. Therefore, $B \subset \Sigma e_j' R = F$, so B is an R-lattice in Λ. Moreover, $\tilde{B} = \cap B_{\underline{p}}$ ($\underline{p} \in Ht_1(R)$) is a divisorial R-lattice, and hence it is an R-order containing B (see (8.2)). By (8.3) the divisorial R-orders in F satisfy the ascending chain condition, so there is a maximal one containing A. The remarks above imply that it must be a maximal R-order.

If A is a maximal R-order then we have seen that A must be a divisorial R-lattice. If $A_{\underline{p}_0}$ is not maximal over $R_{\underline{p}_0}$ for some $\underline{p}_0 \in Ht_1(R)$ then we can use (8.2) to construct a divisorial R-order B such that $B_{\underline{p}} = A_{\underline{p}}$ if $\underline{p} \neq \underline{p}_0$ and $B_{\underline{p}_0}$

properly contains $A_{\underline{p}_0}$. This contradicts the maximality of A.

Conversely, suppose A is a divisorial R-order and that $A_{\underline{p}}$ is maximal for all $\underline{p} \in Ht_1(R)$. Then if B is an R-order containing A we have $B_{\underline{p}} = A_{\underline{p}}$ for all $\underline{p}$ so $B \subset \tilde{A} = A$.

Note that the arguments in the last two paragraphs used only the fact that every R-order in Λ is an R-lattice.

General case. Write $\Lambda = \Pi\Lambda_i$ where Λ_i is simple with center C_i, and let R_i' be the integral closure of R in C_i. If A_i is the projection of A in Λ_i then clearly $R_i'[A_i]$ is an R-order in Λ_i (see (5.5)). By (7.19) R_i' is a Krull ring which is a finite R-algebra if R is a finitely generated algebra over a field. By Case 1, we can embed $R_i'[A_i]$ in a maximal R_i'-order B_i in Λ_i, and B_i is an R_i'-lattice. Evidently, B_i is a maximal R-order in Λ_i, and B_i is an R-lattice in case R_i is a finite R-algebra. It is now easy to see that $B = \Pi B_i$ is a maximal R-order containing A. For any order containing B must decompose into a product of orders containing B_i, which are maximal. If R is a finitely generated algebra over a field we have seen that each B_i, and hence also B and A are R-lattices. By virtue of the remark at the end of case 1, this proves the theorem.

For the remainder of this section we make the following assumptions:

(2) R *is a Krull ring with field of fractions* L.
Λ *is a semi-simple finite* L-*algebra*.
Every R-*order in* Λ *is an* R-*lattice*.

Let A be an R-order in Λ and let $V \in \underline{\underline{M}}(\Lambda)$. We shall call an R-lattice in V an A-*lattice* if it is an A-submodule of V. These always exist. For let $M \subset \Lambda$ be a finitely generated R-module containing A (which exists by (2) above) and let $(e_i)_{1 \leq i \leq n}$ be an L-basis for V. Then $\Sigma e_i A$ is an

A-submodule of V containing a basis and contained in the finitely generated R-module $\Sigma e_i M$. In Λ itself we shall speak of left, right, and two-sided A-lattices, in the obvious sense.

(8.6) THEOREM. Assume R is a Dedekind ring, and let A be a maximal R-order in Λ. Let max(A) denote the set of maximal two sided ideals in A. Then the set of two sided A-lattices in Λ is, under multiplication, a free Abelian group with max(A) as a basis.

REMARKS. We shall use only the fact that R is a noetherian integrally closed integral domain of dimension ≤ 1. Thus, in the special case $\Lambda = L$, this theorem will imply that R is Dedekind, thus proving the implication (4) $\Rightarrow$ (1) of (7.12) which was postponed until now.

We shall call M ε mod-R a torsion (resp., torsion free) module if the map $M \longrightarrow M \otimes_R L$ is zero (resp., a monomorphism), and we shall apply these terms, in particular, to A-modules. If $M \in \underline{\underline{M}}(A)$ is torsion then $\mathrm{ann}_R(M) \neq 0$, so supp(M) is a proper closed subset of spec(R). The latter is irreducible and of dimension ≤ 1 so supp(M) must be a finite set of maximal ideals. We conclude therefore, from (5.8) that: If $M \in \underline{\underline{M}}(A)$ is torsion then M has finite length as an R-module. From (5.9), moreover, we see that $\underline{p} \in \max(A)$ $\Leftrightarrow$ $\underline{p}$ is prime and $A/\underline{p}$ is torsion. But if $\underline{\alpha} \subset A$ is an ideal then clearly $A/\underline{\alpha}$ is torsion $\Leftrightarrow$ $A \otimes_R L = \underline{\alpha} \otimes_R L$ $\Leftrightarrow$ $\underline{\alpha}$ is an R-lattice. Thus: If $\underline{p} \subset A$ is a two sided A-lattice then $\underline{p} \in \max(A)$ $\Leftrightarrow$ $\underline{p}$ is prime. We now go to the proof.

Proof of (8.6). We carry it out in several steps. All lattices referred to are in Λ.

(i) If $\underline{\alpha}$ is a left A-lattice then $A = \{x \in \Lambda \mid x\underline{\alpha} \subset \underline{\alpha}\}$, and similarly for right lattices.

For $\{x \in \Lambda \mid x\underline{\alpha} \subset \underline{\alpha}\}$ is evidently an R-order containing A, and A is maximal.

(ii) If $\underline{a} \subset A$ is a two sided A-lattice then $\underline{a}$ contains a product of elements of max(A).

If $\underline{a} \in \max(A)$ there is nothing to prove. If not then, as remarked above, $\underline{a}$ is not prime. Thus there are two sided ideals $\underline{b}$ and $\underline{c}$ properly containing $\underline{a}$ (and hence lattices) such that $\underline{b}\ \underline{c} \subset \underline{a}$. By a noetherian induction argument we can assume that $\underline{b}$ and $\underline{c}$ contain products of elements of max(A). Hence the same holds for $\underline{a}$.

If $\underline{a}$ is a two sided A-lattice we shall write

$$\bar{\underline{a}} = \{x \in \Lambda \mid x\underline{a} \subset A\}$$

Since $(A\bar{\underline{a}}\ A)\underline{a} = A\bar{\underline{a}}\ \underline{a} \subset AA = A$ we see that $\bar{\underline{a}}$ is also a two sided A-lattice.

(iii) If $\underline{p} \in \max(A)$ then $\bar{\underline{p}} \neq A$

Choose $a \neq 0$ in R such that $aA \subset \underline{p}$, and choose $\underline{p}_1, \ldots, \underline{p}_n \in \max(A)$ such that $\underline{p}_1 \cdots \underline{p}_n \subset aA$ (using (ii)). Let us assume also that n is as small as possible. Since $\underline{p}$ is prime it contains some $\underline{p}_i$, and hence $\underline{p} = \underline{p}_i$ because $\underline{p}_i$ is maximal. Therefore, we can write $\underline{a}\ \underline{p}\ \underline{b} \subset aA$ where $\underline{a} = \underline{p}_1 \cdots \underline{p}_{i-1}$ and $\underline{b} = \underline{p}_{i+1} \cdots \underline{p}_n$. Now we reason:

$$a^{-1}\ \underline{a}\ \underline{p}\ \underline{b} \subset A \Rightarrow \underline{b}\ a^{-1}\ \underline{a}\ \underline{p}\ \underline{b} \subset \underline{b}$$

$$\Rightarrow \underline{b}\ a^{-1}\ \underline{a}\ \underline{p} \subset A \qquad \text{(step (i))}$$

$$\Rightarrow \underline{b}\ a^{-1}\ \underline{a} \subset \bar{\underline{p}}$$

Since $\underline{b}\ \underline{a}$ is a product of n - 1 primes, the minimality of n implies $\underline{b}\ \underline{a} \not\subset aA$, so $a^{-1}\ \underline{b}\ \underline{a} \not\subset A$. Thus $\bar{\underline{p}} \neq A$. q.e.d.

(iv) If $\underline{p} \in \max(A)$ then $\bar{\underline{p}}\ \underline{p} = A = \underline{p}\ \bar{\underline{p}}$.

Since $A \subset \bar{\mathfrak{p}}$ we have $\mathfrak{p} \subset \bar{\mathfrak{p}}\,\mathfrak{p} \subset A$, so the maximality of $\mathfrak{p}$ implies $\bar{\mathfrak{p}}\,\mathfrak{p} = A$ or $\bar{\mathfrak{p}}\,\mathfrak{p} = \mathfrak{p}$. If $\bar{\mathfrak{p}}\,\mathfrak{p} = \mathfrak{p}$ then (i) implies $\bar{\mathfrak{p}} \subset A$, contradicting (iii). Thus $\bar{\mathfrak{p}}\,\mathfrak{p} = A$. Now $\mathfrak{p}\,\bar{\mathfrak{p}}\,\mathfrak{p} = \mathfrak{p}A = \mathfrak{p}$ so (i) implies $\mathfrak{p}\,\bar{\mathfrak{p}} \subset A$, and clearly $\mathfrak{p} \subset \mathfrak{p}\,\bar{\mathfrak{p}}$. Arguing just as before we conclude now that $\mathfrak{p}\,\bar{\mathfrak{p}} = A$.

(v) *If* $\mathfrak{p}_1, \mathfrak{p}_2 \in \max(A)$ *then* $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1$.

Let $\mathfrak{a} = \bar{\mathfrak{p}}_1\mathfrak{p}_2\mathfrak{p}_1$. Since $\mathfrak{p}_2\mathfrak{p}_1 \subset \mathfrak{p}_1$ it follows that $\mathfrak{a} \subset A$. Since $\mathfrak{p}_1\mathfrak{a} = \mathfrak{p}_2\mathfrak{p}_1 \subset \mathfrak{p}_2$, and since $\mathfrak{p}_2$ is a prime not containing $\mathfrak{p}_1$, it follows that $\mathfrak{a} \subset \mathfrak{p}_2$. Therefore, $\mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{a} \subset \mathfrak{p}_1\mathfrak{p}_2$, and the reverse inclusion follows by symmetry.

(vi) *A two sided A-lattice* $\mathfrak{a} \subset A$ *is uniquely, up to order, a product of elements of* $\max(A)$.

We can assume $\mathfrak{a} \neq A$, so choose $\mathfrak{p} \in \max(A)$ containing $\mathfrak{a}$. Then $\mathfrak{a} \subset \bar{\mathfrak{p}}\,\mathfrak{a} \subset A$, and (i) and (iii) imply $\mathfrak{a} \neq \bar{\mathfrak{p}}\,\mathfrak{a}$. By a noetherian induction we can assume $\bar{\mathfrak{p}}\,\mathfrak{a}$ is a product of elements of $\max(A)$, and therefore $\mathfrak{a} = \mathfrak{p}(\bar{\mathfrak{p}}\,\mathfrak{a})$ is also.

Suppose $\mathfrak{p}_1 \cdots \mathfrak{p}_n = q_1 \cdots q_m$, with the $\mathfrak{p}$'s and q's in $\max(A)$. Since $\mathfrak{p}_1$ is prime and contains $q_1 \cdots q_m$ it must contain some q_i. Using (v) to rearrange terms we can assume $\mathfrak{p}_1 \supset q_1$. Since q_1 is maximal we have $\mathfrak{p}_1 = q_1$. Multiply the equation above by $\bar{\mathfrak{p}}$, and one obtains $\mathfrak{p}_2 \cdots \mathfrak{p}_n = q_2 \cdots q_m$, and the uniqueness follows by induction on n (the case n = 1 being obvious).

Finally, if $\mathfrak{a}$ is any two sided A-lattice then $b\,\mathfrak{a} \subset A$ for some $b \neq 0$ in R, so $\mathfrak{a} = (\overline{bA})(b\mathfrak{a})$ is a product of elements of $\max(A)$ and their inverses. If there were a relation $A = \prod \mathfrak{p}^{n_\mathfrak{p}}$ ($\mathfrak{p} \in \max(A)$, $n_\mathfrak{p} = 0$ for most $\mathfrak{p}$) then we could put all factors with $n_\mathfrak{p} < 0$ on the left and obtain a relation in A contradicting (vi). Thus $\max(A)$ is a free basis for the group of two sided A-lattices. q.e.d.

(8.7) THEOREM. Assume R is a Dedekind ring and let A be an R-order in Λ.

(a) A is right hereditary (see §6) $\Longleftrightarrow$ every p ε max(A) is a projective right A-module. In this case $hd_A(M) = hd_R(M)$ for all M ε $\underline{\underline{M}}(A)$, and M ε $\underline{\underline{P}}(A)$ $\Longleftrightarrow$ M is torsion free. Moreover, every M ε $\underline{\underline{M}}(A)$ is the direct sum of its torsion submodule and of modules isomorphic to right ideals.

(b) A is a maximal order $\Longleftrightarrow$ A is right hereditary and every P ε $\underline{\underline{P}}(A)$ which is faithful (i.e., $ann_A(p) = 0$) is faithfully projective (i.e., in this case, a generator of mod-A).

Proof. (a) If M ε $\underline{\underline{M}}(A)$ is torsion free then M(= M$\otimes$1) is an A-lattice in $M\otimes_R L = V$. Since Λ is semi-simple we can solve $V \oplus W \simeq \Lambda^n$ for some $n \geq 0$. Let N be an A-lattice in in W, so that $M \oplus N$ is an A-lattice in Λ^n. Since $A^n \subset \Lambda^n$ is is another such A-lattice we can find $\alpha \neq 0$ in R such that $(M \oplus N)\alpha \subset A^n$. Of course, $(M \oplus N)\alpha \simeq M \oplus N$. In conclusion, we have shown that a torsion free M ε $\underline{\underline{M}}(A)$ is isomorphic to a submodule of A^n for some $n \geq 0$, and the converse is evident. It follows that A is right hereditary if and only if all such M are A-projective.

Assume now that A is right hereditary. Since a module which is projective over R or over A must be torsion free we conclude that $\underline{\underline{P}}(A)$ consists of the torsion free modules in $\underline{\underline{M}}(A)$, and similarly for R. Since the only homological dimensions are 0 and 1 (and -1 for the zero module) we have $hd_A(M) = hd_R(M)$ for M ε $\underline{\underline{M}}(A)$. Moreover, if M ε $\underline{\underline{M}}(A)$ has torsion submodule T then M/T is projective, so $M \simeq T \oplus M/T$. According to (6.2) M/T is a direct sum of modules isomorphic to right ideals.

Conversely, assuming every $\underline{p} \in \max(A)$ is projective as a right A-module, we must show that A is right hereditary, i.e., $\mathrm{hd}_A(M) \leq 1$ for all $M \in \underline{\underline{M}}(A)$.

Let $T \in \underline{\underline{M}}(A)$ be torsion. The T has finite length. Moreover, if $0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$ is exact then by (I, 6.8), $\mathrm{hd}_A(T) \leq \sup(\mathrm{hd}_A(T'), \mathrm{hd}_A(T''))$. Therefore, by induction on length, it suffices to show that $\mathrm{hd}_A(T) \leq 1$ when T is simple. Let $\underline{p} = \mathrm{ann}_A(T)$. By (5.9) $\underline{p} \in \max(A)$, so $A/\underline{p}$ is a simple ring. It follows that T is a direct summand of $A/\underline{p}$. Moreover, the exact sequence $0 \longrightarrow \underline{p} \longrightarrow A \longrightarrow A/\underline{p} \longrightarrow 0$, plus our assumption that $\underline{p} \in \underline{\underline{P}}(A)$, shows that $\mathrm{hd}_A(A/\underline{p}) \leq 1$. Hence $\mathrm{hd}_A(T) \leq 1$.

Now let $M \in \underline{\underline{M}}(A)$ be torsion free. It suffices to show that $\mathrm{hd}_A(M) \leq 0$ for all such M. The first part of the proof showed that we could find an embedding $M \oplus N \subset A^n$ so that $M \oplus N$ is a lattice in Λ^n. This leads to an exact sequence $0 \longrightarrow M \oplus N \longrightarrow A^n \longrightarrow T \longrightarrow 0$ where the cokernel T must be torsion. We have proved that $\mathrm{hd}_A(T) \leq 1$, so $M \oplus N$ is projective. q.e.d.

(b) Let A be a maximal order. If $\underline{p} \in \max(A)$ then $\overline{\underline{p}}\,\underline{p} = A = \underline{p}\,\overline{\underline{p}}$, so we can find $a_i \in \underline{p}$ and $b_i \in \overline{\underline{p}}$ such that $\Sigma a_i b_i = 1$. Define $h_i\colon \underline{p} \longrightarrow A$ by $h_i(x) = b_i x$. Then if $x \in \underline{p}$ we have $x = \Sigma a_i b_i x = \Sigma a_i h_i(x)$. Hence, by (II, 4.5), $\underline{p}$ is a projective right A-module. It follows now from part (a) that A is right hereditary.

Suppose $P \in \underline{\underline{P}}(A)$ is faithful, and let $\underline{a} = \Sigma hP$ ($h \in P^* = \mathrm{Hom}_A(P, A)$). Since $P_L = P \otimes_R L$ is a faithful Λ-module and Λ is semi-simple it follows that $\Lambda = \Sigma hP_L$ ($h \in P_L$), and since $\mathrm{Hom}_\Lambda(P_L, \Lambda) = \mathrm{Hom}_A(P, A) \otimes_R L$, we conclude that $\underline{a} \subset A$ is an R-lattice. Moreover, it follows from (II, 4.5) that $\underline{a}$ is an idempotent two sided ideal. But

Theorem (8.6) asserts unique factorization for the two sided A-lattices in Λ, so we must have $\underline{a} = A$, i.e., P is a generator. Since $P \in \underline{\underline{P}}(A)$ it follows from (II, 1.2) that P is a faithfully projective right A-module.

Suppose, conversely, that A is right hereditary and that every faithful $P \in \underline{\underline{P}}(A)$ is faithfully projective. Let B be an R-order containing A. Then $B \in \underline{\underline{P}}(A)$ because A is right hereditary, using part (a), and B is clearly faithful. Viewing B as a right A-module we can identify $\mathrm{Hom}_A(B, A)$ with $\overline{B} = \{x \in \Lambda \mid xB \subset A\}$. Our assumptions imply B is a faithfully projective right A-module, and so $\overline{B}\,B = A$. But then $A = \overline{B}\,B = \overline{B}(BB) = (\overline{B}B)B = AB = B$. This proves that A is a maximal order, and completes the proof of (8.7).

(8.8) THEOREM. _Keeping the assumptions of_ (2) _above, let_ A _be a maximal_ R-_order in_ Λ. _Let_ V _be a faithful finitely generated left_ Λ-_module, and let_ P _be a divisorial_ A-_lattice in_ V. _Then_ $A' = \mathrm{End}_A(P)$ _is a maximal_ R-_order in_ $\Lambda' = \mathrm{End}_\Lambda(V)$, _and_ $A = \mathrm{End}_{A'}(P)$.

Proof. Since V is faithful we can view Λ as a subalgebra of $E = \mathrm{End}_L(V)$, and then Λ' is just the centralizer in E of Λ. Moreover, $A \subset (P: P) = \{h \in E \mid hP \subset P\} \simeq \mathrm{End}_R(P)$, and $A' = \Lambda' \cap (P: P)$ because an $h \in E$ commutes with A if and only if it commutes with Λ. (Recall $\Lambda = A \cdot L$). Since P is divisorial it follows from (8.4) that (P: P) is a divisorial R-lattice in E, and hence A′ is a divisorial R-order in Λ'.

Our hypotheses imply V is a faithfully projective Λ-module, and hence $\Lambda = \mathrm{End}_{\Lambda'}(V)$, i.e., Λ is the centralizer in E of Λ'. Reasoning as above, we see that $\Lambda \cap (P: P) = \mathrm{End}_{A'}(P)$, and that $\Lambda \cap (P: P)$ is an R-order in Λ, containing A. Since A is maximal we have $A = \mathrm{End}_{A'}(P)$.

It remains only to be shown that A′ is a maximal

R-order in Λ'. We have seen that A' is a divisorial R-lattice so, by (8.5), it suffices to show that $A_{\underline{p}}$ is a maximal $R_{\underline{p}}$-order for each $\underline{p} \in Ht_1(R)$. Using (8.4) again we have $A_{\underline{p}} = \Lambda \cap (P: P)_{\underline{p}} = \Lambda \cap (P_{\underline{p}} : P_{\underline{p}}) = End_{A_{\underline{p}}}(P_{\underline{p}})$. Since $A_{\underline{p}}$ is a maximal $R_{\underline{p}}$-order, and since $R_{\underline{p}}$ is a Dedekind ring (in fact, a DVR) it follows from (8.7) that $P_{\underline{p}}$ is a faithfully projective $R_{\underline{p}}$-module. Therefore

$$Hom_{A_{\underline{p}}}(P_{\underline{p}}, \cdot\,): A_{\underline{p}}\text{-mod} \longrightarrow A'_{\underline{p}}\text{-mod}$$

is an R-equivalence of R-categories (see (II, §§1-2)). Since $A_{\underline{p}}$ is hereditary (see (8.7)), $A'_{\underline{p}}$ is also. Moreover, every object of $\underline{\underline{P}}(A'_{\underline{p}})$ is isomorphic to $Hom_{A_{\underline{p}}}(P_{\underline{p}}, Q)$ for some $Q \in \underline{\underline{P}}(A_{\underline{p}})$. If $Hom_{A_{\underline{p}}}(P_{\underline{p}}, Q)$ is a faithful $A'_{\underline{p}}$-module then Q is a faithful $A_{\underline{p}}$-module, by (II, 8.3 (7)). In this case, therefore, Q is a faithfully projective $A_{\underline{p}}$-module, by (8.7), and hence $Hom_{A_{\underline{p}}}(P_{\underline{p}}, Q)$ is a faithfully projective $A'_{\underline{p}}$-module. We have thus established the criterion of (8.7)(b), which shows that $A'_{\underline{p}}$ is a maximal $R_{\underline{p}}$- order in Λ'. This completes the proof of (8.8).

(8.9) COROLLARY. Keeping the assumptions of (2), let A be a maximal R-order in Λ. Then $A \simeq \Pi End_{A_i'}(P_i)$ where each A_i' is a maximal order in a divisor algebra D_i and where P_i is a divisorial A_i'-lattice in a finite dimensional D_i-module.

Proof. Let $S_1, \ldots, S_n$ represent the distinct simple left Λ-modules, let $D_i = \mathrm{End}_\Lambda(S_i)$, let P_i be a divisorial A-lattice in S_i, and let $P = \amalg P_i \subset V = \amalg S_i$. Then D_i is a division algebra (Schur's lemma), $\Lambda' = \Pi D_i$, and $A' = \Pi A_i'$, where $A_i' = \mathrm{End}_A(P_i)$, in the notation of (8.8). The corollary now follows from (8.8).

We close this section with the following proposition, which gives a method for reducing certain questions for arbitrary finite R-algebras to the case of orders in semi-simple algebras.

(8.10) PROPOSITION. *Let* R *be a Dedekind ring and let* B *be a finite* R-*algebra. Then there is a largest two sided nilpotent ideal* N *in* B. *If* T *is the* (R-) *torsion submodule of* B/N *then* T *is a semi-simple ring (of finite length as an* R-*module) and* B/N $\simeq$ T x A *where* A *is an* R-*order in a semi-simple* L-*algebra*.

Proof. If $N_1^{n_1} = 0 = N_2^{n_2}$ then $(N_1 + N_2)^{n_1+n_2} = 0$, clearly, for two sided ideals N_1 and N_2. Since B is noetherian (e.g., as an R-module) it follows that a largest nilpotent two sided ideal exists. Evidently, B/N has no nilpotent ideals. Since T has finite length as an R-module we can apply (1.6) and conclude that B $\simeq$ T x A, for some A, and T is semi-simple. It follows that A is torsion free and has no non-zero nilpotent ideals. Hence A is an R-order in $A \otimes_R L$, and the latter has no non-zero nilpotent ideals. (They would have to intersect A.) Now (1.5) implies that $A \otimes_R L$ is semi-simple. q.e.d.

HISTORICAL REMARKS

The following are a few of the many possible references for the material of this chapter:

§§1-2: Artin-Nesbitt-Thrall [1], Bourbaki [2], Curtis-Reimer [1], Deuring [1].

§§3-5: Bourbaki [4], Serre [2].

§6 : Cartan-Eilenberg [1], Kaplansky [1].

§7 : Bourbaki [7].

§8 : Deuring [1], Fossum [1].

Part 2

THE STABLE STRUCTURE OF PROJECTIVE MODULES AND OF THEIR AUTOMORPHISM GROUPS

Chapter IV
THE STABLE STRUCTURE OF PROJECTIVE MODULES

This chapter contains the two basic "stability theorems" for projective modules. The first is Serre's Theorem (§2) which says that a projective module of "large" rank has a free direct summand. The second is the "Cancellation Theorem" which gives similar conditions for the uniqueness of the complementary summand.

In §5 and §6 we present what are, essentially, the only known "non-stable" structure theorems. The first is a theorem of P.M. Cohn which asserts that projective modules over a free algebra (the algebra of "non-commuting polynomials") over a field are free. The same is true of the group algebra of a free group. These results are deduced from a general theorem on free products of augmented algebras.

The second non-stable theorem is the theorem of Seshadri. It implies that the projective modules over $R\pi$ are free when R is a principal ideal domain, and where π is a free (non-commutative) monoid or group. When π has one generator this implies that projective modules are free over a polynomial ring in two commuting variables over a field. The case of more than two variables remains unsettled; this is "Serre's Problem".

§1. PROJECTIVE MODULES OVER SEMI-LOCAL RINGS

In this section we fix a ring A with radical

J = rad A. Eventually we shall assume that A is semi-local, i.e., that A/J is semi-simple.

Let P ε mod-A and let $\alpha \in P$. We shall write:

$$P^* = \mathrm{Hom}_A(P, A) \quad \text{and}$$

$$o_P(\alpha) = \{h\alpha \mid h \in P^*\}$$

The latter is a left ideal. We say α is unimodular in P if $o_P(\alpha) = A$. This is evidently equivalent to the condition that h: $A \longrightarrow P$ ($h(a) = \alpha a$) is a split monomorphism.

(1.1) PROPOSITION. Let σ, τ: $Q \longrightarrow P$ be morphisms in mod-A, and assume $Q \in \underline{\underline{P}}(A)$.

(a) σ is a split monomorphism $\Longleftrightarrow$ $\sigma : P^* \longrightarrow Q^*$ is an epimorphism.

(b) If $\mathrm{Im}(\sigma - \tau) \subset PJ$ then σ is a split monomorphism $\Longleftrightarrow$ τ is.

Proof. (a). If σ has a left inverse, then σ^* has a right inverse, so σ^* is surjective. Conversely, since $Q^* \in \underline{\underline{P}}(A^0)$, σ^* has a right inverse if it is surjective. Therefore, σ^{**}: $P^{**} \longrightarrow Q^{**}$ has a left inverse, say σ'. If h_P: $P \longrightarrow P^{**}$ is the canonical map then $h_Q^{-1}(\sigma')^* h_P$ is a left inverse for σ.

(b) The inclusion $JQ^* \subset \mathrm{Hom}_A(Q, J)$ is an equality when Q = A, clearly, and hence also when $Q \in \underline{\underline{P}}(A)$, by additivity. If $h \in \mathrm{Im}(\sigma - \tau)^*$ then $h(Q) \subset J$, so $h \in JQ$, by the remark just made. Hence σ^*, τ^*: $P^* \longrightarrow Q^*$ agree mod JQ . By Nakayama's lemma, therefore, σ^* is surjective $\Longleftrightarrow$ τ^* is. Now (b) follows from (a). q.e.d.

Henceforth, we assume A is semi-local. If $P \in$ mod-A and if S is a subset of P, we denote by (S) the submodule of P generated by S. The non-negative integer (or infinity)

$$\text{f-rank}_A(S; P)$$

is the supremum of all $r \geq o$ such that (S) contains a direct summand of P isomorphic to A^r. Since A here is fixed, we shall often drop the subscript.

(1.2) PROPOSITION. f-rank(S; P) = f-rank((S) + PJ; P).

Proof. It clearly suffices to show the left side dominates the right. Let $\sigma\colon A^r \longrightarrow P$ be a split monomorphism with $\mathrm{Im}(\sigma) \subset (S) + PJ$. Choose $\tau\colon A^r \longrightarrow (S)$ such that $\mathrm{Im}(\sigma - \tau) \subset PJ$. Then (1.1)-(b) implies τ is a split monomorphism. q.e.d.

(1.3) PROPOSITION. Let $P \in$ mod-A and let $\alpha, \beta \in P$ be unimodular. Then there is a $\phi \in \mathrm{Aut}_A(P)$ such that (i) $\phi(\alpha A) = \beta A$, and (ii) ϕ leaves invariant all submodules containing α and β.

Proof. Write $P = \beta A \oplus P'$ and $\alpha = \beta b + \alpha_{P'}$ ($b \in A$, $\alpha_{P'} \in P'$). Then $A = o_P(\alpha) = Ab + o_{P'}(\alpha_{P'})$. According to (III, 2.8) there is an $a \in o_{P'}(\alpha_{P'})$ such that $u = b + a \in U(A)$. Choose $f' \in P'^*$ such that $a = f'(\alpha_{P'})$ and define $f\colon P \longrightarrow P$ by $f(\beta x + \gamma) = \beta f'(\gamma)$ for $x \in A$, $\gamma \in P'$. Then $f^2 = o$ so $\phi_1 = 1_P + f$ is an automorphism such that $\phi_1(\alpha) = \beta u + \alpha_{P'}$. Define $g\colon P \longrightarrow P$ by $g(\beta x + \gamma) = \alpha_{P'} u^{-1} x$. Again $g^2 = o$ so $\phi_2 = 1_P - g$ is an automorphism, and $\phi_2\phi_1(\alpha) = \beta u$. It is clear now that $\phi = \phi_2\phi_1$ satisfies (i) and (ii). q.e.d.

(1.4) COROLLARY. *Suppose* P, P' ε mod-A *and* $Q \varepsilon \underline{\underline{P}}(A)$. *Then* $P \oplus Q \simeq P' \oplus Q \Rightarrow P \simeq P'$.

Proof. Writing $Q \oplus Q' \simeq A^n$, and using induction on n, reduces us to the case $Q = A$. Then we have an equality of modules $P \oplus \alpha A = P' \oplus \beta A$ with α and β unimodular (after using the isomorphism to identify). Choose ϕ as in (1.3). Then $P \simeq (P \oplus \alpha A)/\alpha A \simeq (P \oplus \alpha A)/\phi(\alpha A) = (P' \oplus \beta A)/\beta A \simeq P'$.

(1.5) COROLLARY. *If* M *is a submodule of* P ε mod-A *then*

$$\text{f-rank}(A^r \oplus M;\ A^r \oplus P) = r + \text{f-rank}(M;\ P).$$

Proof. The left side clearly dominates the right. To prove the converse it suffices, by an easy induction, to treat the case $r = 1$. Let $\alpha_1,\ldots,\alpha_s$ ε $\beta A \oplus M$ be a basis for a free direct summand of $\beta A \oplus P$. Choose ϕ as in (1.3) with respect to α_1 and β. Then condition (1.3)(ii) implies $\phi(\alpha_i)$ ε $\beta A \oplus M$ for all i. Moreover $\beta A \oplus M = \phi(\alpha_1) A \oplus M$, by (1,3)(i), so we can write $\phi(\alpha_i) = \phi(\alpha_1)\, a_i + \beta_i$, with β_i ε M $(2 \leq i \leq s)$. It is now evident that $\beta_2,\ldots,\beta_s$ are a basis for a free direct summand of P. Thus we have shown that:

$$\text{f-rank}(A \oplus M;\ A \oplus P) \geq s \Rightarrow \text{f-rank}(M;\ P) \geq s - 1. \text{ q.e.d.}$$

(1.6) COROLLARY. *Let* P ε mod-A *and let* α *and* S *be an element and subset, respectively, of* P. *Then*

$$\text{f-rank}(S, \alpha;\ P) \leq 1 + \text{f-rank}(S;\ P).$$

Proof. Map $A \oplus P$ onto P by sending A onto αA. A split monomorphism $\sigma: A^n \longrightarrow P$ with image in $\alpha A + (S)$ lifts to a homomorphism $\sigma': A^n \longrightarrow A \oplus P$ with image in $A \oplus (S)$. Therefore f-rank$(S, \alpha; P) \leq$ f-rank$(A \oplus (S); A \oplus P)$. Now apply (1.5).

(1.7) PROPOSITION. Let $P \in$ mod-A and let $\alpha_1, \ldots, \alpha_r \in P$. Suppose, for some $t < r$, that f-rank$(\alpha_1, \ldots, \alpha_r; P) \geq t$. Then there exist $\beta_i = \alpha_i + \alpha_r a_i (a_i \in A)$ $(1 \leq i \leq t)$ such that f-rank$(\beta_1, \ldots, \beta_t, \alpha_{t+1}, \ldots, \alpha_{r-1}; P) \geq t$.

Proof. Induction on t; the case $t = 0$ is trivial. $t = 1$. Choose a unimodular $\beta \in (\alpha_1, \ldots, \alpha_r)$ and write $P = \beta A \oplus Q$. Write $\alpha_i = \beta b_i + \alpha_i'$ $(b_i \in A, \alpha_i' \in Q)$ $(1 \leq i \leq r)$. Writing $\beta = \Sigma\, \alpha_i c_i$ shows that we have $\Sigma\, b_i c_i = 1$. With the aid of (III, 2.8), applied to $b_1 A + \Sigma_{2 \leq i \leq r} b_i A = A$, we can solve $u = b_1 + \Sigma_{i \geq 2} b_i a_i \in U(A)$. Hence $\alpha = \alpha_1 + \Sigma_{i \geq 2} \alpha_i a_i = \beta u + (\alpha_1' + \Sigma_{i \geq 2} \alpha_i' a_i)$ is unimodular. Therefore f-rank$(\alpha_1 + \alpha_r a_r, \alpha_2, \ldots, \alpha_{r-1}; P) \geq 1$.

$t > 1$. By (1.6) we have f-rank$(\alpha_2, \ldots, \alpha_r', P) \geq t - 1$. By induction, therefore, we can find $\beta_i = \alpha_i + \alpha_r a_i$ $(2 \leq i \leq t)$ such that f-rank$(\beta_2, \ldots, \beta_t, \alpha_{t+1}, \ldots, \alpha_{r-1}; P) \geq t - 1$. Let $P' \subset (\beta_2, \ldots, \beta_t, \alpha_{t+1}, \ldots, \alpha_{r-1})$ be a direct summand of P isomorphic to A^{t-1}, and write $P = P' \oplus P''$. Write $\alpha_i = \alpha_i' + \alpha_i''$ and $\beta_i = \beta_i' + \beta_i''$ in these coordinates. Then

$$\begin{aligned} t &\le \text{f-rank}(\alpha_1,\ldots,\alpha_r;\ P) \\ &= \text{f-rank}(\alpha_1,\beta_2,\ldots,\beta_t,\ \alpha_{t+1},\ldots,\alpha_r;\ P) \\ &= \text{f-rank}(P' \oplus (\alpha_1'',\beta_2'',\ldots,\beta_t'',\alpha_{t+1}'',\ldots,\alpha_r'');\\ &\quad P' \oplus P'') \\ &= (t-1) + \text{f-rank}(\alpha_1'',\beta_2'',\ldots,\beta_t'',\\ &\quad \alpha_{t+1}'',\ldots,\alpha_r'';\ P''), \end{aligned}$$

using (1.5). By the case $t = 1$, therefore, we can find a_1 so that f-rank$(\alpha_1'' + \alpha_r''a_1,\ \beta_2'',\ldots,\beta_t'',\ \alpha_{t+1},\ldots,\alpha_{r-1}'';\ P'') \ge 1$. If we set $\beta_1 = \alpha_1 + \alpha_r a_1$ then $\beta_1,\ldots,\beta_t$ clearly solve our problem.

§2. SERRE'S THEOREM

For the next two sections we shall fix the following data:

$$\text{(2.1)}\qquad \begin{aligned} R &= \underline{\text{a commutative ring such that}} \\ X &= \max(R)\ \underline{\text{is a noetherian space}} \\ A &= \underline{\text{a finite}}\ R\text{-}\underline{\text{algebra}}. \end{aligned}$$

If $M \in$ mod-A recall that

$$\text{supp}_m(M) = \{\underline{m} \in X \mid M_{\underline{m}} \ne 0\}.$$

If M is a finitely generated A-module then $\text{supp}_m(M) = V(\text{ann}_R(M)) = \{\underline{m} \in X \mid \underline{m} \supset \text{ann}_R(M)\}$, a closed set. (see (III, §3) for a discussion of these matters). Since A is a finite R-algebra it follows that $A_{\underline{m}}$ is semi-local for each $\underline{m} \in X$, and hence we can define, for $P \in$ mod-A, and $S \subset P$,

$$\text{f-rank}_A(S;\ P) = \inf_{\underline{m} \in X} \text{f-rank}_{A_{\underline{m}}}(S;\ P_{\underline{m}}),$$

and

$$\text{f-rank}_A(P) = \text{f-rank}_A(P;\ P).$$

The following is an immediate consequence of (1.5) and the definition:

(2.2) PROPOSITION. Let M be a submodule of $P \in$ mod-A. Then $\text{f-rank}_A(A^r \oplus M;\ A^r \oplus P) = r + \text{f-rank}_A(M;\ P)$.

Now if $P \in$ mod-A and if S is a subset of P then we define the "singular sets" of S in P, for each $j \geq 0$:

$$F_j(S;\ P) = \{\underline{m} \in X \mid \text{f-rank}_{A_{\underline{m}}}(S;\ P_{\underline{m}}) < j\}.$$

For example $F_0(S;\ P) = \phi$ for all S, and $F_j(\phi;\ P) = X$ for all $j > 0$.

(2.3) PROPOSITION. Suppose $P \in$ mod-A is a direct summand of a direct sum of finitely presented modules. Then for any $S \subset P$, and for any $j \geq 0$, $F_j(S;\ P)$ is a closed set in X.

Proof. Suppose $\underline{m} \in F_j(S;\ P)$. Then there is a split monomorphism $h^1: A_{\underline{m}}^j \longrightarrow P_{\underline{m}}$. We can even arrange that $h^1 = h_{\underline{m}}$ for some $h: A^j \longrightarrow P$. If we show that $U = \{\underline{n} \mid h_{\underline{n}}$ is a split monomorphism$\}$ is open then U will be a neighborhood of $\underline{m}$ not meeting $F_j(S;\ P)$, showing thus that $F_j(S;\ P)$ is closed.

Write $\underline{P} \oplus \underline{P}' \approx \amalg_i Q_i$ where each Q_i is finitely presented. Choose a finite sum, Q, of the Q_i's so that $\mathrm{Im}(h) \subset Q$. Then $h: A^j \longrightarrow P$ has a left inverse <=> the induced homomorphism $A^j \longrightarrow Q$ has one, clearly. Therefore there is no loss in assuming P itself is finitely presented.

In this case it follows from (III, 4.5) that the natural map $(\underline{P}^*)_{\underline{n}} \longrightarrow (\underline{P}_{\underline{n}})^*$ is an isomorphism for each $\underline{n} \in X$. The same applies, of course, to A^j. Now, using (1.1) (a), we have

$$\begin{aligned} U &= \{\underline{n} \in X \mid h_{\underline{n}} \text{ is split monomorphism}\} \\ &= \{\underline{n} \in X \mid \mathrm{Coker}\,((h_{\underline{n}})^*) = 0\} \\ &= \{\underline{n} \in X \mid \mathrm{Coker}\,((h^*)_{\underline{n}}) = 0\} \\ &= X - \mathrm{supp}(\mathrm{Coker}\,(h^*)). \end{aligned}$$

Since $\mathrm{Coker}\,(h : P^* \longrightarrow (A^j)^*)$ is finitely generated it has closed support. q.e.d.

The last part of the proof above showed that h splits if and only if $h_{\underline{n}}$ splits for all $\underline{n}$. If $\alpha_1, \ldots, \alpha_j$ is the image of the basis of A^j, therefore we obtain the following conclusion:

(2.4) COROLLARY. Let P be as in (2.3) and let $\alpha_1, \ldots, \alpha_j \in P$. Then $\alpha_1, \ldots, \alpha_j$ is a basis for a free direct summand of P if and only if $F_j(\alpha_1, \ldots, \alpha_j; P) =$. In particular $\alpha \in P$ is unimodular if and only if $F_1(\alpha; P) = \phi$.

We now come to the main theorem of this section.

(2.5) THEOREM (Serre). Let $V = V(\underline{a})$ be a closed set *in* X *such that* $X - V$ *is a disjoint union of a finite number of subspaces, each of dimension* $\leq d$. *Let* $P \in$ mod-A *be a direct summand of a direct sum of finitely presented modules, and let* $\gamma \in P$ *be such that its image in the* $(A/A\underline{a})$-*module* $P/P\underline{a}$ *is unimodular. Then, if* $\text{f-rank}_A(P) > d$, *there is a unimodular element* $\alpha \in P$ *such that* $\alpha \equiv \gamma \bmod P\underline{a}$.

(2.6) COROLLARY. *Let* V *and* P *be as above. Assume that* $\text{f-rank}_A(P) \geq d + r$ *and that* $P/P\underline{a}$ *has a direct summand isomorphic to* $(A/A\underline{a})^r$. *Then* P *has a direct summand isomorphic to* A^r.

Proof. Choose $\gamma \in P$ to reduce to part of an $(A/A\underline{a})$-basis for a direct summand of $P/P\underline{a}$ isomorphic to $(A/A\underline{a})^r$. Then the theorem gives us a unimodular $\alpha \in P$ such that $P = \alpha A \oplus P'$ and $\alpha \equiv \gamma \bmod P\underline{a}$. The last condition guarantees the induction hypothesis for P', so we finish by induction on r.

(2.7) COROLLARY. *Let* P *be as above, assume* X *is a disjoint union of a finite number of subspaces each of dimension* $\leq d$, *(e.g. if* $\dim X \leq d$*). Then, if* $\text{f-rank}_A(P) > d$, P *contains a unimodular element*.

Proof. Take $\underline{a} = R$, so $V = \phi$.

Remark. The hypotheses of (2.7) do not imply dim $X \leq d$. For example let R be a semi-local noetherian ring of dimension $d > 0$, and let $A = R[t]$, t an indeterminate. Then $\dim \max(A) = d + 1$, while $\max(A)$ is a union of a closed set and an open set each of dimension $\leq d$ (see III, 3.13).

The proof of (2.5) will be based on two lemmas. Let $Y_1, \ldots, Y_N$ be disjoint subspaces of $X - V$. Recall from

(III, 3.8) that all subspaces of X are noetherian. Moreover, by virtue of our hypothesis on P, it follows from (2.3) that $F_j(S; P)$ is closed in X for all $S \subset P$ and all $j \geq 0$. Here are the lemmas:

(2.8) LEMMA. _Suppose_ $\text{f-rank}_A(P) \geq r$. _Then_ given $\gamma_1, \ldots, \gamma_r \in P$, _there exist_ $\alpha_1, \ldots, \alpha_r \in P$ _such that_ $\alpha_1, \ldots, \alpha_r \in P$ such that $\alpha_i \equiv \gamma_i \bmod P\underline{a}$ $(1 \leq i \leq r)$ _and such that_

$$\text{codim}_{Y_i}(Y_i \cap F_j(\alpha_1, \ldots, \alpha_r; P)) \geq r + 1 - j$$
$$(j \geq 0;\ 1 \leq i \leq N).$$

(2.9) LEMMA. _Suppose_ $\alpha_1, \ldots, \alpha_r \in P$ $(r \geq 1)$ _and_ $k \geq 0$ _are such that_

$$\text{codim}_{Y_i}(Y_i \cap F_j(\alpha_1, \ldots, \alpha_r; P)) \geq k - j \quad (1 \leq j \leq r)$$
$$(1 \leq i \leq N).$$

Then there exist $\beta_i = \alpha_i + \alpha_r a_i (a_i \in A)$ $(1 \leq i \leq r)$ _such that_

$$\text{codim}_{Y_i}(Y_i \cap F_j(\beta_1, \ldots, \beta_{r-1}; P)) \geq k - j$$
$$(1 \leq j \leq r - 1)$$
$$(1 \leq i \leq N).$$

Proof that (2.8) and (2.9) imply (2.5). By hypothesis we can choose Y_i's as above so that $X = V \cup Y_1 \cup .. \cup Y_N$. Moreover we can apply (2.8) with $r = d + 1$. In doing so we take $\gamma_1 = \gamma$ (given in (2.5)) and $\gamma_i = 0$ for $i > 1$. Then (2.8) gives us $\alpha_1, \ldots, \alpha_r \in P$ such that

$$(*)\quad \begin{aligned} \alpha_1 &\equiv \gamma \bmod P\underline{a} \\ \alpha_i &\equiv 0 \bmod P\underline{a} \qquad (\alpha \le i \le r), \end{aligned}$$

and such that

$$\operatorname{codim}_{Y_i}(Y_i \cap F_j(\alpha_1,\ldots,\alpha_r; P)) \ge r + 1 - j \qquad (j \ge 0,\ 1 \le i \le N).$$

Now we apply (2.9) to these data, (r - 1) times in succession. The result will be a single element, α, such that

$$\operatorname{codim}_{Y_i}(Y_i \cap F_1(\alpha; P)) \ge r + 1 - 1 = r = d + 1$$

$(1 \le i \le N)$. Since $\dim Y_i \le d$ (by hypothesis) this implies $F_1(\alpha; P) \cap Y_i = \phi$ $(1 \le i \le N)$. Moreover, the transformation $\alpha_i \longmapsto \beta_i = \alpha_i + \alpha_r a_r$, used in (2.9) will leave the congruences (*) above in tact for the β's: i.e. $\beta_1 \equiv \gamma \bmod P\underline{a}$ and $\beta_i \equiv 0 \bmod P\underline{a}$ $(1 < i < r)$. Thus, in the end, we have $\alpha \equiv \gamma \bmod P\underline{a}$. Since γ, by hypothesis, is unimodular mod $P\underline{a}$, it follows that α is also, and hence $F_1(\alpha; P) \cap V(\underline{a}) = \phi$. Since X is the union of V and of the Y_i's this proves that $F_1(\alpha; P) = \phi$. Hence α is unimodular, by (2.4). q.e.d.

<u>Proof of (2.8)</u>. Induction on r; the case r = o is trivial. Suppose now that $r \ge 0$ and $\text{f-rank}_A(P) \ge r + 1$. Given $\gamma_1,\ldots,\gamma_{r+1} \in P$, we can construct $\alpha_1,\ldots,\alpha_r \in P$ as in (2.8), by induction, and we seek α_{r+1}. Recall that

$$\alpha_i \equiv \gamma_i \bmod P\underline{a} \qquad (1 \le i \le r)$$

and

$$\operatorname{codim}_{Y_i}(Y_i \cap F_j) \ge r + 1 - j \quad (j \ge 0;\ 1 \le i \le N),$$

where $F_j = F_j(\alpha_1,\ldots,\alpha_r; P)$. Fix a j, $0 \leq j \leq r$, and let C be an irreducible component of $Y_i \cap F_{j+1}$ such that $\mathrm{codim}_{Y_i}(C) = (r + 1) - (j + 1) = r - j$. Since $\mathrm{codim}_{Y_i}(Y_i \cap F_j) \geq r + 1 - j > r - j$ it follows that $C \not\subset F_j$. By varying C now we see that there are finite sets D_j $(0 \leq j \leq r)$ such that

$$D_j \subset F_{j+1}, \quad D_j \cap F_j = \phi, \quad D_j \cap V = \phi,$$

and such that, for each i, D_j contains a point in each component of $Y_i \cap F_{j+1}$ of codimension $r - j$ in Y_i $(1 \leq i \leq N)$.

If $\underline{m} \in D_j$ then $\alpha_1,\ldots,\alpha_r$ have f-rank $\geq j$ in $P_{\underline{m}}$. Therefore, since f-rank$(P) \geq r + 1 > r \geq j$, there is an $\alpha(\underline{m}) \in P_{\underline{m}}$, which we can even take to be in P, such that $\alpha_1,\ldots,\alpha_r, \alpha(\underline{m})$ have f-rank $\geq j + 1$ in $P_{\underline{m}}$. This follows easily from (1.5). Let $D = \cup D_j$ $(0 \leq j \leq r)$. By the Chinese Remainder Theorem (III, 2.14) we can choose $\alpha_{r+1} \in P$ to satisfy $\alpha_{r+1} \equiv \gamma_{r+1} \bmod P\underline{a}$ and $\alpha_{r+1} \equiv \alpha(\underline{m}) \bmod P\underline{m}$ for each $\underline{m} \in D$. Then, if $\underline{m} \in D_j$, it follows from (1.2) that $\underline{m} \in F'_{j+1} = F_{j+1}(\alpha_1,\ldots,\alpha_{r+1}; P)$. This uses the fact that $A_{\underline{m}} \cdot \underline{m} \subset \mathrm{rad}\, A_{\underline{m}}$. Evidently $F'_{j+1} \subset F_{j+1}$, and we know that $\mathrm{codim}_{Y_i}(Y_i \cap F_{j+1}) \geq (r + 1) - (j + 1) = r - j$. Since F'_{j+1} excludes one point from each component of codimension $r - j$ in $Y_i \cap F_{j+1}$ we conclude that

$$\mathrm{codim}_{Y_i}(Y_i \cap F'_{j+1}) \geq r - j + 1 = (r + 1) + 1 - (j + 1)$$

$(0 \leq j \leq r)$. Since $F_0' = \phi$ and since $(r + 1) + 1 -(j + 1) \leq 0$ for $j > r$, the above inequality persists for all $j \geq 0$ $(1 \leq i \leq N)$. q.e.d.

Proof of (2.9). We are given $\alpha_1,\ldots,\alpha_r \in P$ such that $\text{codim}_{Y_i}(Y_i \cap F_j) \geq k - j(1 \leq i \leq N;\ 0 \leq j \leq r)$, where $F_j = F_j(\alpha_1,\ldots,\alpha_r;\ P)$. Suppose $0 \leq j < r$. Then, for each i, the components of $Y_i \cap F_{j+1}$ whose codimension equals $k -(j + 1)$ cannot be contained in F_j. Therefore, we can choose a finite set $D_j \subset F_{j+1}$ such that $D_j \cap F_j = \phi$ and such that D_j contains, for each i, one point from each irreducible component of $Y_i \cap F_{j+1}$ of codimension $k -(j + 1)$. It follows that if $\underline{m} \in D_j$ then $\alpha_1,\ldots,\alpha_r$ have f-rank $j < r$ in $P_{\underline{m}}$. Therefore we can apply (1.7), according to which there are $\beta_i(\underline{m}) = \alpha_i + \alpha_r\, a_i(\underline{m})$ $(a_i(\underline{m}) \in A_{\underline{m}})$ $(1 \leq i < r)$, such that $\text{f-rank}_{A_{\underline{m}}}(\beta_i(\underline{m}),\ldots,\beta_{r-1}(\underline{m});\ P_{\underline{m}}) \geq j$. Since $\underline{m}$ is maximal we have $A_{\underline{m}}/\underline{m} \cdot A_{\underline{m}} = A/\underline{m} \cdot A$. Hence, by the Chinese Remainder Theorem, we can find $a_i \in A$ such that $a_i \equiv a_i(\underline{m})$ mod $\underline{m} \cdot A_{\underline{m}}$ for each $\underline{m} \in \bigcup_j D_j (0 \leq j < r)$. Now set $\beta_i = \alpha_i + \alpha_r\, a_i$ $(1 \leq i < r)$. Then the submodules of P, $(\alpha_1,\ldots,\alpha_{r-1},\alpha_r)$ and $(\beta_1,\ldots,\beta_{r-1},\alpha_r)$ are equal, so it follows from (1.6) that $F_j' = F_j(\beta_1,\ldots,\beta_{r-1};\ P) \subset F_{j+1}(\beta_1,\ldots,\beta_{r-1},\alpha_r;\ P) = F_{j+1}$. On the other hand, if $\underline{m} \in D_j$, then $\beta_i \equiv \beta_i(\underline{m})$ mod $P_{\underline{m}} \cdot \underline{m}$, due to the congruences on the a_i, so it follows from (1.2) that f-rank$_{A_{\underline{m}}}(\beta_1,\ldots,\beta_{r-1}; P_{\underline{m}})$ = f-rank$(\beta_1(\underline{m}),\ldots,\beta_{r-1}(\underline{m}); P_{\underline{m}}) \geq j$. Thus $F_j' \cap D_j = \phi$, so F_j' excludes, for each i, one point from each component of $Y_i \cap F_{j+1}$ of codimension k −(j + 1) in Y_i. It follows therefore that $\text{codim}_{Y_i}(Y_i \cap F_j') \geq k - j$. q.e.d.

§3. CANCELLATION; ELEMENTARY AUTOMORPHISMS

Serre's Theorem gives a criterion for a module P ε mod-A to be of the form $P \simeq A \oplus P'$. The results of this section give a similar criterion for the uniqueness (up to isomorphism) of P′. We retain the notation and assumptions of (2.1). We shall assume, moreover, that X *is the union of a finite number of subspaces whose dimension are each* < d.

(3.1) THEOREM. *Let* P, Q ε mod-A *be projective and assume* f-rank$_A$(P) > d. *Let* $\alpha = \alpha_Q + \alpha_P \in Q \oplus P$ $(\alpha_Q \in Q, \alpha_P \in P)$, *and let* $\underline{a}$ *be a left ideal in* A *such that* $\underline{a} + o_P(\alpha) = A$. (See §1 for definition of $o_P(\alpha)$). *Then there is a homomorphism* f: Q ⟶ P *such that* $\underline{a} + o_P(f(\alpha_Q) + \alpha_P) = A$.

Proof. We use induction on d; the case d = o will be subsumed in the general induction step.

Thanks to Serre's Theorem (2.7) we can write $P = \overline{\beta}A \oplus \overline{P}$ for some unimodular $\overline{\beta} \in P$; write $\alpha_P = \overline{\beta}b + \overline{\alpha}$ $(\overline{\alpha} \in \overline{P})$. Then we have $A = \underline{a} + o(\alpha) = \underline{a} + o(\alpha_Q) + Ab + o(\overline{\alpha})$. Let $D \subset X$ be a finite set containing one point (at least) from each irreducible component of each of the subspaces of which X is assured, above, to be the union. Then if $\underline{q} = \Pi\underline{m}$ $(\underline{m} \in D)$ the ring $A/\underline{q}A$ is semi-local. Hence it follows from

(III, 2.8) that we can find $c \in \underline{a}$, $\alpha_Q \in o(\alpha_Q)$, and $\overline{\alpha} \in o(\overline{\alpha})$ such that $c + b + \alpha_Q + \overline{\alpha}$ maps onto a unit in $A/A\underline{q}$. By definition of $o(\overline{\alpha})$ there is a homomorphism $g: \overline{P} \longrightarrow \overline{\beta}A$ such that $g(\overline{\alpha}) = \overline{\beta\alpha}$. Extend g to an endomorphism of P by $g(\overline{\beta}) = o$. Then $g^2 = o$, so $\sigma = 1_P + g$ is an automorphism, and $\sigma(\alpha_P) = \overline{\beta}(b + \overline{\alpha}) + \overline{\alpha}$. Set $\beta = \sigma^{-1}(\overline{\beta})$, $P_1 = \sigma^{-1}(\overline{P})$, and $\alpha_1 = \sigma^{-1}(\overline{\alpha}) \in P_1$. Then we have $P = \beta A \oplus P_1$ and $\alpha_P = \sigma^{-1}(\sigma(\alpha_P)) = \beta(b + \overline{\alpha}) + \alpha_1$.

By definition of $o(\alpha_Q)$ there is a homomorphism $f_1: Q \longrightarrow \beta A \subset P$ such that $f_1(\alpha_Q) = \beta\, \alpha_Q$. Then

$$(*) \qquad f_1(\alpha_Q) + \alpha_P = \beta b_1 + \alpha_1,$$

where $b_1 = b + \alpha_Q + \overline{\alpha}$. We saw above that $c + b_1$ maps to a unit in $A/A\underline{q}$. If we set $S = R - (\bigcup_{\underline{m} \in D} \underline{m})$ then $S^{-1} R$ is semi-local (its maxial ideals correspond to those in D) and $(S^{-1} A)/\underline{q} \cdot (S^{-1} A) = A/\underline{q} \cdot A$ (recall $\underline{q} = \prod_{\underline{m} \in D} \underline{m}$). Moreover $q \cdot (S^{-1} A) \subset \operatorname{rad}(S^{-1} A)$ so it follows that $c + b_1 \in U(S^{-1}A)$. If $d = 0$ then $D = X$ so $c + b_1 \in U(A)$, and the proof is complete in this case. If not we still have $S^{-1}(\underline{a} + Ab_1) = S^{-1} A$ so we can find a $t \in S$ such that

$$(**) \qquad At \subset \underline{a} + Ab_1.$$

Write $R' = R/Rt$, $A' = A/At$, $\underline{a}'$= image of $\underline{a}$ in A', etc. Then $X' = \max(R') = V(Rt)$ is disjoint from D, so it is a closed set in X containing no irreducible component of any of the given subspaces of which X is the union. Therefore, X′ is the union of its intersections with these subspaces, and the intersections have strictly smaller dimension than their counterparts in X. Thus X′ is a finite union of subspaces each of dimension $\leq d - 1$. Moreover we see, with the aid of (2.2), that $\text{f-rank}_{A'}(P_1') \geq \text{f-rank}_A(P_1) = \text{f-rank}_A(P) - 1 > d - 1$. Consider $\gamma' = \alpha_Q' + \alpha_1' \in Q' \oplus P_1'$. Since $A = \underline{a} + o(\alpha) = \underline{a} + o(\alpha_Q) + Ab_1 + o(\alpha_1)$ it follows that $A' = \underline{a}' + o(\gamma') + A'b_1'$. Now we are in a position to apply the induction hypothesis to $\gamma' \in Q' \oplus P_1'$ and the left ideal $\underline{a}' + Ab_1'$. We obtain a homomorphism $h'\colon Q' \longrightarrow P_1'$ such that $\underline{a}' + A'b_1' + o'_{P_1'}(h'(\alpha_Q') + \alpha_1') = A'$, (where $o'_M(\delta) = \{g\delta \mid g \in \text{Hom}_A'(M, A')\}$ for $M \in$ mod-A' and $\delta \in M$). Since Q is projective we can cover h′ by a homomorphism $h\colon Q \longrightarrow P_1$ $(\subset P)$. Now, for the theorem, we take

$$f = \begin{pmatrix} f_1 \\ h \end{pmatrix}\colon Q \longrightarrow P = \beta A \oplus P_1.$$

It remains to be shown that $\underline{a} + \underline{b} = A$, where $\underline{b} = o(f(\alpha_Q) + \alpha_P)$. Using (*) above we see that $f(\alpha_Q) + \alpha_P = (h(\alpha_Q) + f_1(\alpha_Q)) + \alpha_P = h(\alpha_Q) + (\beta b_1 + \alpha_1) = \beta b_1 + (h(\alpha_Q) + \alpha_1) \in \beta A \oplus P_1$. Since P_1 is projective the natural map $o_{P_1}(h(\alpha_Q) + \alpha_1) \longrightarrow o'_{P_1'}(h'(\alpha_Q') + \alpha_1')$ is surjective. We have constructed h′ so that $\underline{a}' + A'b_1' + o'_{P_1'}(h'(\alpha_Q') + \alpha_1') = A' = A/At$. Hence we conclude that $\underline{a} + \underline{b} + At = \underline{a} + Ab_1 + o_{P_1}(h(\alpha_Q) + \alpha_1) + At = A$. Since $At \subset \underline{a} + Ab_1$ (see (**)) $\subset \underline{a} + \underline{b}$ it follows that $\underline{a} + \underline{b} = A$. q.e.d.

(3.2) COROLLARY. *In the setting of* (3.1) *assume* $Q = \gamma A$ *for some unimodular* γ; *say* $\alpha = \gamma q + \alpha_P$.

(a) $P = \beta A \oplus p'$ *for some unimodular* $\beta \in P$.

(b) *Suppose, for some two sided ideal* $\underline{q}$, that $\alpha \equiv \beta$ mod $(\gamma A \oplus P)\underline{q}$. *Then there is a* $\gamma' \in P\,\underline{q}$ *such* that $o_P(\gamma' q + \alpha_P) + \underline{a} = A$.

Proof. (a). follows from Serre's Theorem.

(b). Since $\alpha \equiv \beta(\in P)$ modulo $(\gamma A \oplus P)\underline{q}$ it follows that $q \in \underline{q}$. By assumption (see (3.1)) there is an $h: \gamma A \oplus P \longrightarrow A$, and an $a \in \underline{a}$ such that $1 = h(\gamma)q + h(\alpha_P) + a$. Hence $q = r + qh(\alpha_P) + qa$, where $r = qh(\gamma)q$. Set $\alpha' = \gamma r + \alpha_P$. Then $q \in o(\alpha') = Ar + o(\alpha_P) \subset Aq + o(\alpha_P) = o(\alpha)$, so $o(\alpha') = o(\alpha)$. Hence we can apply (3.1) to α' and $\underline{a}$ to obtain an $f: \gamma A \longrightarrow P$ such that $o(f(\gamma r) + \alpha_P) + \underline{a} = A$. Since $f(\gamma r) = f(\gamma)\, qh(\gamma)q$ we see that $\gamma' = f(\gamma)\, qh(\gamma) \in P\,\underline{q}$ solves our problem. q.e.d.

In preparation for the next theorem we shall introduce now some notation which will also be used in the next chapter. For these definitions our hypotheses (2.1) on A are irrelevant.

Let $M \in$ mod-A have a direct sum decomposition $M = M_1 \oplus \ldots \oplus M_n$. Then $\mathrm{End}_A(M)$ is the direct sum of the $\mathrm{Hom}_A(M_i, M_j)$, where we identify $h \in \mathrm{Hom}_A(M_i, M_j)$ with its extension to M by $h(M_k) = 0$ for $k \neq i$. In case $i \neq j$ then $gh = 0$ whenever $g, h \in \mathrm{Hom}_A(M_i, M_j)$, and hence $(1_M + g)(1_M + h) = 1_M + g + h$, and we deduce a homomorphism $\mathrm{Hom}_A(M_i, M_j) \longrightarrow \mathrm{Aut}_A(M)$ for each $\neq j$. The group generated by the images of these homomorphisms, for all $i \neq j$, will be denoted

$$E(M_1,\ldots,M_n).$$

If $h \in \mathrm{Hom}_A(M_i, M_j)$ $(i \neq j)$, then we shall call $1_M + h$ an elementary automorphism (with respect to the decomposition $M = M_1 \oplus \ldots \oplus M_n$). If $\underline{q}$ is a two sided ideal in A we shall call $1_M + h$ $\underline{q}$ - elementary if $\mathrm{Im}(h) \subset M\underline{q}$. We denote by

$$E(M_1,\ldots,M_n;\ \underline{q})$$

the normal subgroup of $E(M_1,\ldots,M_n)$ generated by all $\underline{q}$-elementary automorphisms.

(3.3) PROPOSITION. Let $P = P_1 \oplus \ldots \oplus P_n$ be a projective right A - module, let $\underline{q}$ be a two sided ideal in A, and let $f\colon A \longrightarrow A'$ be a surjective ring homomorphism. Then the induced homomorphism,

$$E(P_1,\ldots,P_n;\ \underline{q}) \longrightarrow E(P_1{}',\ldots,P_n{}';\ \underline{q}'),$$

is surjective, where $\underline{q}' = f(\underline{q})$ and $P_i{}' = P_i \otimes_A A'$ $(1 \leq i \leq n)$.

Proof. Since $P_j\underline{q} \longrightarrow P_j{}'\underline{q}'$ is surjective, any homomorphism $h'\colon P_i{}' \longrightarrow P_j{}'\underline{q}'$ lifts to a homomorphism $h\colon P_i \longrightarrow P_j\underline{q}$, because P_i is projective. This shows that $\underline{q}'$ - elementary automorphisms can be lifted. Taking $\underline{q} = A$ this shows that $E(P_1,\ldots,P_n) \longrightarrow E(P_1{}',\ldots,P_n{}')$ is surjective. Now $E(P_1{}',\ldots,P_n{}';\ \underline{q}')$ is generated by elements of the form $\sigma'\tau'\sigma'^{-1}$ where $\sigma' \in E(P_1{}',\ldots,P_n{}')$ and τ' is $\underline{q}'$ - elementary. We can lift τ' to a $\underline{q}$ - elementary τ, and we can lift σ' to a $\sigma \in E(P_1,\ldots,P_n)$. Hence $\sigma\tau\sigma^{-1} \in E(P_1,\ldots,P_n;\ \underline{q})$

is the required lifting of $\sigma'\tau'\sigma'^{-1}$. q.e.d.

Now we return to our standing hypotheses (2.1). Moreover d has the same meaning as in (3.1).

(3.4) THEOREM. Let $M = \gamma A \oplus M_1$ where $M \in$ mod-A, γ is unimodular in M, and M_1 has a projective direct summand P of f-rank > d. Let $\underline{q}$ be a two sided ideal in A and let α, $\alpha' \in M$ be unimodular elements such that $\alpha \equiv \alpha' \bmod M\underline{q}$. Then there is an automorphism $\tau \in E(\gamma A, M_1; \underline{q})$ such that $\tau\alpha = \alpha'$.

Proof. We have $M_1 = P \oplus N$ for some N, and $P = \beta A \oplus P'$ for some unimodular $\beta \in P$ by Serre's Theorem.

Case 1: $\alpha' = \beta$. Write $\alpha = \gamma q + \alpha_{M_1}$ $(\alpha_{M_1} \in M_1)$ and $\alpha_{M_1} = \alpha_P + \alpha_N (\alpha_P \in P, \alpha_N \in N)$. According to (3.2) (b) there is $\gamma' \in P\underline{q}$ such that $o(\gamma' q + \alpha_P) + o(\alpha_N) = A$.

Remark. It is only at this point, to apply (3.2)(b) to $\gamma q + \alpha_P$ (with $\underline{a} = o(\alpha_N)$), and above to write $P = \beta a \oplus P'$, that the hypothesis on $\text{f-rank}_A(P)$ is used. If we accept these conclusions from (3.2), our standing assumptions on A and P (vis-a-vis R and X) do not otherwise intervene. This observation will be used in the next chapter.

Define $g_1: M \longrightarrow M$ by $g_1(\gamma) = \gamma'$ and $g_1(M_1) = o$. Then evidently $\tau_1 = 1_M + g_1 \in E(\gamma A, M_1; \underline{q})$. Moreover $\tau_1(\alpha) = \gamma q + (f(\gamma q) + \alpha_{M_1}) = \gamma q + (\gamma' q + \alpha_P) + \alpha_N$. Write $\gamma' q + \alpha_P = \beta b + \alpha' \in P = \beta A \oplus P'$ $(\alpha' \in P')$. By construction, $\delta = \gamma' q + \alpha_P + \alpha_N = \beta b + \alpha' + \alpha_N$ is unimodular in $P \oplus N = M_1$, so we can write $M_1 = \delta A \oplus M_1'$. Let $g_2: M \longrightarrow M$ by $g_2(\delta) = \gamma(1 - b - q)$ and $g_2(\gamma A) = g_2(M_1') = 0$. Since $\alpha \equiv \beta \bmod M\underline{q}$ we must have $b \equiv 1 \bmod \underline{q}$, and hence $\tau_2 = 1_M + g_2 \in E(\gamma A, M_1; \underline{q})$.

Moreover $\tau_2\tau_1(\alpha) = \tau_2(\gamma q + \delta) = \gamma(1 - b) + \delta = \gamma(1 - b) + \beta b + \alpha' + \alpha_N$.

Define $g_3, g_4 \colon M \longrightarrow M$ by $g_3(\gamma) = \beta$, $g_3(M_1) = 0$, and $g_4(\beta) = \gamma(b - 1)$, $g_4(\gamma A) = 0 = g_4(P \oplus N)$. Then $\tau_3 = 1_M + g_3 \in E(\gamma A, M_1)$, and $\tau_4 = 1_M + g_4 \in E(\gamma A, M_1; \underline{q})$. Moreover, $\sigma = \tau_3^{-1} \tau_4\tau_3\tau_2\tau_1 \in E(\ A, M_1; \underline{q})$, and $\sigma(\alpha) = \tau_3^{-1} \tau_4\tau_3(\gamma(1 - b) + \beta b + \alpha' + \alpha_N) = \tau_3^{-1} \tau_4(\gamma(1 - b) + \beta + \alpha' + \alpha_N) = \tau_3^{-1}(\beta + \alpha' + \alpha_N) = \beta + \alpha' + \alpha_N$. Finally, define $g_5, g_6 \colon M \longrightarrow M$ by $g_5(\beta) = \gamma$, $g_5(\gamma A) = 0 = g_5(P' \oplus N)$, and $g_6(\gamma) = -(\alpha' + \alpha_N)$, $g_6(M_1) = 0$. Then $\tau_5 = 1_M + g_5 \in E(\gamma A, M_1)$ and $\tau_6 \in 1_M + g_6 \in E(\gamma A, M_1; \underline{q})$, so $\tau_5^{-1} \tau_6\tau_5 \in E(\gamma A, M_1; \underline{q})$. Moreover $\tau_5^{-1} \tau_6\tau_5\ \sigma(\alpha) = \tau_5^{-1} \tau_6(\gamma + \beta + \alpha' + \alpha_N) = \tau_5^{-1}(\gamma + \beta) = \beta$. This proves case 1.

General case. Apply case 1 with $\underline{q} = A$ to obtain a $\sigma \in E(\gamma A, M_1)$ such that $\sigma\ \alpha' = \beta$. Now apply case 1 to $\sigma\ \alpha' \equiv \beta \bmod M\underline{q}$ to find $\tau \in E(\gamma A, M_1; \underline{q})$ such that $\tau\ \sigma\ \alpha = \beta = \sigma\ \alpha'$. Then $\sigma^{-1}\ \tau\ \sigma \in E(\gamma A, M_1; \underline{q})$ solves our problem. q.e.d.

(3.5) COROLLARY. ("Cancellation") *Suppose* $M \in$ mod-A *has a projective direct summand* of f-rank $> d$. Then *if* $M' \in$ mod-A *and if* $Q \in \underline{\underline{P}}(A)$,

$$Q \oplus M \simeq Q \oplus M' \Longrightarrow M \simeq M'.$$

Proof. After writing $Q \oplus Q' \simeq A^n$ an induction on n reduces this to the case $Q = A$. If we use the isomorphism to identify the modules we obtain $\alpha A \oplus M = \alpha' A \oplus M'$ where α and α' are unimodular. We can now apply (3.4) (with $\alpha = \gamma$, $M = M_1$, in the notation of (3.4)) to obtain an automorphism σ such that $\sigma\ \alpha = \alpha'$. Therefore $M \simeq (\alpha A \oplus M)/(\alpha A) \simeq (\alpha A \oplus M)/\sigma(\alpha A) = (\alpha' A \oplus M')/\alpha' A \simeq M'$. q.e.d.

(3.6) COROLLARY. Let M be as in (3.4), and let $\underline{a}$ be a two sided ideal in A. Write $A' = A/\underline{a}$ and $M' = M/A\underline{a}$. If α' is a unimodular element in M' (as A'-module) then there is a unimodular element α in M whose image mod $M\underline{a}$ is α'.

Proof. Apply (3.4) to M' over A' (the hypotheses are clearly still valid) to obtain a $\tau' \in E(\gamma' A', M_1')$ such that $\tau'\gamma' = \alpha'$. Now use (3.3) to lift τ' to $\tau \in E(\gamma A, M_1)$. Then $\alpha = \tau\gamma$ solves the problem. q.e.d.

(3.7) COROLLARY. Let M and $\underline{q}$ be as in (3.4), and suppose $M = \gamma' A \oplus M_1'$ for some unimodular element γ'. Then $E(\gamma' A, M_1'; \underline{q}) = E(\gamma A, M_1; \underline{q})$.

Proof. From (3.4) we obtain a $\sigma \in E(\gamma A, M_1; \underline{q})$ such that $\sigma\gamma = \gamma'$. It follows from the definitions that

$$E(\gamma A, M_1; \underline{q}) = \sigma E(\gamma A, M_1; \underline{q})\sigma^{-1} = E(\gamma' A, \sigma M_1; \underline{q}).$$

Therefore we may assume $\gamma = \gamma'$. Define $g: M \longrightarrow M$ by $g(\gamma) = 0$ and $g|M_1 = p|M_1$, where p is the projection of $\gamma A \oplus M_1'$ on γA. Then $\tau = 1_M - g \in E(\gamma A, M_1)$, and $\tau M_1 = M_1'$. Hence $E(\gamma A, M_1; \underline{q}) = \tau E(\gamma A, M_1; \underline{q})\tau^{-1} = E(\gamma A, M_1'; \underline{q})$. q.e.d.

(3.8) COROLLARY. Suppose $P \in \underline{\underline{P}}(A)$ is such that, for each $\underline{m} \in X$, $P_{\underline{m}}$ can be generated (over $A_{\underline{m}}$) by $\leq r$ elements. Then P can be generated by $\leq r + d$ elements.

Proof. Write $P \oplus Q \simeq A^{r+n}$ for some $n \geq 0$. It suffices to shown we can do this with $n \leq d$, so suppose otherwise. If $\underline{m} \in X$ then, by hypothesis, we can write $P_{\underline{m}} \oplus P' \simeq A_{\underline{m}}^{\ r}$ for some P'. Since $A_{\underline{m}}$ is semi-local it follows from (1.4) that $Q_{\underline{m}} \simeq P' \oplus A_{\underline{m}}^{\ n}$. Thus $\text{f-rank}_A(Q) \geq n > d$, so Serre's Theorem (2.7) implies $Q \simeq Q' \oplus A$. Since $P \oplus Q' \oplus A \simeq A^{r+n}$ it follows that $(P \oplus Q')_{\underline{m}} \simeq A_{\underline{m}}^{\ r+n-1}$ for each $\underline{m} \in X$, again by

(1.4). If $r = 0$ then $P = 0$ and there is nothing to prove. Otherwise $r + n - 1 > d$ so we can apply cancellation, (3.5) above, to conclude that $P \oplus Q' \simeq A^{r+n-1}$. The conclusion now follows by induction. q.e.d.

Remark. Swan [4] has recently shown that (3.8) above is valid without the assumption that P is projective.

§4. THE AFFINE GROUP OF A MODULE

It is convenient to make here a few simple observations on the groups of elementary automorphisms introduced in §3. These results will be used in the next chapter.

We fix a ring A.

(4.1) PROPOSITION. *Let* $P_1, \ldots, P_n \in \underline{\underline{P}}(A)$, *and assume that at least two of the* P_i*'s are faithfully projective. Let* $P = P_1 \oplus \ldots \oplus P_n$.

(a) *The additive group generated by* $E(P_1, \ldots, P_n)$ *is all of* $End_A(P)$.

(b) *The centralizer in* $Aut_A(P)$ *of* $E(P_1, \ldots, P_n)$ *is*

$$\text{center }(Aut_A(P)) = \{c\, 1_P \mid c \in (\text{center }(A))\}.$$

(c) *An additive subgroup of* P *invariant under* $E(P_1, \ldots, P_n)$ *is of the form* $P\mathfrak{a}$ *for a unique left ideal* $\mathfrak{a}$ *in* A, *and* $P\mathfrak{a}$ *is also invariant under* $End_A(P)$.

Proof. Let $B = End_A(P)$ and let B_o be the additive group generated by $E = E(P_1, \ldots, P_n)$. Then E and B_o have the same centralizer in B, and $\mathfrak{a}$ subgroup of P invariant under E is a B_o-module. Therefore (a) implies (b) and (c). For, since P is faithfully projective, it follows from (III, 3.5) and (II, 4.4) that center (B) = center (A) and that every B - submodule of P has the form described in (c).

It remains to prove (a). It is clear from the definition of $E(P_1,\dots,P_n)$ that the additive group B_o it generates is generated by $I (= 1_P)$ and by all $\mathrm{Hom}_A(P_i, P_j)$ $(i \neq j)$. Therefore we need only show how to recover $\mathrm{Hom}_A(P_i, P_i)$ for each i. Suppose we have an endomorphism $f = gh$ of P_i which factors as $P_i \xrightarrow{h} P_j \xrightarrow{g} P_i$ for some $j \neq i$. Then $(I + g)(I + h) = I + g + h + gh \in E(P_1,\dots,P_n)$ and $I, g, h \in B_o$. Our hypothesis guarantees that we can choose a $j \neq i$ so that P_j is faithfully projective. Therefore it will suffice to show that $\mathrm{End}_A(P_i)$ is additively spanned by endomorphisms which admit a factorization through P_j. If $f \in \mathrm{End}_A(P_i)$ factors through P_j^n then it is a sum of n endomorphisms that factor through P_j. For n large enough P_j^n has a direct summand $\approx A$. Since $P_i \in \underline{\underline{P}}(A)$, it follows from (II, 4.4(a)) that $\mathrm{End}_A(P_i)$ is additively generated by endomorphisms which factor through A, and hence through P_j^n. q.e.d.

Before introducing the affine group we shall establish some group theoretic conventions.

Let G be a group, and let $x, y, z \in G$. Then we shall write $x^y = y^{-1} x y$ and $[x, y] = x^{-1} y^{-1} x y = x^{-1} x^y$. The following formulas are familiar, and easily checked:

$$(x^y)^z = x^{yz} = (x^z)^{y^z}$$

$$[x, y]^{-1} = [y, x]$$

$$[x, yz] = [x, z]\,[x, y]^z \tag{4.2}$$

$$[xy, z] = [x, z]^y\,[y, z]$$

If H and H´ are subgroups of G then [H, H´] denotes the subgroup generated by all [x, x´] ($x \in H$, $x' \in H'$).

Let P be a group on which G operates as a group of automorphisms ($x \mapsto \alpha(x)$, for $x \in P$, $\alpha \in G$). (This structure is equivalent to a homomorphism $G \longrightarrow \mathrm{Aut}(P)$). Then we can form the <u>semi-direct product</u>

$$P \times_{s-d} G,$$

whose underlying set is P x G and whose multiplication is defined by

$$(x, \alpha)\ (y, \beta) = (x \cdot \alpha(y),\ \alpha\beta).$$

For example $(x, \alpha)^{-1} = (\alpha^{-1}(x)^{-1}, \alpha^{-1})$. We can identify $x \in P$ with $(x, 1)$ and $\alpha \in G$ with $(1, \alpha)$. As such, P is a normal subgroup of $P \times_{s-d} G$, and we have a "split group extension"

$$1 \longrightarrow P \longrightarrow P \times_{s-d} G \longrightarrow G \longrightarrow 1.$$

Suppose now that P is an additive abelian group. Then it is suggestive to use matrix notation, writing

$$\begin{pmatrix} 1 & 0 \\ x & \alpha \end{pmatrix} \text{ in place of } (x, \alpha).$$

Then the group law becomes

$$\begin{pmatrix} 1 & 0 \\ x & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x + \alpha(u) & \alpha\beta \end{pmatrix},$$

i.e. matrix multiplication. The following formulas are easily checked, where we write I for the identity element in G.

(1)

$$\begin{pmatrix} 1 & 0 \\ x & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\alpha^{-1}(x) & \alpha^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ y & I \end{pmatrix}^{\begin{pmatrix} 1 & 0 \\ x & \alpha \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ \alpha(y) & I \end{pmatrix}$$

Finally, if $P \in$ mod-A, the <u>affine group</u> of P is

$$\mathrm{Aff}_A(P) = P \underset{s-d}{\times} \mathrm{Aut}_A(P) \quad " = \begin{pmatrix} 1 & 0 \\ P & \mathrm{Aut}_A(P) \end{pmatrix} "$$

When $P = A^n$ we denote this group by

$$\mathrm{Aff}_n(A) = \begin{pmatrix} 1 & 0 \\ x & \alpha \end{pmatrix} \quad x \in A^n, \ \alpha \in GL_n(A)$$

It is a subgroup of $GL_{n+1}(A)$.

(4.3) PROPOSITION. *Let $P \in$ mod-A and let H be a subgroup of $\mathrm{Aff}_A(P)$ with projection L in $\mathrm{Aut}_A(P)$.*

(a) $[H, P] = [L, P] = \Sigma_{\alpha \in L} \mathrm{Im}(\alpha - 1P)$

(b) *If H is normalized by P then $H \cap P = \{1\} \Rightarrow H = \{1\}$.*

(c) *If $P = P_1 \oplus \ldots \oplus P_n$ is as in (4.1), and if H is normalized by $E(P_1, \ldots, P_n)$, then there are unique left ideals $\mathfrak{a}$, $\mathfrak{b}$ in A such that $H \cap P = P\mathfrak{a}$ and $[H: P] = P\mathfrak{b}$. If $L \neq \{1\}$ then $\mathfrak{b} \neq 0$.*

Proof. (a) follows immediately from the last formula in (1) above.

(b) If P normalizes H then $[H, P] \subset H \cap P$. Therefore if $H \cap P = \{1\}$ formulas (1) show that $H \subset P$, and this proves (b).

(c) If $E(P_1, \ldots, P_n)$ normalizes H then $H \cap P$ and $[H, P]$ are additive subgroups of P invariant under $E(P_1, \ldots, P_n)$. Therefore (c) follows from (4.1)(c) together with part (a).

§5. FREE PRODUCTS OF FREE IDEAL RINGS; COHN'S THEOREM

If $A = R[t]$ is a polynominal ring in one variable t over a field R then A is a principal ideal ring. This is a direct consequence of the euclidean (division) algorithm in A. When we pass to a polynomial ring in several variables, $R[t_1,...,t_n]$, this situation no longer prevails. If, on the other hand, we consider a polynomial ring in "non commuting variables", i.e. the free associative algebra on $t_1,...,t_n$, then the case of general n behaves very much like the case $n = 1$. Of course the ideals are no longer principal, but they are free as modules. Moreover, this property can be deduced from a generalization of the division algorithm.

These results are due to P. M. Cohn [1]. His point of view is to regard the free algebra as a "free product" of polynomial rings in one variable, and then to show that free products of algebras whose ideals are free again have this property. This theorem applies equally well to free products of copies of $R[t, t^{-1}]$, and these are just group algebras of free (non abelian) groups.

Since the material of this section is lengthy and rather technical it is perhaps useful to mention that it is not required elsewhere in these notes except in §6 (Corollary (6.4)) and in Chapter XII, §11.

(5.1) DEFINITION. Let n be an integer ≥ 1. A ring A is called an n-fir (fir = "free ideal ring") if it satisfies:

(a_n) Every basis for A^n has cardinality n; and

(b_n) Every right ideal with at most n generators is a free A-module.

Each condition implies the corresponding conditions for smaller values of n. Condition (a_n) asserts that, for all $m \geq 0$, $A^n \simeq A^m \Rightarrow n = m$. Taking duals, i.e., $\mathrm{Hom}_A(\ \ , A)$ we deduce the same condition for free left A-modules, so (a_n) is left-right symmetric. We shall see below that the notion of n-fir is likewise left-right symmetric.

(5.2) PROPOSITION. Let A be an n-fir. Then:

(a_n') Every epimorphism $f: A^n \longrightarrow A^n$ is an isomorphism; and

(b_n') For each $m \geq 0$, the image of every homomorphism $f: A^n \longrightarrow A^m$ is free.

Proof. $(b_n) \Rightarrow (b_n')$. We use induction on m, the case $m = 1$ being just (b_n). If $M = \text{Im}(f)$ we have an exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ where M'' is the projection of M on the last coordinate, and $M' \subset A^{m-1}$. Since M'' has $\leq n$ generations it is free, by (b_n). Hence $M \simeq M'' \oplus M'$ so M' also has $\leq n$ generators. By induction M' is also free. Therefore M is free.

$[(a_n)$ and $(b_n)] \Rightarrow (a_n')$. We have $A^n \simeq \text{Ker}(f) \oplus A^n$ if $f: A^n \longrightarrow A^n$ is surjective. Therefore (b_n') (which follows from (b_n)) implies Ker(f) is free, say $\simeq A^r$. Then (a_n) implies $n + r = n$, so $r = 0$, i.e. $\text{Ker}(f) = 0$.

It is easy to see that $(a_n') \Rightarrow (a_n)$, so that (5.2) actually characterizes n-firs. The condition (a_n') can sometimes be verified with the aid of the following useful proposition.

(5.2) PROPOSITION. Let A be a ring, let $M \in \underline{\underline{M}}(A)$, and let $h: M \longrightarrow M$ be an epimorphism. Assume that either (i) M is noetherian, or (ii) A is commutative. Then H is an isomorphism.

Proof. (i) The chain $\text{Ker}(h^n)$ terminates; say $\text{Ker}(h^n) = \text{Ker}(h^{n+1})$ for some $n > 0$. If $x \in \text{Ker}(h)$ write $x = h^n(y)$ (note that h^n is surjective). Then $h^{n+1}(y) = h(x) = 0$ so

$y \in \mathrm{Ker}(h^{n+1}) = \mathrm{Ker}(h^n)$, and hence $x = h^n(y) = 0$. q.e.d.

(ii) It suffices to show that $h_{\underline{m}}$ is surjective for each $\underline{m} \in \max(A)$, so we can assume A is local, say with maximal ideal $\underline{m}$. Choose $g: A^n \longrightarrow M$ so that $g \otimes_A (A/\underline{m})$ is an isomorphism. Then g is surjective, by Nakayama's Lemma. Therefore we can find $f: A^n \longrightarrow A^n$ covering h (i.e. $gf = hg$). Again Nakayama implies f is surjective, because $f \otimes_A (A/m) \simeq h \otimes_A (A/\underline{m})$. Let $P(t) = a_0 + .. + a_{n-1}t^{n-1} + t^n$ be the characteristic polynomial of f. Then $a_0 = (-1)^n \det(f)$ is a unit, being non zero modulo $\underline{m}$. By the Cayley-Hamilton Theorem, $P(f) = 0$, so $f^{-1} = a_0^{-1}(a_1 + .. + a_{n-1} f^{n-2} + f^{n-1})$. Since f leaves Ker(g) invariant so also does f^{-1}, being a polynomial in f. Therefore f^{-1} induces an endomorphism h' of M, and evidently $h' = h^{-1}$. q.e.d.

Let $e_1, \ldots, e_n$ be the standard basis of $A^n = \coprod e_i A$. We can identify $GL_n(A)$ with $\mathrm{Aut}_A(A^n)$ where $\sigma \in GL_n(A)$ operates on $\alpha = (a_1, \ldots, a_n)$ $(= \Sigma e_i a_i) \in A^n$ by ${}^t(\sigma {}^t\alpha)$. Here the "t" denotes transpose, so that ${}^t\alpha$ is a column vector. We have the group

$$E_n(A) = E(e_1A, \ldots, e_nA)$$

introduced in §3. If e_{ij} denotes the matrix with 1 in the (i, j) coordinate and zeros elsewhere (so $e_{ij}\, e_k = \delta_{jk}\, e_i$) then $E_n(A)$ can be identified with the group generated by all <u>elementary matrices</u>, $I_n + a\, e_{ij}$ $(a \in A,\ i \neq j)$. The group of all diagonal matrices, $\mathrm{diag}(u_1, \ldots, u_n) = \Sigma u_i e_{ii}$ $(u_i \in U(A),\ 1 \leq i \leq n)$ will be denoted

$D_n(A)$.

Since the diagonal matrices normalize the set of elementary matrices we can write

$$GE_n(A) = D_n(A) \cdot E_n(A)$$

for the group generated by $D_n(A)$ and $E_n(A)$. When $n = 1$ we have $E_1(A) = \{1\}$ and $GL_1(A) = GE_1(A) = D_1(A) = U(A)$.

(5.4) DEFINITION. A ring A is said to be a generalized n-euclidean ring if A is an n-fir such that $GE_r(A) = GL_r(A)$ $(1 \leq r \leq n)$. If this is so for all $n \geq 1$ we call A generalized euclidean.

The motivation for this terminology will appear in Proposition (5.9) below.

We shall view $GL_n(A)$ as a subgroup of $GL_{n+1}(A)$ by identifying $\sigma \in GL_n(A)$ with $\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(A)$. Suppose now that we are given a family of subgroups $GL'_n(A) \subset GL_n(A)$ containing $E_n(A)$ and such that $GL'_{n+1}(A) \cap GL_n(A) \supset GL'_n(A)$. Relative to this family of subgroups we can formulate condition: $(C_n)_{GL'}$. If $r \leq n$ and if $a_1, \ldots, a_r$ are linearly dependent elements in a free right A-module F, then there is a $\sigma \in GL'_r(A)$ such that $(a_1, \ldots, a_r)\sigma$ has at least one zero coordinate.

By induction on r it follows that there is a $\sigma \in GL'_r(A)$ such that the non zero coordinates of $(a_1, \ldots, a_r)\sigma$ are a basis for the A-module generated by $a_1, \ldots, a_r$. In particular, a submodule of F with $\leq n$ generators is free, thus showing that $(C_n)_{GL'} \Rightarrow (b_n)$.

(5.5) PROPOSITION. The ring A satisfies condition $(C_n)_{GL'}$ if and only if A is an n-fir such that $GL'_r(A) = GL_r(A)$ for each $r \leq n$. For this it suffices even that A satisfy $(C_n)_{GL'}$ only for the free module $F = A$.

Proof. Let $(C'_n)_{GL'}$ denote the special case of $(C_n)_{GL'}$ when $F = A$. The remarks above show that $(C'_n)_{GL'} \Rightarrow (b_n)$. We further prove (a_n) and the fact that $GL'_r(A) = GL_r(A)$ for $r \leq n$.

Let $\alpha_1,\ldots,\alpha_s$ be a basis for the left A-module A^r. By induction on r we will show that there is a $\sigma \in GL'_r(A)$ such that $\alpha_1\sigma,\ldots,\alpha_s\sigma$ is the standard basis. This implies $r = s$, and hence condition (a_n) (or, rather, its left hand analogue, with which it is equivalent), as well as the fact that $GL'_r(A) = GL_r(A)$.

Since α_1 is unimodular its coordinates generate the unit right ideal. It follows therefore form $(C'_n)_{GL'}$ that $\alpha_1\sigma_1 = (u,0,\ldots,0)$ for some u, necessarily a unit, and we can arrange that $u = 1$ using an element of $D_n(A)$. Choose $\tau \in E_r(A)$ so that $\alpha_i\,\sigma_1\,\tau = \alpha_i\sigma_1 - \alpha_1\sigma_1 a_i$, where a_i is the first coordinate of $\alpha_i\sigma_1$, $(1 < i \leq s)$. Then $\beta_i = \alpha_i\sigma_1\tau$ has first coordinate zero, so $\beta_2,\ldots,\beta_s$ can be viewed as a basis for A^{r-1}. By induction we can transform these to the standard basis of A^{r-1} with some $\sigma_2 \in GL'_{r-1}(A)$, and then

$$\sigma = \sigma_1\,\tau\begin{pmatrix}1 & 0\\ 0 & \sigma_2\end{pmatrix} \text{ solves our problem.}$$

For the converse, we will show that if A is an n-fir and if $GL'_r(A) = GL_r(A)$ for $r \leq n$ then A satisfies $(C_n)_{GL'}$.

Given $a_1,\ldots,a_r \in F$ as in $(C_n)_{GL'}$ define $f\colon A^r \longrightarrow F$ by $f(e_i) = a_i$ $(1 \leq i \leq r)$. Condition (b_n') implies Im(f) is free, so $A^n \simeq \text{Ker}(f) \oplus \text{Im}(f)$. Therefore Ker(f) has at most r ($\leq$ n) generators so it likewise is free. Say $\text{Ker}(f) \simeq A^s$ and $\text{Im}(f) \simeq A^t$. Condition (a_n) implies $s + t = r$ so there is a $\sigma \in GL_r(A)$ (= $GL_r'(A)$) such that $\sigma e_1,\ldots,\sigma e_s$ is a basis for Ker(f). Since $a_1,\ldots,a_r$ are assumed to be linearly dependent we have $s > 0$. If $\sigma e_i = \Sigma_j e_j b_{ji}$ then $0 = f(\sigma e_1) = \Sigma_j a_j b_{j1}$ so $(a_1,\ldots,a_r)(b_{ji})_{1\leq i,j\leq r}$ has first coordinate zero, and $\sigma = (b_{ji}) \in GL_r'(A)$. This concludes the proof of $(C_n)_{GL'}$, and hence of the proposition.

(5.6) COROLLARY. The ring A is generalized n-euclidean if and only if it satisfies $(C_n)_{GE}$.

(5.7) COROLLARY. If A is an n-fir then so also is A^o. (i.e. the notion of an n-fir is left right symmetric).

Proof. According to (5.5) it suffices to show that if $b_1,\ldots,b_r \in A$ $(r \leq n)$ are left linearly dependent then there is a $\sigma \in GL_r(A)$ such that $\sigma^t\beta$ has a zero coordinate, where $\beta = (b_1,\ldots,b_r)$. We can clearly assume that none of the b_i are already zero. Let $\Sigma a_i b_i = 0$ be a dependence relation. According to (5.5), and our hypothesis, there is a $\sigma \in GL_r(A)$ such that the non zero coordinates of $\alpha\sigma$ are right linearly independent, where $\alpha = (a_1,\ldots,a_r)$. Since $0 = \alpha\, {}^t\beta = \alpha\sigma\, \sigma^{-1}\, {}^t\beta$ it follows that the i^{th} coordinate of $\sigma^{-1}\, {}^t\beta$ is zero whenever the i^{th} coordinate of $\alpha\sigma$ is not zero. Since $\alpha \neq 0$, by assumption, there exists at least one such i. q.e.d.

(5.8) DEFINITION. A euclidean algorithm on a ring A is a function $|\ |: A \longrightarrow \underline{\underline{R}}$ satisfying: (i) $|A|$ is a closed discrete subset of $\underline{\underline{R}}$; (ii) $|a| \geq 0$ and $|a| = 0 \Longleftrightarrow a = 0$ for $a \in A$; (iii) $|ab| > |a|\ |b|$ for a, $b \in A$; and (iv). If a, $b \in A$ and $a \neq 0$ then $b = aq + r$ for some $q, r \in A$ such that $|r| < |a|$. A is called euclidean if it possesses a euclidean algorithm.

The main examples of euclidean rings are $A = \underline{\underline{Z}}$ ($|\ |$ = ordinary absolute value) and $A = k[t]$, a polynomial ring over a field ($|f| = \exp(\text{degree}(f))$).

(5.9) PROPOSITION. *If* A *is a euclidean ring then* A *is a generalized euclidean ring and every right ideal in* A *is principal*.

Proof. Let $\underline{a}$ be a right ideal in A; we claim $\underline{a}$ is principal. We can assume $\underline{a} \neq 0$, and, thanks to (i), we can choose $a \neq 0$ in $\underline{a}$ so that $|a|$ is minimal. If $b \in \underline{a}$ write $b = aq + r$ as in (iv). Then $r = b - aq \in \underline{a}$ and $|r| < |a|$ so $r = 0$. Therefore $\underline{a} = aA$ is principal.

In particular A is right noetherian so (5.3) implies A satisfied condition (a_n) (cf (5.2)) for all $n \geq 1$. Condition (iii) implies A is an integral domain. Since right ideals are principal they are therefore free, so we have condition (b_n) for all $n \geq 1$.

It remains to be shown that $GE_n(A) = GL_n(A)$ for each n. This is an easy consequence of the following fact: If $\alpha = (a_1, \ldots, a_n) \in A^n$ there is an $\varepsilon \in E_n(A)$ such that $\alpha\varepsilon = (a, 0, \ldots, 0)$ for some $\alpha \in A$. For it suffices, by induction on n, to make a single coordinate of $\alpha\varepsilon$ equal zero. For this we can, thanks to (i), use induction on $m(\alpha)$ = the minimum of $|a_i|$ $(1 \leq i \leq n)$. If $|a_i| = m(\alpha)$ and $a_i \neq 0$ then we can apply (iv) to $a_j (j \neq i)$ and write $a_j = a_i q_j + r_j$ with

$|r_j| < |a_i|$. Putting $a_j' = r_j$ if $j \neq i$ and $a_i' = a_i$ we can find ε so that $\alpha\varepsilon = (a_1', \ldots, a_n') = \alpha'$, and $m(\alpha') < m(\alpha)$. The proof concludes now by induction.

(5.10) EXAMPLES. When n = 1 the notion of a 1-fir reduces simply to the notion of an integral domain, i.e. a ring without proper divisors of zero, though not necessarily commutative.

Two elements a,b in a commutative ring A can never be linearly independent: $ab - ba = 0$. Therefore a free ideal must have a basis of cardinality at most one. It follows easily that, *if A is commutative the following conditions are equivalent*.

(i) A *is an* n-*fir for some* $n > 1$.

(ii) A *is an integral domain in which every finitely generated ideal is principal*.

(iii) A *is an* n-*fir for all* $n \geq 1$.

Moreover the extra condition for A to be generalized n-euclidean can be restated as: $E_r(A) = SL_r(A)$ for all $r \leq n$.

Let A be a Dedekind ring, and let S be a multiplicative set in A. If $A' = S^{-1}A$ then, since $\text{Pic}(A) \longrightarrow \text{Pic}(A')$ is surjective, it follows that A is principal if A is. Moreover it follows from the remark after (VI, 1.5) in Chapter VI below that $SL_n(A')$ is generated by $SL_n(A)$ together with $E_n(A')$. We conclude therefore that A′ is generalized n-euclidean if A is. As a special case, the group ring $k[t, t^{-1}]$ over a field k of an infinite cyclic group is a generalized n-euclidean ring for all $n \geq 1$, thanks to (5.9).

Let R be a commutative ring. An augmented R-*algebra* is an R-algebra, $e: R \longrightarrow A$, together with an R-algebra homomorphism $\varepsilon = \varepsilon_A: A \longrightarrow R$; note that $\varepsilon e = 1_R$, the only R-algebra endomorphism of R, so that $A = R \oplus \overline{A}$ as R-module, where $\overline{A} = \text{Ker}(\varepsilon)$ is called the *augmentation ideal* of A. The augmented R-algebras the objects of a category in which a morphism $f: A \longrightarrow B$ is an R-algebra homomorphism such that $\varepsilon_B f = \varepsilon_A$. In this category coproducts (sometimes called free products) exist. We shall give a description of them, due to Stallings, and then prove Cohn's Theorem stating that a coproduct of n-firs over a field is again an n-fir.

(5.11) EXAMPLE. The functor

$$\begin{pmatrix}\text{augmented} \\ \text{R-algebras}\end{pmatrix} \xrightarrow{\text{augmentation ideal}} \text{R-mod}$$

has an adjoint, $T(= T_R)$, called the tensor algebra. Thus, if $M \in$ R-mod and A is an augmented R-algebra, then

$$\text{Hom}_{\text{aug.R-alg.}}(T(M), A) = \text{Hom}_{\text{R-mod}}(M, \bar{A}).$$

$T(M)$ is actually a graded R-algebra, with $T^n(M) = M^{\otimes n} = M \otimes_R \ldots \otimes_R M$, and with the obvious multiplication. The augmentation sends $T^n(M)$ to zero for all $n > 0$. If we denote the coproduct of A and B in (aug.R-alg.) by $A * B$ then the adjointness formula shows that

(1) $\qquad T(M \oplus N) = T(M) * T(N).$

Moreover T commutes with base change, $R \longrightarrow R'$, in the obvious sense. If M is a free module with basis $(x_i)_{i \in I}$ then $T(M)$ is called the "polynomial algebra in non commuting indeterminates $(x_i)_{i \in I}$".

(5.12) EXAMPLE. The functor

$$\begin{pmatrix}\text{augmented} \\ \text{R-algebras}\end{pmatrix} \xrightarrow{A \longrightarrow (1 + \bar{A})} \text{(monoids)}$$

has an adjoint called the monoid algebra. If π is a monoid the monoid algebra, $R\pi$, is the free R-module with basis π, and with multiplication extended R-bilinearly from the multiplication in π. If $a = \Sigma\, a_x\, x$ $(x \in \pi)$ then $\varepsilon(a) = \Sigma\, a_x$, so the augmentation ideal $\overline{R\pi}$, is, as an R-module, generated by all $1 - x$ $(x \in \pi)$. If A is an augmented R-algebra the adjointness is expressed by

$$\mathrm{Hom}_{\text{aug.R-alg.}}(R\pi, A) = \mathrm{Hom}_{\text{monoid}}(\pi, 1 + \bar{A}).$$

Again it follows that

$$R[\pi_1 * \pi_2] = R\pi_1 * R\pi_2,$$

where $\pi_1 * \pi_2$ denotes coproduct (or free product) in the category of monoids. When π is a free monoid with basis $(x_i)_{i\varepsilon I}$ then $R\pi$ is a new representation of the ring of "non commuting polynomials" encountered above. If π is a free abelian monoid we recover the ordinary commutative polynomial algebra. In particular, if π_o is a free monoid with generator t, and if π_1 is the free group with generator t then $R\pi_o = R[t]$ and $R\pi_1 = R[t, t^{-1}]$ are euclidean if R is a field. Thus if π is a free monoid or group and if R is a field, then $R\pi$ is a coproduct of euclidean rings.

We now come to the construction of coproducts. In describing them we shall use the notion of a π-graded R-algebra A where π is a not necessarily commutative monoid. Such a grading consists of an R-module decomposition. $A = \coprod A^w$ $(w \in \pi)$ such that $A^u A^v \subset A^{uv}$ $(u, v \in \pi)$.

Let A and B be augmented R-algebras. We propose to describe $C = A * B$.

To begin with the R-module homomorphism $\bar{A} \subset A$ induces an algebra epimorphism, $p_A: T(\bar{A}) \longrightarrow A$, from the tensor algebra of $\bar{A}$. Similarly we have $p_B: T(\bar{B}) \longrightarrow B$, and these induce an epimorphism $p = p_A * p_B: \tilde{C} \longrightarrow \tilde{C}$, where $\tilde{C} = T(\bar{A}) * T(\bar{B})$. Since

$$T(\bar{A}) * T(\bar{B}) = T(\bar{A} \oplus \bar{B})$$

it follows that $\tilde{C}$ has a natural $\tilde{\pi}$-grading, where $\tilde{\pi}$ is the free monoid on two generators $\tilde{\alpha}$ and $\tilde{\beta}$. Specifically,

$$C^1 = R$$

$$C^{w\tilde{\alpha}} = C^w \otimes \bar{A}$$

$$C^{w\tilde{\beta}} = C^w \otimes \bar{B} \qquad (w \in \tilde{\pi})$$

If $w \in \tilde{\pi}$ write $|w|$ for its "length", i.e. the number of factors $\tilde{\alpha}$ and $\tilde{\beta}$ in w. In passing from $\tilde{C}$ to C this grading collapses, and we are led to introduce the monoid π with generators α and β subject only to the relations $\alpha^2 = \alpha$ and $\beta^2 = \beta$. We map $\tilde{\pi} \longrightarrow \pi$ by $\tilde{\alpha} \longmapsto \alpha$ and $\tilde{\beta} \longmapsto \beta$. In π every element $w \neq 1$ has a unique representation of the form $w = \alpha\beta\alpha\beta\ldots$ or $w = \beta\alpha\beta\alpha\ldots$ Thus, if $w \in \pi$ then there is a unique preimage $\tilde{w} \in \pi$ for which $|\tilde{w}|$ is minimal. We then define the length $|w|$ of w to be the length of $\tilde{w}$.

Now we shall construct C and exhibit a π-grading of C. If $w \in \pi$ we set $C^w = C^{\tilde{w}}$. Next define

$$f_{w,\alpha}\colon C^w \otimes \bar{A} \longrightarrow C^{w\alpha}$$

as follows. If w terminates with β let $f_{w,\alpha}$ be the identity; this makes sense because $\widetilde{w\alpha} = \tilde{w}\tilde{\alpha}$ in this case. If $w = u\alpha$ for some $u \in \pi$ of smaller length then we have $w\alpha = w$ and $C^w = C^u \otimes \bar{A}$, and we define $f_{w,\alpha} = C^u \otimes m_A$, where $m_A\colon \bar{A} \otimes \bar{A} \longrightarrow \bar{A}$ is induced by the multiplication in A.

Similarly we define $f_{w,\beta}\colon C^w \otimes \bar{B} \longrightarrow C^{w\beta}$. By induction on $|v|$ we can then define an associative multiplication $f_{w,v}\colon C^w \otimes C^v \longrightarrow C^{wv}$ which makes C a π-graded R-algebra. It is augmented by $\varepsilon_C(C^w) = 0$ for all $w \neq 1$.

The inclusion $A = R \oplus \bar{A} = C^1 \oplus C^\alpha \subset C$ is an inclusion of augmented R-algebras, and we have a similar inclusion $B \subset C$. To show that the C just constructed is indeed $A * B$ consider the projection $p\colon T(\bar{A} \oplus \bar{B}) \longrightarrow C$ which exists because C is generated by $\bar{A} \oplus \bar{B} \subset \bar{C}$. Clearly $p(\tilde{C}^u) = C^{p(u)}$ where we also write $p\colon \tilde{\pi} \longrightarrow \pi$ for the projection $\tilde{\alpha} \longmapsto \alpha$, $\tilde{\beta} \longmapsto \beta$.

Algebra homomorphisms $h_A\colon A \longrightarrow D$ and $h_B\colon B \longrightarrow D$ induce module homomorphisms or $\bar{A}$ and $\bar{B}$ from which we obtain an R-algebra homomorphism $\tilde{h}\colon \tilde{C} \longrightarrow D$. Since $\tilde{h}$ induces h_A and h_B on $\bar{A}$ and $\bar{B}$, respectively, and since h_A and h_B are algebra homomorphisms, it is clear that $\tilde{h}$ factors uniquely through a homomorphism $h\colon C \longrightarrow D$. This shows that C is the free product of A and B.

We shall occasionally omit the symbol $\otimes$ when writing the multiplication in C. Thus, for example,

$$C = R \oplus (\overline{A} \oplus \overline{B}) \oplus (\overline{AB} \oplus \overline{BA}) \oplus (\overline{ABA} \oplus \overline{BAB}) \oplus \ldots$$

For each n > 0 there are precisely two elements of length n in π. (An element of length n is uniquely determined by either its initial or terminal factor (α or β). We filter C by

$$F^nC = \coprod_{|w| \leq n} C^w$$

and this clearly makes C a filtered ring, i.e. $F^nC \cdot F^mC \subset F^{n+m}C$. If $a \in C$ we shall write $h(a) = n$ if $a \in F^nC$ but $a \notin F^{n-1}C$ with the convention that $F^nC = \{0\}$ for $n < 0$, and hence that $h(0) = -\infty$. For example, $h(a) \leq 0 \iff a \in R$. If $h(a) = n$ we write

$$\overline{a} \in gr_n C = F^nC/F^{n-1}C,$$

and the $grC = \coprod_{n \geq 0} gr_nC$ is a graded R-algebra in the usual sense. Clearly $gr_0C = R$. If n > 0 then the projection $F^nC \longrightarrow gr_nC$ induces an isomorphism of R-modules, $C^u \oplus C^v \longrightarrow gr_nC$, where u and v are the two elements of length n in π. We shall often use this isomorphism to identify the two modules. If $\overline{a}$ lies in C^u or C^v, say in C^u, we shall say that a <u>is pure of type</u> u, and write $u = w(a)$. Thus a is pure of type u if $a \in F^{n-1}C \oplus C^u$ but $a \notin F^{n-1}C$.

(5.13) PROPOSITION. <u>Let</u> A′ <u>denote the</u> R-<u>algebra with the same underlying</u> R-<u>module and augmentation as</u> A, <u>but with multiplication defined by</u> $\overline{A}^2 = 0$. <u>Define</u> B′ <u>similarly. Then</u> grC = A′ * B′. <u>If</u> a, b $\in$ C <u>and if</u> $\overline{a}\,\overline{b} \neq 0$, <u>then</u> $\overline{ab} = \overline{a}\,\overline{b}$, and $h(ab) = h(a) + h(b)$.

<u>Proof</u>. If u, v $\in \pi$ then $C^uC^v \subset C^{uv}$. If u and v "interact", i.e. if the terminal factor of u coincides with

the initial factor of v, then $|uv| = |u| + |v| - 1$. If not then $|uv| = |u| + |v|$. Projecting C^u into $gr_{|u|}C$ and C^v into $gr_{|v|}C$, therefore we see that, in grC, $C^u C^v = 0$ if u and v interact, and otherwise $C^u \otimes C^v \longrightarrow C^{uv}$ in grC coincides with the corresponding multiplication in C. Thus we see from the construction above of C that grC is obtained by the same construction, but applied to A′ and B′ instead.

If $\bar{a}\,\bar{b} \neq 0$ then, by definition, $h(ab) = h(a) + h(b)$ and $\bar{a}\,\bar{b} = \overline{ab}$.

(5.14) PROPOSITION. *Let* R $\longrightarrow$ R′ *be a homomorphism of commutative rings. Then there is a natural isomorphism*

$$(A *_R B) \otimes_R R' = (A \otimes_R R') *_{R'} (B \otimes_R R').$$

I.e. base change preserves coproducts of augmented algebras.

Proof. This follows easily from the following adjointness property of base change: If D is an R′-algebra then $Hom_{R\text{-}alg}(A, D) = Hom_{R'\text{-}alg}(A \;\; _R R', D)$.

Now we come to the main result of this section.

(5.15) THEOREM. (P.M. Cohn) *Let* R *be a field and let* A *and* B *be augmented R-algebra which are* N-*firs. Then* C = A * B *is also an* N-*fir, and, for each* $n \leq N$, *the group* $GL'_n(C)$, *generated by* $GL_n(A)$, $GL_n(B)$, *and* $E_n(C)$, *is all of* $GL_n(C)$.

(5.16) COROLLARY. *If* A *and* B *above are generalized* N-*euclidean rings so also is* C.

(5.17) COROLLARY. *Let* G *be a free monoid or a free group, and let* A = R[G] *be the monoid algebra of* G *over a field* R. *Then* A *is a generalized euclidean ring*.

Proof. If X is a basis for G we can write X as a direct limit of finite subsets, and A is then a corresponding direct limit. We can use this device to reduce to the case when X is finite, and then argue by induction on card X. If X has one element then $A=R[t]$, or $R[t, t^{-1}]$, is generalized euclidean (see (5.1)). Otherwise $X = X_1 \cup X_2$ (disjoint), $G = G_1 * G_2$ (where X_i is the basis of G_i) and $A = R[G_1 * G_2] = R[G_1] * R[G_2]$. By induction $R[G_i]$ is generalized euclidean ($i = 1, 2$), so A is generalized euclidean thanks to (5.15).

The rest of this section is devoted to the proof of (5.15). We fix the notation and hypothese of (5.15). Further, we shall use the π-grading of C discussed above, where π is the monoid generated by α and β with relations $\alpha^2 = \alpha$ and $\beta^2 = \beta$.

Since $N \geq 1$ in (5.15) both A and B are integral domains.

(5.18) LEMMA. *If $a_1, \ldots, a_r, b$ are non zero elements of* A *such that $a_i b \in R$ $(1 \leq i \leq r)$ then there is a* $u \in U(A)$ *such that $a_i u$, $u^{-1} b \in R$ $(1 \leq i \leq r)$.*

Proof. Since A is an integral domain $a_1 b \neq 0$ is a unit, and hence a_1 and b are units, so we can take $u = a_1^{-1}$. By induction on r we can find v such that $a_i v$, $v^{-1} b \in R$ $(2 \leq i \leq r)$. Then $a_1 v = u^{-1} v \in R$ because $u^{-1} v = (u^{-1} b)(b^{-1} v) = (u^{-1} b)(v^{-1} b)^{-1} \in R$. q.e.d.

We shall call $c \in C$ *left* (resp., *right*) *reduced* if $h(c) \leq h(uc)$ (resp., $h(c) \leq h(cu)$) for all $u \in U(A) \cup U(B)$.

(5.19) PROPOSITION. *Let a and b be non zero elements of* C *such that either a is right reduced or* b *is left reduced. Assume that $\overline{a\,b} = 0$ in* grC. *Then a is pure of type* $w(a)$, b *is pure of type* $w(b)$, *and $a\,b$ is pure of type* $w(a)\,w(b)$. *In particular* $h(ab) = h(a) + h(b) - 1$.

Proof. Since $\overline{a\,b} = 0$ neither a nor b can be in R. Say

$r = h(a)$ and $s = h(b)$, and write $\overline{a} = \overline{a}_A + \overline{a}_B$ and $\overline{b} = {}_A\overline{b} + {}_B\overline{b}$. The summands here correspond to the decomposition $gr_r C = C^{u\alpha} \oplus C^{u\beta}$ and $gr_s C = C^{\alpha x} \oplus C^{\beta y}$. Then $\overline{a}\,\overline{b} = \overline{a}_A\,{}_A\overline{b} + \overline{a}_A\,{}_B\overline{b} + \overline{a}_B\,{}_A\overline{b} + \overline{a}_A\,{}_B\overline{b} = \overline{a}_A\,{}_B\overline{b} + \overline{a}_B\,{}_A\overline{b} \in C^{u\alpha\beta y} \oplus C^{v\beta\alpha x}$. Therefore $\overline{a}_A\,{}_B\overline{b} = \overline{a}_A \otimes {}_B\overline{b} = 0$, so $\overline{a}_A = 0$ or ${}_B\overline{b} = 0$. Similarly $\overline{a}_B = 0$ or ${}_A\overline{b} = 0$. It follows that a and b are pure of interacting types: either $\overline{a} = \overline{a}_A$ and $\overline{b} = {}_A\overline{b}$ or $\overline{a} = \overline{a}_B$ and $\overline{b} = {}_B\overline{b}$. Assume the former is the case, and say b is left reduced. (The other cases will follow by symmetry).

Write $a = a_{u\alpha} + a_u + \ldots$ and $b = {}_{\alpha x}b + {}_x b + \ldots$ in π-homogeneous coordinates. Then $ab = (a_{u\alpha} \cdot {}_x b + a_u \cdot {}_{\alpha x}b) + \ldots$ where all the undenoted terms lie in $F^{r+s-2}C$. Therefore the assertion that ab is pure of type $w(a)w(b) = u\alpha x$ is equivalent to the assertion that $h(ab) = r + s - 1$, and in any case it is $\leq r + s - 1$.

If $r = s = 1$ then $h(ab) = 1$ thanks to (5.18), because a or b is reduced. If $r > 1$ write $a \equiv \Sigma\, c_i\, d_i \bmod F^{r-1}C$, where (c_i) is an R-basis for C^u and $d_i \in A$. Then if $ab \in F^{r+s-2}C$ we conclude also that $h(\Sigma\, c_i\, d_i b) \leq r + s - 2$. Since $d_i b \in F^{s-1}C \oplus C^{\alpha x}$ and since the sum $\Sigma\, c_i\, C^{\alpha x}$ is direct it follows that $d_i b \in F^{s-1}C$ for each i. This is impossible if $d_i \neq 0$ and $d_i \in R$. Choosing an i for which $d_i \neq 0$ we can replace a by that d_i and reduce to the case $r = 1$.

Now apply the same reasoning to b, and we find that

$b \equiv \Sigma\, e_i f_i \bmod F^{s-1}C$ where (f_i) is an R-basis for C^x and $e_i \in A$. It follows as above that $ae_i \in F^{r-1}C = R$ for all i. According to (5.18) there is $au \in U(A)$ such that au^{-1}, $ue_i \in R$ for all i. Hence $ub \equiv \Sigma\, ue_i f_i \bmod F^{s-1}C$. But $ue_i f_i \in Rf_i \subset C^x \subset F^{s-1}C$ so $ub \in F^{s-1}C$, contradicting the assumption that b was reduced. q.e.d.

Remark. The proof of (5.19) shows that $\bar{a}\,\bar{b}$ depends only on a mod $F^{h(a)-2}C$ and on b mod $F^{h(b)-2}C$.

Proof of (5.15) when N = 1. We must show that C is an integral domain and that U(C) is generated by U(A) and U(B). Suppose a and b are not zero. Let u be a product of units in U(A) and U(B) so that au is right reduced: If $ab = 0$ then $au\, u^{-1}b = 0$ whereas (5.19) implies $h(ab) \geq h(u^{-1}b)$. Therefore C is an integral domain. If $a \in U(C)$ choose u as above. If $au \notin R$ then the equation $(au)\,(au)^{-1} = 1$ contradicts (5.19) again. Therefore $au \in U(R) \subset U(A)$ so a is in the subgroup generated by U(A) and U(B). q.e.d.

(5.20) PROPOSITION. *Suppose* $c_1,\ldots,c_n \in C$ $(n \leq N)$ *are such that there is a relation* $\Sigma\, c_i d_i \in F^{s-1}C$ *with* $h(c_i d_i) = s$ $(i \leq i \leq n)$. *Then there is a* $\sigma \in GL_n'(C)$ *such that* $h(\gamma\sigma) < h(\gamma)$, *where* $\gamma = (c_1,\ldots,c_n)$ *and we write* $h(\gamma) = \Sigma\, h(c_i)$ *and similarly for* $h(\gamma\sigma)$.

Proof that (5.20) ⟹ (5.15). We prove (5.15) by induction on N, the case N = 1 being accounted for above. According to (5.5) it suffices to show that, if $c_1,\ldots,c_n$ are right linearly dependent, we can find $\sigma \in GL_n'(C)$ such that $\gamma\sigma$ has a zero coordinate. Choose σ so that $h(\gamma\sigma)$ is minimal, and let $\gamma' = \gamma\sigma = (c_1',\ldots,c_n')$. If no c_i' is zero let $\Sigma\, c_i' d_i = 0$ be a dependence relation. Say $h(c_i' d_i) = s$

for $i \leq m$ and $h(c_i' \, d_i) < s$ for $i > m$ after relabeling. Then by (5.20) there is a $\sigma' \in GL_m'(C)$ such that $h(c_1', \ldots c_m')\sigma') < h(c_1', \ldots, c_m')$. Putting $\sigma'' = \begin{pmatrix} \sigma' & 0 \\ 0 & I \end{pmatrix}$ we have $h(\gamma' \, \sigma'') < h(\gamma')$, contradicting minimality. q.e.d.

Proof of (5.20). We shall argue by induction on n, the case $n = 1$ being vacuous. This permits us to assume that no proper subset of $c_1, \ldots, c_n$ satisfies a relation of the type given. We can also assume that all the c_i are right reduced.

From the fact that $h(c_i \, d_i) = s$ we conclude using (5.13) and (5.19), that $h(c_i) \leq s$ for each i. We shall assume the c_i's listed so that $h(c_1) \geq h(c_2) \geq \ldots \geq h(c_n)$.

Case 1. In grC, $\bar{c}_i \, \bar{d}_i \neq 0$ for each i.

We claim $\bar{c}_1$ is a right linear combination of $\bar{c}_2, \ldots, \bar{c}_n$. Lifting such an expression to C it will follow that we can subtract a linear combination of $c_2, \ldots, c_n$ from c_1 and lower $h(c_1)$.

If some $\bar{d}_i \in R$ then we must have $d_1 \in R$, and the conclusion is clear. If each $\bar{d}_i$ has positive degree we can write it as $\bar{d}_i = d_{iA} + d_{iB}$ in π-homogeneous coordinates, where the terminal factor of the π-degree of d_{iA} is α, and that of d_{iB} is β. Then the two sums in $\Sigma \, \bar{c}_i \, \bar{d}_{iA} + \Sigma \, \bar{c}_i \, \bar{d}_{iB} = 0$ are independent, so each separately equals zero. Either d_{1_A} or d_{1_B} is not zero, say $d_{1_A} \neq 0$. Write $d_{iA} = \Sigma_j \, e_{ij} \, a_j$ where e_{ij} has lower degree and where (a_j) is an R-basis for $\bar{A}$. The terms of the sum $\Sigma_j(\Sigma_i \, \bar{c}_i \, e_{ij})a_j = 0$ are independent, so we have $\Sigma_i \, \bar{c}_i \, e_{ij} = 0$ for each j. The degrees of the coefficients have been reduced so our conclusion follows by applying induction (on the minimum of the degrees of the coefficients) to an equation for which $e_{1j} \neq 0$.

Assume $h(c_i) = r$ for $i \leq m$ and $h(c_i) < r$ for $i > m$.

Case 2. In grC, $\bar{c}_i \bar{d}_i \neq 0$ for some $i \leq m$.

We can assume $\bar{c}_1 \bar{d}_1 \neq 0$, and again we will show that $h(c_1)$ can be reduced by subtracting a linear combination of $c_2,\ldots,c_n$.

Suppose $i > 1$ and $\bar{c}_i \bar{d}_i = 0$. Then c_i and d_i are pure, say of types $w(c_i) = u\alpha$ and $w(d_i) = \alpha v$ (see (5.19)). Then we can write $d_i = \Sigma x_{ij} e_{ij}$ so that $x_{ij} \in A$, $h(e_{ij}) < h(d_i)$, and either $\overline{c_i x_{ij}} \bar{e}_{ij} \neq 0$ or else $h(c_i x_{ij} e_{ij}) < s$. In the congruence $\Sigma c_i d_i \equiv 0 \bmod F^{s-1}C$ leave the terms for which $\bar{c}_i \bar{d}_i \neq 0$ unchanged, and replace the others by the expressions $\Sigma_j' c_i x_{ij} e_{ij}$, where Σ_j' means summation over only those terms such that $h(c_i x_{ij} e_{ij}) = s$. Then by case 1 we can reduce $h(c_1)$ by subtracting a (right) linear combination of the c_i for which $\bar{c}_i \bar{d}_i \neq 0$ together with the $c_i x_{ij}$ for the other $i > 1$. Altogether the latter is a right linear combination of $c_2,\ldots,c_n$, so case 2 is established.

Case 3. We have $\bar{c}_i \bar{d}_i = 0$ $(1 \leq i \leq m)$.

The remark after the proof of (5.19) shows that $\overline{c_i d_i}$ depends on d_i only modulo $F^{h(d_i)-2}$. If $h(d_i) > 1$ for all i, therefore, we can modify the d_i's to have no constant term and then write $d_i = d_{iA} + d_{iB}$ in $C\bar{A} \oplus C\bar{B}$. Then the congruence $\Sigma c_i d_i \equiv 0 \bmod F^{s-1}C$ breaks up into two congruences permitting us to assume, say, that each $d_i \in C\bar{A}$. Then we can write $d_i = \Sigma d_{ij} a_j$ where (a_j) is an R-basis for $\bar{A}$ and $h(d_{ij}) < h(d_i)$. Then we conclude that $\Sigma_i c_i d_{ij} \equiv 0 \bmod F^{s-2}C$ for each j, and at least one $c_i\, d_{ij}$ has height $\geq s - 1$. By induction on s, therefore, we can assume that $h(d_i) \leq 1$ for some i. If some $d_i \in R$ then that c_i is a linear combination of the remaining c_j. Hence we can assume that $h(d_i) = 1$ $(1 \leq i \leq m)$, and $h(d_i) > 1$ for $i > m$.

It follows from (5.19) that $c_i\, d_i$ is pure for $1 \leq i \leq m$. If we segregate the terms with terminal factor α and β, respectively, for their types, and then write $d_i = d_{iA} + d_{iB}$ for $i > m$ (as above), we can obtain two separate congruences from $\Sigma\, c_i\, d_i \equiv 0 \bmod F^{s-1}C$. If both types occur among $c_1\, d_1, \ldots, c_m\, d_m$ then the two resulting congruences will each involve fewer than n terms, and the proof concludes by induction on n.

Therefore we may assume $c_1\, d_1, \ldots, c_m\, d_m$ are all pure of the same type. It follows that $d_1, \ldots, d_m$ all lie in either A or B, say in A. Moreover each c_i is pure of type $w(c_i) = u\alpha$ for some u.

Let J denote the R-module generated by elements of the form $c_i d$ such that $i > m$ and $c_i d \in F^{s-1}C + C^{u\alpha}$. We claim that $J + F^{s-1}C = V\bar{A} + F^{s-1}C$ where V is an R-submodule of C^u. For let V be the largest R-submodule of C^u such that $V\bar{A} \subset J + F^{s-1}C$. To show that every $c_i d$ as above lies in $V\bar{A} + F^{s-1}C$ we can of course assume $h(c_i d) = s$.

If $\overline{c_i d} \neq 0$ then we can modify d, without changing $\overline{c_i d} = \bar{c}_i \bar{d}$, so that $d \in C\bar{A}$. Then we can write $d = \Sigma\, e_j\, a_j$ where $h(e_j) < h(d)$ and (a_j) is an R-basis for $\bar{A}$. From $c_i d = \Sigma_j\, c_i\, e_j\, a_j$ we conclude that $\overline{c_i d} = \Sigma_j\, \overline{c_i\, e_j}\, \bar{a}_j$. Therefore, $c_i\, e_j \in V + F^{s-2}C$ for each j, so $c_i d \in V\bar{A} + F^{s-1}C$.

Next suppose $\bar{c}_i\bar{d} = 0$. Since c_i is reduced it follows from (5.19) that $s = h(c_i d) = h(c_i) + h(d) - 1 \leq s - 1 + h(d) - 1$, so we have $h(d) \geq 2$. Further, since $c_i d$ is pure of type $u\alpha$ it follows that d is pure of type $w(d) = v\alpha$ for some v. Without changing $c_i d$ mod $F^{s-1}C$ we can further assume that $d = {}_{v\alpha}d + {}_{v'\alpha}d$ where v' is v with its initial factor removed. Therefore $c_i d \in c_i C^v\bar{A} + c_i C^{v'}\bar{A}$, and the modules $c_i C^v$ and $c_i C^{v'}$ project, modulo $F^{s-2}C$, into $V \subset C^u$. Therefore again $c_i d \in V\bar{A} + F^{s-1}C$.

Now we return to our congruence (*) $(c_1 d_1 + \ldots + c_m d_m) + (c_{m+1} d_{m+1} + \ldots + c_n d_n) \equiv 0 \bmod F^{s-1}C$. The second term in parenthesis lies in $F^{s-1}C \oplus V\bar{A}$, as we have just proved, and $c_1, \ldots, c_m \in F^{s-1}C \oplus C^u\bar{A}$. Passing to

$$(F^{s-1}C \oplus C^u\bar{A}) \;/\; (F^{s-1}C \oplus V\bar{A}) \simeq (C^u/V) \otimes \bar{A}$$

the congruence (*) becomes a linear dependence relation over A between the images of $c_1, \ldots, c_m$. Since $(C^u/V) \otimes \bar{A}$ is a submodule of the free A-module $(C^u/V) \otimes A$, it follows from the fact that A is an n-fir that there is a $\sigma \in GL_m(A)$ such that, if $(c_1, \ldots, c_m)\sigma = (c_1', \ldots, c_m')$, we have $c_m' \in F^{s-1}C \oplus V\bar{A}$. Hence, modulo $F^{s-1}C$, c_m' is a linear combination of $c_{m+1}, \ldots, c_n$, so we can reduce $h(c_m')$ to something $\leq s - 1$. since we still have $h(c_i') \leq s$ $(1 \leq i \leq m)$ we have succeeded in reducing $\Sigma\, h(c_i)$, as claimed. q.e.d.

§6. SESHADRI'S THEOREM

It states, under suitable hypotheses, the following: Let R be a commutative ring, let S be a multiplicative set in R, let A be an R-algebra, and let $P \in \underline{\underline{P}}(A)$. Then if $S^{-1}P$ is a free S^{-1} A-module, P is a free A-module.

In the original version of Seshadri, R was a principal ideal domain, S was R-{0}, and A was R[t], a polynomial ring in one variable. In this case $S^{-1}P$ is automatically free, clearly, so he deduced that all $P \in \underline{\underline{P}}(R[t])$ are free. Seshadri's argument applies to somewhat more general situations, as many authors have observed, and we shall present such a generalization. While the hypotheses are necessarily quite restrictive, they allow certain non commutative R-algebras A. Moreover it is useful to further allow a more general type of multiplicative set than heretofore considered, and we begin by taking up this point.

Let R be a commutative ring, and let S be a multiplicative set of invertible ideal in R. We propose to construct a localization functor $M \longrightarrow S^{-1}M$ from mod-R to mod-$S^{-1}R$ with all the properties of ordinary localization, with which it coincides when the ideals in S are principal. Let L be the full ring of fractions of R. Then if $\underline{a} \in S$ we have $\underline{a}^{-1} \subset L$, and

$$S^{-1}R = \bigcup \underline{a}^{-1} \qquad (\underline{a} \in S)$$

is clearly a subring of L. It is more convenient to write

$$S^{-1}R = \varinjlim \underline{a}^{-1} \qquad (\underline{a} \in S),$$

the maps in the direct system being the inclusions. If $M \in$ mod-R then we set

$$S^{-1}M = \varinjlim (M \otimes_R \underline{a}^{-1})$$

$$= M \otimes_R (\varinjlim \underline{a}^{-1})$$

$$= M \otimes_R S^{-1} R.$$

We used here the fact that $\otimes_R$ and $\varinjlim$ commute. Since $\otimes_R \underline{a}^{-1}$

is exact ($\underline{a}^{-1}$ is projective) and since $\underrightarrow{\lim}$ is exact, we see that S^{-1} is an exact functor mod-R $\longrightarrow$ mod - $S^{-1}R$.

There is a natural homomorphsim

$$M = M \otimes_R R \longrightarrow S^{-1}M,$$

and (see (I, 8.2)) its kernel is the union of the $\operatorname{Ker}(M \otimes_R R \longrightarrow M \otimes_R \underline{a}^{-1})$ ($\underline{a} \in S$). We claim the latter is just $\operatorname{ann}_M(\underline{a}) = \{x \in M \mid x\,\underline{a} = 0\}$. Since $\underline{a}$ is finitely generated (being invertible) it suffices to check this locally. Therefore we may assume $\underline{a} = \alpha R$ is principal. But then the homomorphisms $M \otimes_R R \longrightarrow M \otimes_R \alpha^{-1}R$ and $M \xrightarrow{\alpha} M$ are isomorphic, and the latter clearly has kernel $\operatorname{ann}_M(\alpha)$.

Let A be an R-algebra. Then evidently $S^{-1}A$ is an $S^{-1}R$-algebra, and S^{-1} induces an exact functor

$$S^{-1}\colon \text{mod-}A \longrightarrow \text{mod - } S^{-1}A.$$

We claim:

If $M \in \underline{\underline{M}}(A)$ then $S^{-1}M = 0 \iff M\underline{a} = 0$ for some $\underline{a} \in S$. The implication $\Leftarrow$ is clear. For the converse we apply the conclusion of the last paragraph to each of a finite set of A-generators of M, and let $\underline{a}$ be the product of the annihilating ideals so obtained.

We have the natural homomorphism of $S^{-1}R$-modules

$$h_P\colon S^{-1}\operatorname{Hom}_A(P, M) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}P, S^{-1}M)$$

for P, M $\in$ mod-A. Evidently h_A is an isomorphism, so it follows from additivity that h_P is also for all $P \in \underline{\underline{P}}(A)$. Using half exactness and the 5-lemma it now follows by a standard argument that h_P is an isomorphism whenever P is a finitely presented A-module.

(6.1) THEOREM (Seshadri,...). *Let* R *be a commutative ring, and let* S *be the multiplicative monoid generated by a set* S_o *of invertible prime ideals in* R. *Let* A *be an* R-*algebra which is faithful and flat as an* R-*module and such that for each* $\underline{p} \in S_o$ *and* $\underline{a} \in S$, $A/A\underline{p}$ *is generalized* n-*euclidean ring* (see (5.4)) *and* $A\underline{a}/A\underline{a}\underline{p} \simeq A/A\underline{p}$. Let $P, L_1,\ldots,L_n \in \underline{\underline{P}}(A)$ *be such that* $L_i/L_i\underline{p} \simeq A/A\underline{p}$ $(1 \leq i \leq n)$ *for each* $\underline{p} \in S_o$, *and such that* $S^{-1}P \simeq S^{-1}L$, *where* $L = L_1 \oplus \ldots \oplus L_n$. *Then there is an* $\underline{a}$ *in the group* $\overline{S}$ *generated by* S *such that*

(1) $$P \simeq L_1\underline{a} \oplus L_2 \oplus \ldots \oplus L_n.$$

Moreover, if $\underline{a}_1,\ldots,\underline{a}_n \in \overline{S}$ *are such that* $\underline{a}_1\ldots\underline{a}_n \simeq R$, *then*

(2) $$L \simeq L_1\underline{a}_1 \oplus L_2\underline{a}_2 \oplus \ldots \oplus L_n\underline{a}_n.$$

(6.2) COROLLARY. *Suppose above that* $A/A\underline{p}$ *is generalized euclidean for each* $\underline{p} \in S_o$ *and that every module in* $\underline{\underline{P}}(S^{-1}A)$ *is* $S^{-1}A$-*free. Then if* $P \neq 0$ *and* $P \in \underline{\underline{P}}(A)$ *we have* $P \simeq A\underline{a} \oplus A^{n-1}$ *for some* $\underline{a} \in \overline{S}$ *and some* $n > 0$.

Proof. By hypothesis $S^{-1}P \simeq S^{-1}(A^n)$ for some $n > 0$, so we can take each $L_i = A$ above. q.e.d.

(6.3) EXAMPLES. Suppose A and B are augmented R-algebras for which the hypotheses of (6.1), vis-a-vis S_o and S, hold. Then they hold also for $A *_R B$; this follows from (5.15).

Let R be a Dedekind ring and let $A = R\pi$, where π is a free monoid or group. Then it follows from (5.17) that $A/A\underline{p}$ is generalized euclidean for all $\underline{p} \in \max(R)$. Moreover the same is true of $S^{-1}A$, where $S^{-1}R$ is the field of fractions of R. Thus we can apply (6.2).

(6.4) COROLLARY. Let R be a Dedekind ring and let π be a free monoid or free group. If $P \in \underline{\underline{P}}(R\pi)$, $P \neq 0$, then $P \simeq (R\pi \otimes_R L) \oplus (R\pi)^{n-1}$ for some $L \in \underline{\underline{Pic}}(R)$ and some $n > 0$.

Proof of (6.1). We shall carry out the proof in several steps.

(i) If $H \in \underline{\underline{P}}(A)$, $\underline{a} \in \overline{S}$, and $\underline{p} \in S_0$, then $H\underline{a} \simeq H \otimes_R \underline{a}$, and $H\underline{a}/H\underline{ap} \simeq H/H\underline{p}$. (We regard $H\underline{a} \subset S^{-1}H$).

Since A is R-flat so also is $H(\in \underline{\underline{P}}(A))$. Therefore $H \otimes_R$ preserves the exactness of $0 \longrightarrow \underline{a} \longrightarrow S^{-1}R$, so $H \otimes_R \underline{a}$ is thus identified with $H\underline{a} \subset H \otimes_R S^{-1}R = S^{-1}H$. (In case $\underline{a} = R$ we see that H is embedded in $S^{-1}H$).

Suppose $\underline{a} \in S$. Then $H\underline{a}/H\underline{ap} \simeq H \otimes_R (\underline{a}/\underline{ap}) \simeq (H \otimes_A A) \otimes_R (\underline{a}/\underline{ap}) \simeq H \otimes_A (A\underline{a}/A\underline{ap}) \simeq H \otimes_A (A/A\underline{p})$ (by hypothesis) $\simeq H/H\underline{p}$. In general we can write $\underline{a} = \underline{b}\,\underline{c}^{-1}$ with $\underline{b}, \underline{c} \in S$. Then $H\underline{a}/H\underline{ap} \simeq H\underline{c}^{-1}/H\underline{c}^{-1}\underline{p} \simeq H/H\underline{p}$, applying the special case above to ($H\underline{c}^{-1}$ and $\underline{b}$) and to ($H\underline{c}^{-1}$ and $\underline{c}$), respectively. This proves (i).

A splitting of an $H \in \underline{\underline{P}}(A)$ will mean a direct sum decomposition $H = H_1 \oplus \ldots \oplus H_n$ such that $H_i/H_i\underline{p} \simeq A/A\underline{p}$ for all $\underline{p} \in S_0$ $(1 \leq i \leq n)$.

(ii) Let $H = H_1 \oplus \ldots \oplus H_n$ be a split submodule of $Q \in \underline{\underline{P}}(A)$, and assume $Q\underline{bp} \subset H$ for some $\underline{p} \in S_0$ and $\underline{b} \in S$. Then $(H \cap Q\underline{p})/H\underline{p} \simeq (A/A\underline{p})^r$ for some r $(0 \leq r \leq n)$, and there is a module, $\overline{H}$, such that $Q\underline{b} \subset \overline{H} \subset Q$, and with a splitting $\overline{H} \simeq H_1\underline{p}^{-r} \oplus H_2 \oplus \ldots \oplus H_n$.

Consider the exact sequence of $(A/A\underline{p})$-modules

$$0 \longrightarrow (H \cap Q\underline{p})/H\underline{p} \longrightarrow H/H\underline{p} \xrightarrow{j} Q/Q\underline{p}.$$

We have $Q/Q\underline{p} \in \underline{\underline{P}}(A/A\underline{p})$, and $H/H\underline{p} \simeq (A/A\underline{p})^n$. Thus Im(j) is a submodule with $\leq n$ generators of a projective $(A/A\underline{p})$-module, so (5.2) implies that Im(j) is $(A/A\underline{p})$-free. In particular $H/H\underline{p} \simeq \mathrm{Im}(j) \oplus \mathrm{Ker}(j)$, so Ker(j) also has $\leq n$ generators and is therefore also free; say $\mathrm{Ker}(j) \simeq (A/A\underline{p})^r$ and $\mathrm{Im}(j) \simeq (A/A\underline{p})^s$. Then (5.1) (α_n) implies $r + s = n$.

Write $M' = M/M\underline{p}$ for $M \in$ mod-A. Then there is an $\alpha' \in \mathrm{Aut}_{A'}(H')$ such that $\alpha'(H_1' \oplus \ldots \oplus H_r') = \mathrm{Ker}(j)$, and hence $\alpha'(H_{r+1}' \oplus \ldots \oplus H_n')$ is mapped injectively by j. Now $\mathrm{Aut}_{A'}(H') \simeq GL_n(A') = GE_n(A')$, the latter equality because A' is generalized n-euclidean. Therefore we can write $\alpha' = \varepsilon'\delta'$ where $\varepsilon' \in E(H_1',\ldots,H_n')$ and δ' is represented by a diagonal matrix with respect to a basis consisting of elements in the various H_i. Since $\delta'(H_i') = H_i'$ for each i it follows that $\mathrm{Ker}(j) = \varepsilon'(H_1' \oplus \ldots \oplus H_r')$ also. According to (3.3) there is an $\varepsilon \in E(H_1,\ldots,H_n)$ which reduces modulo $\underline{p}$ to ε'.

Let $G = \varepsilon(H_1 \oplus \ldots \oplus H_r)$, so that $G + H\underline{p} = H \cap Q\underline{p}$. Put $\overline{H} = G\underline{p}^{-1} \oplus \varepsilon(H_{r+1} \oplus \ldots \oplus H_n) = G\underline{p}^{-1} + H = (H \cap Q\underline{p})\underline{p}^{-1}$. Since $Qb\underline{p} \subset H \cap Q\underline{p}$ we have $Q\underline{b} \subset (H \cap Q\underline{p})\underline{p}^{-1} = \overline{H}$. There now remains only to be shown that

$$(*) \qquad \overline{H} \simeq H_1\underline{p}^{-r} \oplus H_2 \oplus \ldots \oplus H_n.$$

We have $\overline{H}\underline{p} \subset H$ and $\overline{H}\underline{p}/H\underline{p} \simeq (A/A\underline{p})^r$ for some $r (0 \leq r \leq n)$. Under these conditions we will show, by induction on r, that (*) holds. If $r = 0$ then $\overline{H}\underline{p} = H\underline{p}$ so $\overline{H} = H$. Assume now that $r > 0$. Choose ε as above and put K =

$\varepsilon(H_1)\underline{p}^{-1} \oplus \varepsilon(H_2 \oplus \ldots \oplus H_n)$. Then, using part (i) of the proof we see that this is a splitting of $K \subset \overline{H}$, and clearly $H\underline{p}/K\underline{p} \simeq (A/A\underline{p})^{r-1}$. By induction, therefore, $\overline{H} \simeq \varepsilon(H_1)\underline{p}^{-r} \oplus \varepsilon(H_2 \oplus \ldots \oplus H_n) \simeq H_1\underline{p}^{-r} \oplus H_2 \oplus \ldots \oplus H_n$.

(iii) The isomorphism (2) holds.

Since $\overline{S}$ is a free abelian group with basis S_0 it will suffice to show that, if $\underline{p} \in S_0$, and if $i \neq j$, then $L \simeq L_i\underline{p}^{-1} \oplus L_j\underline{p} \oplus \coprod_{k\neq i,j} L_k$. For, according to (i), the right side of this is a new splitting of L, and the isomorphism (2) can then be realized as the composite of a finite sequence of isomorphisms of the above type.

To prove the isomorphism above there is no loss in assuming $(i, j) = (1, 2)$, just to simplify writing. Let $H = L_1 \oplus L_2\underline{p} \oplus L_3 \oplus \ldots \oplus L_n \subset L$. Then $L\underline{p} \subset H$ and $L\underline{p}/H\underline{p} \simeq L_2\underline{p}/L_2\underline{p}^2 \simeq L_2/L_2\underline{p} \simeq A/A\underline{p}$, using (i). Thanks to (ii) now we conclude that $L \simeq L_1\underline{p}^{-1} \oplus L_2\underline{p} \oplus L_3 \oplus \ldots \oplus L_n$. q.e.d.

(iv) The isomorphism (1) holds.

By hypothesis we can identify $S^{-1}P$ with $S^{-1}L$. Every element of $S^{-1}P/P$ is annihilated by some element of S. If we apply this to the images in $S^{-1}\underline{p}/\underline{p}$ of a finite generating set of L we obtain an $\underline{a} \in S$ such that $L\underline{a} \subset P$. Put $H = L\underline{a} = L_1\underline{a} \oplus \ldots \oplus L_n\underline{a}$. There is also $\alpha\, \underline{c} \in S$ such that $P\underline{c} \subset H$. It follows from (iii) that $H = H_1 \oplus H_2 \oplus \ldots \oplus H_n$ where $H_i \simeq L_i$ $(1 < i \leq n)$ and $H_1 \simeq L_1\underline{a}^n$. It will therefore suffice to show that $P \simeq H_1\underline{a}' \oplus H_2 \oplus \ldots \oplus H_n$ for some $\underline{a}' \in \overline{S}$, and we shall do this now by induction on the number of prime factors (in S_0) of of $\underline{c}$. If $\underline{c} = R$ then $H = \underline{p}$ and there is nothing to prove. Otherwise we can write $\underline{c} = \underline{pb}$ with $\underline{p} \in S_o$ and $\underline{b} \in S$. We can apply (iii) now to find $\overline{H} \simeq H_1\underline{p}^{-r} \oplus H_2 \oplus \cdots \oplus H_n$ for some r, and such that $P\underline{b} \subset \overline{H} \subset P$. Since $\underline{b}$ has fewer prime factors than $\underline{c}$ the desired isomorphism follows by induction.

This concludes the proof of Theorem (6.1).

We shall close this section by outlining the proof of some further applications of Seshadri's Theorem.

(6.5) COROLLARY. *Let* R *be a commutative noetherian ring of dimension* ≤ 1 *having only finitely many non invertible maximal ideals. Let* $A = R[T]$ *where* T *is a free group or monoid on one generator* t. *Then if* $P \in \underline{\underline{P}}(A)$ *has constant rank* > 0, *it is the direct sum of an invertible module and of a free module*.

Proof. Let $\underline{a}$ be the product of all non-invertible maximal ideals in R, and put $S' = 1 + \underline{a}$. If $\underline{p} \in \max(R)$ and $s \in \underline{p} \cap S'$ then $\underline{p}$ is invertible, so $R_{\underline{p}}$ is a discrete valuation ring. Assuming that spec(R) is connected (which is no essential restriction for the problem at hand) it can be shown that s is not a zero divisor. This derives simply from the fact that $R_{\underline{p}}$ is an integral domain whenever $s \in \underline{p}$.

Let S_o be the set of primes in A generated by the maximal ideals of R which meet S'. Then clearly S_o satisfies the conditions in (6.1), and the ring $S^{-1}A$ in (6.1) coincides with $S'^{-1}A$. If $\underline{m} \in \max(A)$ meets S' then $A_{\underline{m}}$ is a localization of $R_{\underline{p}}[T]$, $\underline{p} = \underline{m} \cap R$. Since $R_{\underline{p}}$ is a DVR it follows that $R_{\underline{p}}[T]$ is a unique factorization domain, so that S' is factorial for A. Hence it follows from (III, 7.17) that

$Pic(A) \longrightarrow Pic(S'^{-1}A)$ is surjective.

With the aid of (6.1), therefore, the corollary will follow once we establish that the conclusion of the corollary is valid for $S^{-1}A$ in place of A. In turn, the latter will follow from Serre's Theorem (see (2.6) and (2.7)) if we can show that $\max(S^{-1}A)$ is a finite union of subspaces of dimension ≤ 1.

Now $S'^{-1}A = R'[T]$ where $R' = S'^{-1}R$ is a semi-local ring of dimension ≤ 1. If $\dim R' = 0$ then $\dim \max(R'[T]) = 1$. On the other hand, if $\dim R' = 1$ then $\dim (R'/\text{rad } R') < \dim (R')$ so it follows from (III, 3.13) that $\max(R'[T])$ is a union of two subspaces of dimension ≤ 1. q.e.d.

(6.6) COROLLARY. Let π be an abelian group of rank one and let $A = \mathbb{Z}\pi$. Then the conclusion of (6.5) is valid for A.

Proof. By a direct limit argument we can reduce to the case when π is finitely generated. Then $\pi = \pi_0 \times T$ where π_0 is finite and T is infinite cyclic. Putting $R = \mathbb{Z}\pi_0$ we have $A = R[T]$, and the hypotheses of (6.5) apply to $R = \mathbb{Z}\pi_0$. (Every maximal ideal of R not containing the "conductor" (see XI, §6)) from $\mathbb{Z}\pi_0$ to its integral closure in $\mathbb{Q}\pi_0$ is invertible).

A refinement of the above methods, due to Endô[1], can be used to prove an analogue of (6.5) for a free abelian group or monoid on two generators. In this case, however, R must be assumed to the semi-local of dimension ≤ 1. The idea is to show that $A = R[T]$ has a "large" set S_0 of invertible primes of the type occurring in (6.1), and then to show, as above, that $\max(S^{-1}A)$ is a union of subspaces of dimension

≤ 1. A broad generalization of Seshadri's theorem has recently been obtained by Murthy [1]. He extends the theorem to the coordinate ring of any affine surface (over an algebraically closed field) which is birationally equivalent to a ruled surface.

HISTORICAL REMARKS

The treatment of the stability theorems here follows closely that of Bass [1]. There are, however, a number of technical improvements of the results as presented in that reference.

The material of §5 is taken from papers of Cohn, especially Cohn [1]. I have used a description of free products which is due to Stallings [1].

Seshadri's Theorem has precedents in a long series of papers by various authors. The exposition here is taken from Bass-Murthy [1] and Bass [2]. The first of these references contains a more extensive bibliography. In particular, as is pointed out there, Endô[1] has contributed greatly to the present form of the theorem.

Chapter V
THE STABLE STRUCTURE OF GL_n

This chapter treats, essentially, the problem of classifying all normal subgroups of $GL_n(A)$, where A is any ring. The theory is satisfactory for "sufficiently large" n. Indeed, if we pass to $GL(A) = \varinjlim_{n\to\infty} GL_n(A)$ (see §1) then one can give a first order solution which is valid for arbitrary A: The normal subgroups are each sandwiched between two groups of the form $E(A, \underline{q}) \subset GL(A, \underline{q})$, for some two sided ideal $\underline{q}$. Here $GL(A, \underline{q}) = \text{Ker}(GL(A) \longrightarrow GL(A/\underline{q}))$ the "congruence group of level $\underline{q}$", and $E(A, \underline{q})$ is the normal subgroup generated by all "$\underline{q}$-elementary" matrices. Moreover we have the commutator formula, $[GL(A), GL(A, \underline{q})] = E(A, \underline{q})$, so that the classification of normal subgroups of GL(A) is reduced to the calculation of the <u>abelian groups</u>

$$K_1(A, \underline{q}) = GL(A, \underline{q})/E(A, \underline{q}).$$

The main results of this chapter (see §4) give conditions, of the type occurring in Chapter IV, for results like those above to hold in GL_n, for finite n. For example, let A be a finite algebra over a commutative noetherian ring of dimension d. Then if $n \geq d + 3$ one can largely reduce the classification of normal subgroups of $GL_n(A)$ to the calculation of certain abelian groups, $GL_n(A, \underline{q})/E_n(A, \underline{q})$, and the latter map isomorphically onto $K_1(A, \underline{q})$.

The proofs of these results are quite long and technical. The exposition is based partly on that of Bass [1], but mainly on Chapter II of Bass-Milnor-Serre [1]. In the latter reference that stability theorem is used to solve the "congruence subgroup problem" for the special linear group over a ring of algebraic integers. This type of application will be discussed below, in Chapter VI, in a rather general setting.

In the basic theorems here we allow A to be non commutative. (This was not the case in Bass-Milnor-Serre). For this reason the results are not completely trivial even in "dimension zero", i.e. when A is a semi-local ring. For example, the theory here touches upon Dieudonné's theory of non commutative determinants, when A is a division ring (cf. §9), and upon work of Klingenberg [1], when A is local.

The groups $K_1(A, \underline{q})$ introduced above will appear in later chapters in a slightly different guise. The methods developed in those chapters will permit us to compute these groups in many interesting cases (cf., for example, the last sections of Chapter XII).

§1. ELEMENTARY MATRICES AND CONGRUENCE SUBGROUPS

Let A be a ring. Then $GL_n(A) = Aut_A(A^n)$ where $A^n = A \oplus \cdots \oplus A$ is the standard free right A-module. We shall write

$$E_n(A) = E(A,\ldots,A),$$

where the notation is that of (IV, §3). If, as we shall do freely, we identify endomorphisms of A^n with the corresponding matrices, then $E_n(A)$ is generated by <u>elementary matrices</u>, $\varepsilon = I_n + ae_{ij}$ ($a \in A$, $i \neq j$), where e_{ij} denotes the matrix with 1 in position (i, j) and zeros elsewhere. We shall say ε is <u>$\underline{q}$-elementary</u>, where $\underline{q}$ is a two sided ideal in A, if $a \in \underline{q}$. The group

$$E_n(A, \underline{q}) = E(A,\ldots,A; \underline{q})$$

(again in the notation of (IV, §3)) is thus the <u>normal</u> subgroup of $E_n(A)$ generated by all $\underline{q}$-elementary matrices.

As a special case of (IV, 3.3) we have.

(1.1) PROPOSITION. _Let q be a two sided ideal in_ A. _and let_ $f: A \longrightarrow B$ _be a surjective ring homomorphism. Then_ f _induces an epimorphsim_ $E_n(A, q) \longrightarrow E_n(B, f(q)$ _for all_ $n \geq 1$.

(1.2) PROPOSITION. _Let_ A _be a ring, let_ a, $b \in A$, _and let_ $u, v \in U(A)$. _Then we have the following formulas in_ $GL_n(A)$.

(a) _If_ $i \neq j$ _then_ $A \longrightarrow GL_n(A)$, $a \longmapsto I + ae_{ij}$, _is a monomorphism of groups_.

(b) _If_ i, j, _and_ k _are distinct (so_ n _must be_ ≥ 3) _then_

$$[I + ae_{ij}, I + be_{jk}] = I + ab\, e_{ik}.$$

(c) $$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{\begin{pmatrix} u & b \\ 0 & v \end{pmatrix}} = \begin{pmatrix} 1 & u^{-1}av \\ 0 & 1 \end{pmatrix},$$

and

$$\left[\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u & b \\ 0 & v \end{pmatrix}\right] = \begin{pmatrix} 1 & u^{-1}av - a \\ 0 & 1 \end{pmatrix}.$$

Proof. Recall that $e_{ij}\, e_{kh} = \delta_{jk}\, e_{ih}$. Hence $e_{ij}^2 = 0$ if $i \neq j$ and (a) follows from this.

(b) $$\begin{aligned}[I + ae_{ij}, I + be_{jk}] &= (I - ae_{ij} - be_{jk} + ab\, e_{ik})(I + ae_{ij} + be_{jk} + ab\, e_{ik}) \\ &= (I - ae_{ij} - be_{jk} + ab\, e_{ik}) + (ae_{ij}) + (be_{jk} - ab\, e_{ik}) + (ab\, e_{ik}) = I + ab\, e_{ik}.\end{aligned}$$

(c) $\begin{pmatrix} u & b \\ 0 & v \end{pmatrix} = \begin{pmatrix} 1 & bv^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, and the left factor commutes with $I + a e_{12}$. Hence

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{\begin{pmatrix} u & b \\ 0 & v \end{pmatrix}} = \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 1 & u^{-1}av \\ 0 & 1 \end{pmatrix},$$

and $$\left[\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u & b \\ 0 & v \end{pmatrix}\right] = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u^{-1}av \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u^{-1}av - a \\ 0 & 1 \end{pmatrix}.$$ q.e.d.

(1.3) COROLLARY. _If_ A _is a finitely generated_ $\underline{\underline{Z}}$-_algebra then_ $E_n(A)$ _is a finitely generated group for all_ $n \geq 3$. _If_ A _is a finite_ $\underline{\underline{Z}}$-_algebra then_ $E_2(A)$ _is likewise finitely generated_.

Proof. $E_n(A)$ is generated by a finite number of subgroups, each isomorphic to the additive group of A, by (1.2) (a); this proves the last assertion.

Suppose $a_0 = 1, a_1, \ldots, a_r$ generate A as a ring. Let $S = \{I + a_i e_{jk} \mid 0 \leq i \leq r,\ 1 \leq j,\ k \leq n,\ j \neq k\}$, a finite set. Since $n \geq 3$ it follows, by induction, from (1.2) (b) that the group generated by S contains all $I + Me_{ij}$ for all $i \neq j$ and all monomials M in $a_0, \ldots, a_r$. These M's generate A additively so it follows now from (1.2) (a) that S generates $E_n(A)$. q.e.d.

(1.4) COROLLARY. *Assume* $n \geq 3$. *Let* $H \subset GL_n(A)$ *be a subgroup normalized by* $E_n(A)$. *Let* T *be a family of elementary matrices contained in* H. *Then* $H \supset E_n(A, \underline{q})$ *where* $\underline{q}$ *is the two sided ideal generated by the coordinates of* $I - \sigma$ *for all* $\sigma \in T$.

Proof. If $\sigma = I + a e_{ij} \in T$ then it follows from (1.2) (b), thanks to the fact that $n \geq 3$, that H contains all matrices of the form $I + bac\, e_{hk}$ $(b, c \in A;\ h \neq k)$. The $E_n(A)$-normalized subgroup generated by these is $E_n(A, AaA)$. Letting σ vary now, the corollary follows easily. q.e.d.

(1.5) COROLLARY. *Assume* $n \geq 3$. *Let* $\underline{q}$ and $\underline{q}'$ *be two sided ideals in* A. *Then*

$$E_n(A, \underline{qq}') \subset [E_n(A, \underline{q}), E_n(A, \underline{q}')].$$

In particular

$$E_n(A, \underline{q}) = [E_n(A), E_n(A, \underline{q})].$$

Proof. The group $[E_n(A, \underline{q}), E_n(A, \underline{q}')]$ is normalized by $E_n(A)$, and (1.2) (b) implies it contains all qq'-elementary matrices; now apply (1.4). The inclusion $E_n(A, \underline{q}) \supset [E_n(A), E_n(A, \underline{q})]$ holds because $E_n(A)$ normalizes $E_n(A, \underline{q})$.

We now introduce some notation which will be used throughout this and the ensuing chapters. Let A be a ring. For $n, m \geq 1$ we shall regard $GL_n(A)$ as a subgroup of $GL_{n+m}(A)$ via the monomorphism

$$\alpha \longmapsto \alpha \oplus I_m = \begin{pmatrix} \alpha & 0 \\ 0 & I_m \end{pmatrix} \quad (GL_n(A) \subset GL_{n+m}(A)).$$

Passing to the limit we obtain

$$GL(A) = \cup_n GL_n(A) \quad (= \varinjlim_n GL_n(A)).$$

We can think of the elements of GL(A) as infinite matrices.

$$\begin{pmatrix} \alpha & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix} \qquad (\alpha \in GL_n(A) \text{ for some } n).$$

In particular we have identified $U(A) = GL_1(A)$ with the set of diagonal matrices, diag(u, 1,...,1) in $GL_n(A)$, for each n. These are a subgroup of

$$D_n(A) = \{\text{diagonal matrices in } GL_n(A)\}.$$

Let $\underline{q}$ be a two sided ideal in A. Then the <u>principal congruence subgroup of level</u> $\underline{q}$ in $GL_n(A)$ is

$$GL_n(A, \underline{q}) = \text{Ker}(GL_n(A) \longrightarrow GL_n(A/\underline{q})).$$

More generally, we shall say that $H \subset GL_n(A)$ is a <u>subgroup of level</u> $\underline{q}$ if H is a subgroup such that

$$E_n(A, \underline{q}) \subset H \subset GL_n(A, \underline{q}).$$

(1) <u>If</u> $n \geq 2$ <u>then the level of, H, is uniquely determined</u>

To see this it suffices to show that if $E_n(A, \underline{q}) \subset GL_n(A, \underline{q}')$ then $\underline{q} \subset \underline{q}'$. Let $f: A \longrightarrow A/\underline{q}'$. Then our assumption and (1.1) imply that $E_n(A/\underline{q}', f(\underline{q})) = \{1\}$. Since $n \geq 2$ this clearly implies $f(\underline{q}) = (0)$, i.e. that $\underline{q} \subset \underline{q}'$.

If α is an m x n matrix over A we shall write

$${}^T\alpha$$

for its <u>transpose</u>. It is an n x m matrix over A^o (not A!). If $\alpha\beta$ is defined then ${}^T\beta\,{}^T\alpha$ is defined and equals ${}^T(\alpha\beta)$. In particular we have a ring antiisomorphism

$$M_n(A) \xrightarrow{\text{transpose}} M_n(A^o).$$

As sets $M_n(A) = M_n(A^o)$, and hence it makes sense, and is true, to say that all the groups introduced above are invariant under transposition.

If A is commutative we have the determinant, and its kernel,

$$SL_n(A) = \mathrm{Ker}(GL_n(A) \xrightarrow{\det} U(A)).$$

We write

$$SL_n(A, \underline{q}) = SL_n(A) \cap GL_n(A, \underline{q}), \text{ and}$$

$$SL(A, \underline{q}) = \cup_n SL_n(A, \underline{q}).$$

The inclusion $U(A) = GL_1(A) \subset GL_n(A)$ is a right inverse for det. Thus we have

$$E_n(A, \underline{q}) \subset SL_n(A, \underline{q}) \subset GL_n(A, \underline{q}) = U(A, \underline{q}) \cdot SL_n(A, q).$$

As before, all of these groups are invariant under transposition.

We shall further write

$$D_n(A, \underline{q}) = D_n(A) \cap GL_n(A, \underline{q}).$$

In case n = 1 we have $U(A, \underline{q}) = GL_1(A, \underline{q}) = D_1(A, \underline{q})$. The group generated by $E_n(A, \underline{q})$ and $D_n(A, \underline{q})$ will be denoted

$$GE_n(A, \underline{q}).$$

The subgroups introduced above are "stable" in the sense that the embedding $GL_n(A) \subset GL_{n+m}(A)$, induces embeddings of these subgroups. Thus we can introduce

$$E(A) = \cup_n E_n(A)$$

$$E(A, \underline{q}) = \cup_n E_n(A, \underline{q})$$

$$GL(A, \underline{q}) = \bigcup_n GL_n(A, \underline{q})$$

$$D(A, \underline{q}) = \bigcup_n D_n(A, \underline{q})$$

$$GE(A, \underline{q}) = \bigcup_n GE_n(A, \underline{q}), \text{ etc.}$$

(1.6) PROPOSITION. For all $n > 1$, $E_n(\underline{\underline{Z}}) = SL_n(\underline{\underline{Z}})$, and $GE_n(\underline{\underline{Z}}) = GL_n(\underline{\underline{Z}})$.

Proof. Since $\underline{\underline{Z}}$ is euclidean this follows from (IV, 5.9).

The next result is a basic tool in what follows. It shows that, modulo the elementary subgroups, the group law in GL and the direct sum coincide.

(1.7) PROPOSITION ("Whitehead Lemma"). Suppose $\alpha \in GL_n(A)$ and $b \in GL_n(A, \underline{q})$, where $\underline{q}$ is a two sided ideal in A. Then

$$\begin{pmatrix} \alpha & 0 \\ 0 & b \end{pmatrix} \equiv \begin{pmatrix} \alpha b & 0 \\ 0 & I \end{pmatrix} \equiv \begin{pmatrix} b\alpha & 0 \\ 0 & I \end{pmatrix} \quad \text{mod } E_{2n}(A, \underline{q})$$

$$\equiv \begin{pmatrix} 0 & \alpha \\ -b & 0 \end{pmatrix} \quad \text{mod } E_{2n}(A).$$

These congruences apply to both left and right cosets.

Proof. We shall give the proof for left cosets. The proof for right cosets is similar. Alternatively one can deduce the latter from the former with the observation that all subgroups involved are invariant under transposition.

First we have $\begin{pmatrix} 0 & \alpha \\ -b & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and it follows from (1.6) that $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in E_{2n}(A)$.

Write $b = I + q$, so that q has all coordinates in $\underline{q}$.

Then direct calculation shows that

$$\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} I & q \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}\begin{pmatrix} I & -b^{-1}q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & 0 \\ -b & I \end{pmatrix}$$

$$= \begin{pmatrix} I & q \\ 0 & I \end{pmatrix}\begin{pmatrix} I & -b^{-1}q \\ 0 & I \end{pmatrix}^{\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}}\begin{pmatrix} I & 0 \\ -q & I \end{pmatrix}$$

$\varepsilon\ E_{2n}(A, \underline{q})$. Therefore

$$\begin{pmatrix} \alpha b & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \equiv \begin{pmatrix} \alpha & 0 \\ 0 & b \end{pmatrix} \mod E_{2n}(A, \underline{q}).$$

Finally, we have

$$\begin{pmatrix} b\alpha & 0 \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \alpha^{-1}b^{-1}\alpha & 0 \\ 0 & b \end{pmatrix}$$

$$= \begin{pmatrix} I & (b\alpha)^{-1}q \\ 0 & I \end{pmatrix}\begin{pmatrix} I & -\alpha^{-1}q \\ 0 & I \end{pmatrix}^{\begin{pmatrix} I & 0 \\ \alpha & I \end{pmatrix}}$$

$$\cdot \begin{pmatrix} I & 0 \\ -b^{-1}q\alpha & I \end{pmatrix} \ \varepsilon\ E_{2n}(A, \underline{q}). \quad \text{q.e.d.}$$

(1.8) COROLLARY. <u>Let $\underline{q}$ be a two sided ideal in</u> A.

(a) <u>If</u> $\alpha_1, \ldots, \alpha_m \ \varepsilon\ GL_n(A, \underline{q})$ <u>then</u> $\mathrm{diag}(\alpha_1, \ldots, \alpha_m) \equiv \mathrm{diag}(\alpha_1 \cdots \alpha_m, I, \ldots, I) \mod E_{nm}(A, \underline{q})$.

(b) $D_n(A)$ <u>normalizes</u> $E_n(A, \underline{q})$, <u>and</u> $GE_n(A, \underline{q}) = U(A, \underline{q}) \cdot E_n(A, \underline{q})$.

(c) $GE_n(A, \underline{q})$ <u>contains all generalized permutation matrices</u> (see proof for definition).

<u>Proof</u>. (a) $\mathrm{diag}(\alpha_1, \ldots, \alpha_m) \equiv \mathrm{diag}(\alpha_1, \ldots, \alpha_m)\ \mathrm{diag}(I, \ldots, I, \alpha_m, \alpha_m^{-1}) = \mathrm{diag}(\alpha_1, \ldots, \alpha_{m-1} \cdot \alpha_m, I) \mod E_{nm}(A, \underline{q})$ by

the Whitehead Lemma. Now (a) follows by induction on m.

(b) If $\delta \in D_n(A)$ and if ε is $\underline{q}$-elementary then it follows from (1.2) (c) that ε^δ is $\underline{q}$-elementary. When $\underline{q} = A$ this shows that δ normalizes $E_n(A)$. In general $E_n(A, \underline{q})$ is generated by elements of the form ε^σ with ε as above and $\sigma \in E_n(A)$. We have just seen that ε^δ is $\underline{q}$-elementary and $\sigma^\delta \in E_n(A)$ so $(\varepsilon^\sigma)^\delta = (\varepsilon^\delta)^{\sigma^\delta} \in E_n(A, \underline{q})$. Thus $D_n(A)$, and hence also $U(A) \subset D_n(A)$, normalize $E_n(A, \underline{q})$. Part (a) implies the group generated by $U(A, \underline{q})$ and $E_n(A, \underline{q})$ contains $D_n(A, \underline{q})$, and this proves (b).

(c) A generalized permutation matrix is one of the form $\delta\pi$, where $\delta \in D_n(A)$ and where π is a permutation matrix (i.e. is invertible and has a single non zero entry, equal to 1, in each row). It follows from (1.6) that $\pi = \text{diag}(\pm 1, 1, \ldots, 1) \cdot \varepsilon$ where $\varepsilon \in E_n(A)$. Hence $\delta\pi \in GE_n(A)$.

(1.9) COROLLARY. _Let_ $\underline{q}$ _be a two sided ideal in_ A. _Then_ $[GL_n(A), GL_n(A, \underline{q})] \subset E_{2n}(A, \underline{q})$.

Proof. Let $\alpha \in GL_n(A)$ and $b \in GL_n(A, \underline{q})$. Then in $GL_{2n}(A)$ we have

$$\left[\begin{pmatrix} \alpha & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & I \end{pmatrix}\right] = \begin{pmatrix} \alpha^{-1}b^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha b & 0 \\ 0 & I \end{pmatrix}$$

$$= \varepsilon_1 \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & b \end{pmatrix} \varepsilon_2$$

$$= \varepsilon_1 \varepsilon_2 \in E_{2n}(A, \underline{q})$$

for suitable $\varepsilon_1, \varepsilon_2 \in E_{2n}(A, \underline{q})$, by the Whitehead Lemma.
q.e.d.

§2. NORMAL SUBGROUPS OF GL(A); $K_1(A, \underline{q})$

(2.1) THEOREM. _Let_ A _be a ring_.

(a) *If* $H \subset GL(A)$ *is a subgroup normalized by* $E(A)$ *then there is a unique two sided ideal* $\underline{q}$ *in* A *such that* H *is of level* $\underline{q}$, *i.e. such that* $E(A, \underline{q}) \subset H \subset GL(A, \underline{q})$.

(b) *Let* q *be a two sided ideal in* A *and let* $H \subset GL(A)$ *be a subgroup of level* $\underline{q}$. *Then*

$$E(A, \underline{q}) = [E(A), H] = [GL(A), H] \ (\subset H).$$

In particular H *is normal in* GL(A).

(c) *Let* $f: A \longrightarrow B$ *be a surjective ring homomorphism, and let* H *be as in* (b). *Then* $E(A, \underline{q}) \longrightarrow E(B, f(\underline{q}))$ *is surjective, and* f(H) *is a normal subgroup of level* $f(\underline{q})$ *in* GL(B).

This theorem shows that, for each two sided ideal $\underline{q}$,

$$K_1(A, \underline{q}) \ \underline{\text{defn}} \ GL(A, \underline{q})/E(A, \underline{q})$$

is an abelian group. Moreover, *the determination of* $K_1(A, \underline{q})$ *for each* $\underline{q}$, *is equivalent to the determination of all normal subgroups of* GL(A). In Chapter IX the group $K_1(A, \underline{q})$ will be introduced from a slightly different point of view, but it will be shown that the definition used there is equivalent with the present one.

In case A is commutative we have det: $GL(A) \longrightarrow U(A)$, whose kernel, SL(A), contains $[GL(A), GL(A)] = E(A)$. Therefore we obtain a split exact sequence

$$0 \longrightarrow SK_1(A, \underline{q}) \longrightarrow K_1(A, \underline{q}) \xrightarrow{\det} U(A, \underline{q}) \longrightarrow 0$$

for each $\underline{q}$, where

$$SK_1(A, \underline{q}) = SL(A, \underline{q})/E(A, \underline{q}).$$

The theorem above shows further that the groups $SK_1(A, \underline{q})$ classify the normal subgroups of SL(A).

When $\underline{q} = A$ we shall write

$$K_1(A) = K_1(A, A) = GL(A)/E(A),$$

and, if A is commutative,

$$SK_1(A) = SK_1(A, A) = SL(A)/E(A).$$

Proof of (2.1). Part (c) follows immediately from (1.1) and parts (a) and (b). For part (b) it clearly suffices to show that

$$E(A, \underline{q}) = [E(A), E(A, \underline{q})] = [GL(A), GL(A, \underline{q})].$$

The first equality follows from (1.5), and, in the second, the inclusion $\subset$ is obvious. Therefore it suffices to show that $[GL(A), GL(A, \underline{q})] \subset E(A, \underline{q})$. This follows, by passing to the limit over n, from (1.9).

It remains to prove (a). The uniqueness of $\underline{q}$ follows from the remark (1) in §1 (or from part (b)).

We first claim that, if $H \neq \{I\}$, then $E(A, \underline{q}) \subset H$ for some $\underline{q} \neq 0$. For let $H_n = H \cap GL_n(A)$ and view this as a subgroup of

$$\begin{pmatrix} GL_n(A) & A^n \\ 0 & I \end{pmatrix} \subset GL_{n+1}(A).$$

This is conjugate to the affine group $Aff_n(A)$, (see IV, §4)). H_n is normalized by $E_n(A)$ and, for large enough n, $H_n \neq \{I_n\}$. Hence it follows from (IV, 4.3 (a) and (c)) that $[H_n, A^n] = A^n\underline{a}$ for some non zero left ideal $\underline{a} \subset A$. I.e. $[H_n, A^n] \subset H$ consists of all matrices $\begin{pmatrix} I_n & x \\ 0 & I \end{pmatrix}$ for which $x \in A^n$ has coordinates in $\underline{a}$. Now it follows from (1.4) that $E(A, \underline{a}A) \subset H$, thus proving our contention.

To conclude the proof now, let $\underline{q}$ be the largest two sided ideal such that $E(A, \underline{q}) \subset H$; this clearly exists. We claim $H \subset GL(A, \underline{q})$. If not let H' be the image of H in $GL(A')$, where $A' = A/\underline{q}$. Since $E(A) \longrightarrow E(A')$ is surjective, H' is normalized by $E(A')$. Since $H' \neq \{I\}$ it follows from the last paragraph that $E(A', \underline{q}'/\underline{q}) \subset H'$ for some $\underline{q}' \neq \underline{q}$. Taking inverse images we conclude that $E(A, \underline{q}') \subset GL(A, \underline{q}) \cdot H$. Hence

$$E(A, \underline{q}') = [E(A), E(A, \underline{q}')] \subset [E(A), GL(A, \underline{q}) \cdot H]$$

If $\varepsilon \in E(A)$, $\alpha \in GL(A, \underline{q})$, and $\beta \in H$ then $[\varepsilon, \alpha\beta] = [\varepsilon, \beta] [\varepsilon, \alpha]^{\beta}$ (see (IV, 4.2)). Since $E(A)$ normalizes H, $[\varepsilon, \beta] \in H$. Moreover $[\varepsilon, \alpha] \in E(A, \underline{q}) \subset H$, by part (b). Hence $[\varepsilon, \alpha\beta] \in H$, and this shows that $E(A, \underline{q}') \subset H$, contradicting the maximality of $\underline{q}$. q.e.d.

§3. THE STABLE RANGE CONDITIONS, $SR_n(A, \underline{q})$

The main theorems of this chapter are stated in the next section. Their formulations involve certain technical hypotheses which we shall define and study in this section. In particular, with the aid of theorems proved in Chapter IV, we shall show that these hypotheses are satisfied by a reasonably large class of rings.

<u>For the following three definitions we fix a ring</u> A <u>and a two sided ideal</u> $\underline{q}$ in A. Recall that $\alpha = (a_1, \ldots, a_n) \in A^n$ is said to be <u>unimodular</u> in A^n if there is a homomorphism $f: A^n \longrightarrow A$ such that $f(\alpha) = 1$. This is evidently equivalent to the condition that $a_1, \ldots, a_n$ generate the unit left ideal: $\Sigma A a_i = A$. If, further, $\alpha \equiv (1, 0, \ldots, 0) \bmod \underline{q}$ then we shall say that α is <u>$\underline{q}$-unimodular</u>.

(3.1) DEFINITION OF

<u>condition</u> $SR_n(A, \underline{q})$:

If $m \geq n$ and if $\alpha = (a_1, \ldots, a_m) \in A^m$ is $\underline{q}$-unimodular then there exist $a_i' = a_i + b_i a_m$ with $b_i \in \underline{q}$ $(1 \leq i < m)$ such that $(a_1', \ldots, a_{m-1}') \in A^{m-1}$ is unimodular.

When $\underline{q} = A$ we will write $SR_n(A)$ in place of $SR_n(A, A)$. Note that $SR_n(A, \underline{q})$ can be fulfilled only for $n \geq 2$. Manifestly,

$$SR_n(A, \underline{q}) \Rightarrow SR_m(A, \underline{q}) \text{ for all } m \geq n.$$

The analogue of $SR_n(A, \underline{q})$, for the right (instead of left) ideal generated by the coordinates of $\alpha \in A^m$, is expressed by $SR_n(A^o, \underline{q})$, where A^o is the opposite ring of A.

(3.1)' DEFINITION OF

<u>condition</u> $SR'_n(A, \underline{q})$:

$GL_n(A, \underline{q})$ operates transitively on the $\underline{q}$-unimodular elements in A^n.

Again, we shall write $SR'_n(A)$ when $\underline{q} = A$. Note that this definition imposes a condition only on A^n, and not on all A^m $(m \geq n)$, as does (3.1).

Before formulating the last condition we must introduce some new notation and terminology. We shall write

$$\overline{E}_m(A, \underline{q})$$

for the group generated by $E_m(A, \underline{q})$ together with $[GE_m(A), GL_m(A, \underline{q})]$. It follows from (1.5) that:

(1) $$\overline{E}_m(A, \underline{q}) = [GE_m(A), GL_m(A, \underline{q})] \text{ for all } m \geq 3.$$

Let $t \in A$. We shall say that $\alpha \in GL_m(A)$ is of <u>type</u> $(\underline{q}, t)$ if α has the form

$$\alpha = \begin{pmatrix} 1+at & \alpha_{12} \\ \alpha_{21}t & \alpha_{22} \end{pmatrix}$$

where $a \in \underline{q}$, $\alpha_{22} \in M_{m-1}(A)$, and α_{12} and ${}^T\alpha_{21}$ have coordinates in $\underline{q}$. Given such a representation of α, we can set

$$\alpha' = \begin{pmatrix} 1+ta & t\alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

and we shall say that an α' obtained in this way is $(\underline{q}, t)$

-related to α. Unfortunately, since t may be a zero divisor, α does not quite determine α'.

(3.1)" DEFINITION OF

condition $SR''_n(A, \underline{q})$:

If $t \in \underline{q}$, if $\alpha \in GL_n(A, \underline{q})$ is of type $(\underline{q}, t)$, and if α' is $(\underline{q}, t)$-related to α, then $\alpha' \alpha^{-1} \in \overline{E}_n(A, \underline{q})$.

This condition is admittedly rather artificial looking, but it is forced on us inescapably by the arguments of §6.

(3.2) PROPOSITION. Let $f: A \longrightarrow A'$ be a surjective ring homomorphism.

(a) If $\underline{q}_0 \subset \underline{q}$ are two sided ideals in A then $SR_n(A, \underline{q}) \Rightarrow SR_n(A, \underline{q}_0)$. In particular $SR_n(A) \Rightarrow SR_n(A, \underline{q})$ for all $\underline{q}$.

(b) If $\underline{q}'$ is a two sided ideal in A' then $SR_n(A, f^{-1}(\underline{q}')) \Rightarrow SR_n(A', \underline{q}')$. In particular, $SR_n(A) \Rightarrow SR_n(A')$.

Proof. (a). Suppose $m \geq n$ and $\alpha = (a_1, \ldots, a_m)$ is $\underline{q}_0$-unimodular. Writing $1 = \Sigma\, c_i a_i$ $(c_i \in A)$ we have $a_m = \Sigma_i a_i c_i a_m$, so a_m is in the left ideal generated by the coordinates of $\alpha' = (a_1, \ldots, a_{m-1}, c_m a_m a_m)$. It follows that α' is unimodular, and hence $\underline{q}$-unimodular. By hypothesis there exist $a_i' = a_i + b_i c_m a_m a_m$ with $b_i \in \underline{q}$ $(1 \leq i < m)$ such that $(a_i', \ldots, a_{m-1}')$ is unimodular. Since $b_i c_m a_m \in \underline{q}_0$ (because $a_m \in \underline{q}_0$) this verifies $SR_n(A, \underline{q}_0)$.

Remark. This proof used only the fact that $\underline{q}_0$ is a left ideal.

(b) Suppose $m \geq n$. We first claim that a $\underline{q}'$-unimodular $\alpha' = (a_1', \ldots, a_m') \in A'^m$ can be lifted to a unimodular $\alpha = (a_1, \ldots, a_m) \in A^m$: $f(\alpha) = \alpha'$. For let α be any lifting. Write $1 = \Sigma\, f(c_i)\, \alpha_i'$ with $c_i \in A$ $(1 \leq i \leq m)$. Then 1 =

$\Sigma\, c_i\, a_i + q$ for some $q \in \mathrm{Ker}(f) \subset f^{-1}(\underline{q}')$. Therefore $(a_1, \ldots, a_m, q)$ is $f^{-1}(\underline{q}')$-unimodular in A^{m+1}, so we can find $b_i = a_i + t_i q$ $(1 \leq i \leq m)$ such that $\beta = (b_1, \ldots, b_m)$ is unimodular, by hypothesis. Clearly $f(\beta) = \alpha'$ so β is $f^{-1}(\underline{q}')$ -unimodular.

Again, by hypothesis, we can find $d_i = b_i + s_i\, b_m$ with $f(s_i) \in \underline{q}'$ $(1 \leq i < m)$ such that $\delta = (d_1, \ldots, d_{m-1})$ is unimodular. Now the $f(d_i) = a_i' + f(s_i)\, a_m'$ solve our problem. q.e.d.

(3.3) THEOREM. *Assume* $SR_n(A, \underline{q})$ *holds, and let* $m \geq n$.

(1) $E_m(A, \underline{q})$ *operates transitively on the* $\underline{q}$*-unimodular elements in* A^m. *In particular* $SR_n(A, \underline{q}) \Rightarrow SR_m'(A, \underline{q})$. *Moreover* $E_m(A, \underline{q})$ *is a normal subgroup of* $GL_m(A, \underline{q})$, *and* $GL_m(A, \underline{q}) = E_m(A, \underline{q}) \cdot GL_{n-1}(A, \underline{q})$.

(2) *Let* $t \in A$ *and let* $\alpha \in GL_m(A, \underline{q})$ *be of type* $(\underline{q}, t)$. *If* α' *is* $(\underline{q}, t)$*-related to* α *then* $\alpha' \in GL_m(A, \underline{q})$ *and* $\alpha' \alpha^{-1} \in E_m(A, \underline{q})$. *In particular,* $SR_n(A, \underline{q}) \Rightarrow SR_m''(A, \underline{q})$.

Proof. We could actually deduce (1) from (IV, 3.4), but the details of the argument are required for part (2). The proof will be carried out in several steps.

(i) *Let* $\alpha, \beta \in GL_m(A)$ *be of type* $(\underline{q}, t)$ *and let* α' *and* β' be $(\underline{q}, t)$*-related to* α *and* β, *respectively. Then* $\alpha\beta$ *is of type* $(\underline{q}, t)$ *and* $\alpha'\beta'$ *is* $(\underline{q}, t)$*-related to* $\alpha\beta$.

Let α' and β' be defined relative to representations

$$\alpha = \begin{pmatrix} 1+at & \alpha_{12} \\ \alpha_{21}t & \alpha_{22} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1+bt & \beta_{12} \\ \beta_{21}t & \beta_{22} \end{pmatrix} \text{ of } \alpha \text{ and } \beta.$$

Then

$$\alpha\beta = \begin{pmatrix} 1 + (a + b + atb + \alpha_{12}\beta_{21})t & \beta_{12} + at\beta_{12} + \alpha_{12}\beta_{22} \\ (\alpha_{21} + \alpha_{21}tb + \alpha_{22}\beta_{21})t & \alpha_{21}\ t\beta_{12} + \alpha_{22}\beta_{22} \end{pmatrix}$$

and

$$\alpha'\beta' = \begin{pmatrix} 1 + t(a + b + abb + \alpha_{12}\beta_{21}) & t(\beta_{12} + at\beta_{12} + \alpha_{12}\beta_{22} \\ \alpha_{21} + \alpha_{21}tb + \alpha_{22}\beta_{21} & \alpha_{21}t\beta_{12} + \alpha_{22}\beta_{22} \end{pmatrix}.$$

(ii) _Suppose_ ${}^T\gamma = (a_1,\dots,a_m) \equiv (1, 0,\dots,0)$ mod $\underline{q}t$ _and that_ γ _is unimodular. Then there is a_ $\tau \in E_m(A, \underline{q})$ _of type_ $(\underline{q}, t)$ _such that_ $\tau\gamma = {}^T(1, 0,\dots,0)$ _and such that there is a_ $\tau' \in E_m(A, \underline{q})$ _which is_ $(\underline{q}, t)$ _related to_ τ.

We first apply (3.2) (a) with $\underline{q}_o = \underline{q}t$; it is remarked there that the proof only requires $\underline{q}_o$ to be a left ideal. Thus we obtain $a_i' = a_i + b_i a_m$ with $b_i \in \underline{q}t$ $(1 \le i < m)$ such that $(a_1',\dots,a_{m-1}')$ is unimodular. Set $\tau_1 = I + \Sigma_{1<i<m} b_i e_{im}$. Since τ_1 has trivial first column we can define a $\tau_1' \in E_m(A, \underline{q})$ (e.g. $\tau_1' = I + tb_1 e_{1m} + \Sigma_{1<i<m} b_i e_{im}$) which is $(\underline{q}, t)$ related to τ_1. Moreover $\tau_1\gamma = {}^T(a_1',\dots, a_{m-1}', a_m) \equiv {}^T(1, 0,\dots,0)$ mod $\underline{q}t$. Write $1 = \Sigma_{i<m} c_i a_i'$ and set $\tau_2 = I + (\Sigma_{i<m} (1 - a_1' - a_m) c_i e_{mi})$. Since $1 - a_1' - a_m \in \underline{q}t$ we see that τ_2 is of type $(\underline{q}, t)$, and again we can define a $\tau_2' \in E_m(A, \underline{q})$ which is $(\underline{q}, t)$-related to τ_2. Moreover $\tau_2\tau_1\gamma = {}^T(a_1',\dots,a_{m-1}', 1 - a_1')$. Now $\varepsilon = I + e_{1m}$

$\varepsilon \in E_m(A)$ has trivial first column, and $\varepsilon' = I + te_{1m} \in E_m(A)$ is $(\underline{q}, t)$-related to ε. Moreover, $\varepsilon\tau_2\tau_1\gamma = {}^T(1, a_2', \ldots, a'_{m-1}, 1 - a_1')$. Set $\tau_3 = I - (\Sigma_{1<i<m} a_i' e_{i1}) - (1 - a_1') e_{m1} \in E_m(A, \underline{q})$. As above we can define a $\tau_3' \in E_m(A, \underline{q})$ which is $(\underline{q}, t)$-related to τ_3. Moreover $\tau_3\, \varepsilon\, \tau_2\tau_1\gamma = {}^T(1, 0,\ldots,0)$, and the latter is fixed by ε. Therefore $\tau = \tau_3^{\varepsilon}\, \tau_2\tau_1 \in E_m(A, \underline{q})$ and $\tau\gamma = {}^T(1,0,\ldots,0)$. It follows from (i) above that τ is of type $(\underline{q}, t)$ and that $\tau' = \tau_3'^{\varepsilon'}\tau_2'\tau_1'$ is $(\underline{q}, t)$-related to τ. Evidently also $\tau' \in E_m(A, \underline{q})$.

(iii) Proof of (1).

In case $t = 1$ we have $\underline{q}t = \underline{q}$ and part (ii) implies $E_m(A, \underline{q})$ acts transitively on the $\underline{q}$-unimodular elements of A^m. If $\sigma \in GL_m(A, \underline{q})$ we can therefore find $\tau_1 \in E_m(A, \underline{q})$ such that $\tau_1\sigma$ has last column ${}^T(0,\ldots,0, 1)$; say $\tau_1\sigma =$

$$\begin{pmatrix}\alpha & 0\\ \rho & 1\end{pmatrix} = \begin{pmatrix} I & 0\\ \rho\alpha^{-1} & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}.$$

The left factor is clearly in $E_m(A, \underline{q})$, and $\alpha \in GL_{m-1}(A, q)$. Thus $GL_m(A, \underline{q}) = E_m(A, \underline{q})\, GL_{m-1}(A, \underline{q})$. By induction on $m - n$ (≥ 0) we conclude that $GL_m(A, \underline{q}) = E_m(A, \underline{q})\, GL_{n-1}(A, \underline{q})$.

Finally, to show that $E_m(A, q)$ is normal in $GL_m(A, \underline{q})$ we take one of the defining generators, τ^{ε}, of $E_m(A, \underline{q})$. Here τ is $\underline{q}$-elementary and $\varepsilon \in E_m(A)$. Since the latter contains all permutation matrices of determinant one, we can, after altering ε, assume τ is of the form $\tau = \begin{pmatrix} I & 0\\ t & 1\end{pmatrix}$.

If $\alpha \in GL_m(A, q)$ then $(\tau^{\varepsilon})^{\alpha} = (\tau^{\alpha^{\varepsilon^{-1}}})^{\varepsilon}$, and ε normalizes $E_m(A, \underline{q})$. Therefore, after replacing α by $\alpha^{\varepsilon^{-1}}$, it suffices to show that $\tau^{\alpha} \in E_m(A, \underline{q})$. Write $\alpha^{-1} = \varepsilon_1^{-1}\alpha_1^{-1}$ with $\varepsilon_1 \in E_m(A, q)$ and $\alpha_1 \in GL_{m-1}(A, \underline{q})$, using the first part of the proof. Then $\tau^{\alpha} = (\tau^{\alpha_1})^{\varepsilon_1}$ so it suffices to show that τ^{α_1} $E_m(A, \underline{q})$. But

$$\begin{pmatrix} I & 0 \\ t & 1 \end{pmatrix}^{\begin{pmatrix} \alpha_1 & 0 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} I & 0 \\ t\alpha_1 & 1 \end{pmatrix} \in E_m(A, \underline{q}). \text{ q.e.d.}$$

(iv) Proof of (2).

Let α and α' be as in part (2). Apply (ii) above to γ_o = first column of α. Then we obtain $\tau \in E_m(A, \underline{q})$ of type (q, t) and $\tau' \in E_m(A, \underline{q})$ and $(\underline{q}, t)$-related to τ, such that $\delta = \tau\alpha$ has the form $\delta = \begin{pmatrix} 1 & \rho \\ 0 & \beta \end{pmatrix}$. Then $\delta' = \begin{pmatrix} 1 & t\rho \\ 0 & \beta \end{pmatrix}$ is $(\underline{q}, t)$-related to δ. and evidently $\delta'\delta^{-1} \in E_m(A, \underline{q})$. Let $\delta_o = \tau'\alpha'$; then (i) implies δ_o is $(\underline{q}, t)$ related to $\tau\alpha = \delta$. To show that $\alpha'\alpha^{-1} \in E_m(A, \underline{q})$ (and thus prove (2)) it suffices to show that $\delta_o\delta^{-1} = \tau'\alpha'\alpha^{-1}\tau^{-1} \in E_m(A, \underline{q})$. Further, since $\delta_o\delta^{-1} = \delta_o\delta'^{-1}\,\delta'\delta^{-1}$, it suffices to show that $\delta_o\delta'^{-1} \in E_m(A, \underline{q})$. Now δ_o and δ' are both $(\underline{q}, t)$-related to the same $\delta = \tau\alpha = \begin{pmatrix} 1 & \rho \\ 0 & \beta \end{pmatrix}$. It follows that δ_o must be of the form $\delta_o = \begin{pmatrix} 1+ta & t\rho \\ \gamma & \beta \end{pmatrix}$ where $at = 0$ and $\gamma t = 0$. Therefore

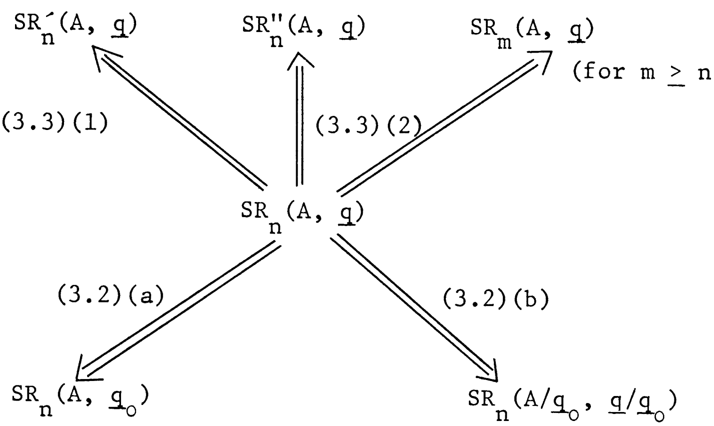
$$\delta_o\delta'^{-1} = \begin{pmatrix} 1+ta & t\rho \\ \gamma & \beta \end{pmatrix}\begin{pmatrix} 1 & -t\rho\beta^{-1} \\ 0 & \beta^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1+ta & 0 \\ \gamma & I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma & I \end{pmatrix}\begin{pmatrix} 1+ta & 0 \\ 0 & I \end{pmatrix}.$$

The left hand factor is in $E_m(A, \mathfrak{q})$, clearly. For the right hand factor we have the following expression in $GL_2(A)$, due to the fact, again, that $\alpha t = 0$:

$$\begin{pmatrix} 1+t\alpha & 0 \\ 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\right] \in E_2(A, \mathfrak{q}). \text{ q.e.d.}$$

We can summarize the logical interdependence of the stable range conditions, which are proved above, in the following diagram. Here A is a ring and $\mathfrak{q}_0 \subset \mathfrak{q}$ are two sided ideals.



Finally, we give two results affirming that the conditions introduced above are satisfied in some generality.

(3.4) PROPOSITION. *Let* A *be a ring and let* $\mathfrak{q}$ *be a two sided ideal in* A.

(a) *If* A *is semi-local or if* $\mathfrak{q} \subset \operatorname{rad} A$ *then* $SR_2(A, \mathfrak{q})$ *is satisfied*.

(b) $SR_1'(A, \mathfrak{q})$ *is always satisfied*. *If* A *is commutative then* $SR_2'(A, \mathfrak{q})$ *is satisfied*.

Proof. (a). Let $(\alpha_1, \ldots, \alpha_m) \in A^m$ be $\mathfrak{q}$-unimodular

$(m \geq 2)$. If $\underline{q} \subset$ rad A then $a_1 \in U(A)$ since $a_1 \equiv 1 \bmod \underline{q}$ so $(a_1, \ldots, a_{m-1})$ is unimodular. If A is semi-local then, since $\Sigma A a_i = A$, it follows from (III, 2.8) that $a_1 + \Sigma_{i \geq 2} b_i a_i \in U(A)$ for some $b_2, \ldots, b_m \in A$. Thus $(a_1 + b_m a_m, a_2, \ldots, a_{m-1})$ is unimodular, and this verifies $SR_2(A)$. By (3.3) (a) this implies $SR_2(A, \underline{q})$.

(b) $SR_1'(A, \underline{q})$ is obvious, so assume A is commutative, and let $\alpha = {}^T(a, b) \in A^2$ be $\underline{q}$-unimodular. Write $1 = ax + by$ for some $x, y \in A$ and then set $c = -by^2 \in \underline{q}$ and $d = x + bxy$. Then $ad - bc = a(x + bxy) + b^2y^2 = ax + by(ax + by) = 1$. Reading this mod $\underline{q}$ shows that $d \equiv 1 \bmod \underline{q}$, and so $\sigma = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is in $SL_2(A, \underline{q})$. Moreover $\sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \alpha$, and this proves (b).

(3.5) THEOREM. Let R be a commutative ring such that max (R) is a noetherian space which is the union of a finite number of subspaces each of dimension $\leq d$. Let A be a finite R-algebra. Then A satisfies $SR_{d+2}(A)$.

Proof. Let $e_1, \ldots, e_m$ be the standard basis for A^m and assume $m \geq d + 2$. Let $\alpha = \Sigma e_i a_i$ be unimodular. By (IV, 3.1) there is a homomorphism $f: e_m A \longrightarrow A^{m-1}$ such that $(a_1, \ldots, a_{m-1}) + f(e_m a_m)$ is unimodular. Then $f(e_m) = (b_1, \ldots, b_{m-1})$ satisfies the requirements of definition (3.1). q.e.d.

§4. THE MAIN THEOREMS

We fix a ring A. If $\underline{q}$ is a two sided ideal we write

$GL_m'(A, \underline{q})$

for the inverse image in $GL_m(A)$ of the center of $GL_m(A/\underline{q})$.

(4.1) THEOREM. *Assume* $SR_n(A)$ (see 3.1)) *and let* $m \geq n$. *Then, for any two sided ideal* $\mathfrak{q}$:

(a) $E_m(A, \mathfrak{q})$ *is a normal subgroup* of $GL_m(A)$, *and* $GL_m(A, \mathfrak{q}) = E_m(A, \mathfrak{q}) \cdot GL_{n-1}(A, \mathfrak{q})$.

(b) $[GE_m(A), GL_m(A, \mathfrak{q})] \subset E_m(A, \mathfrak{q})$. If $m \geq 3$ *this is equality and, moreover,*

$$[E_m(A), GL'_m(A, \mathfrak{q})] = E_m(A, \mathfrak{q}).$$

If $m \geq 2(n - 1)$ *then*

$$[GL_m(A), GL_m(A, \mathfrak{q})] \subset E_m(A, \mathfrak{q}).$$

(c) *Assume* $m \geq 3$. *Then a subgroup* H *of* $GL_m(A)$ *is normalized by* $E_m(A)$ *if and only if there is a two sided ideal* $\mathfrak{q}$ *such that*

$$E_m(A, \mathfrak{q}) \subset H \subset GL'_m(A, \mathfrak{q}).$$

In this case $\mathfrak{q}$ *is determined by*: $E_m(A, \mathfrak{q}) = [E_m(A), H]$.

This theorem will be proved in §5. The proof of parts (a) and (b) will show, more precisely:

(4.1)' PROPOSITION. *Assume only* $SR_n(A, \mathfrak{q})$, *and let* $m \geq n$. *Then* $GL_m(A, \mathfrak{q}) = E_m(A, \mathfrak{q}) \cdot GL_{m-1}(A, \mathfrak{q})$, and

$$[GE_m(A), GL_m(A, \mathfrak{q})] \subset E_m(A, \mathfrak{q}),$$

with equality for $m \geq 3$.

(4.2) THEOREM. *Let* $\mathfrak{q}$ *be a two sided ideal in* A. *Assume that conditions* $SR_n(A, \mathfrak{q})$, $SR_n(A^o, \mathfrak{q})$, $SR'_{n-1}(A, \mathfrak{q})$ *hold. Then*

$$GL_{n-1}(A, \mathfrak{q}) \longrightarrow K_1(A, \mathfrak{q})$$

is surjective, and for $m \geq n$, *the natural homomorphism*

$$GL_m(A, \mathfrak{q})/E_m(A, \mathfrak{q}) \longrightarrow K_1(A, \mathfrak{q})$$

is an isomorphism.

Recall from §2 that $K_1(A, \underline{q}) = GL(A, \underline{q})/E(A, \underline{q})$. Thus the fact that $GL_m(A, \underline{q}) \longrightarrow K_1(A, \underline{q})$ is surjective for $m \geq n - 1$ follows from (4.1) (a) (or, rather, from (4.1)'). The proof of the injectivity assertion is rather technical, and it occupies §§6-8.

When A is commutative it follows immediately from (4.2) that

$$SL_{n-1}(A, \underline{q}) \longrightarrow SK_1(A, \underline{q})$$

is surjective, and that

$$SL_m(A, \underline{q})/E_m(A, \underline{q}) \longrightarrow SK_1(A, \underline{q})$$

is an isomorphism for all $m \geq n$. We also have the following useful corollaries.

(4.3) COROLLARY. *In the setting of* (4.2) *assume further that* $SR_n(A)$ *holds. Let* $m \geq n$, *and let* $H \subset GL_m(A)$ *be a subgroup of level* $\underline{q}$. *Then*

$$E_m(A, \underline{q}) \supset [GL_m(A), H] \supset [E_m(A), H],$$

with equality if $m \geq 3$.

Proof. The equality when $m \geq 3$ follows from (4.1) (b). For the rest it suffices to show that $[GL_m(A), GL_m(A, \underline{q})] \subset E_m(A, \underline{q})$. By (2.1) (b) we have $[GL(A), GL(A, \underline{q})] = E(A, \underline{q})$, and the injectivity part of (4.2) means that $E(A, \underline{q}) \cap GL_m(A) = E_m(A, \underline{q})$. This proves the corollary.

(4.4) COROLLARY. *Suppose, in the setting of* (4.2), *that* $\underline{q} = A$ *and that* A *is a finitely generated* $\underline{\underline{Z}}$*-algebra. If* $GL_m(A)$ *is finitely generated from some* $m \geq n - 1$ *then* $K_1(A)$ *is finitely generated (as an abelian group). Conversely, if* $K_1(A)$ *is finitely generated then* $GL_m(A)$ *is finitely generated for all* $m \geq \max(n, 3)$

Proof. The first assertion follows because $GL_m(A)$

$\longrightarrow K_1(A)$ is surjective for $m \geq n - 1$. According to (1.3) $E_m(A)$ is a finitely generated group for all $m \geq 3$. If, further, $m \geq n$, then (4.2) implies $GL_m(A)/E_m(A) \simeq K_1(A)$. Therefore if $K_1(A)$ is finitely generated so also is $GL_m(A)$. q.e.d.

Remark. If, in (4.4), A is commutative, then we have also the analogue of (4.4) with SL_m in place of GL_m and SK_1 in place of K_1. This follows from the corresponding analogue of (4.2), described above, for SK_1.

(4.5) COROLLARY. *Let* R *be a commutative ring such that* max(R) *is a noetherian space which is the union of a finite number of subspaces each of dimension* $\leq d$. *Let* A *be finite* R-*algebra. Then the conclusions of Theorem* (4.1) *are valid for* A *with* $n = d + 2$. *The conclusions of Theorem* (4.2) *are valid for* A, *and all ideals* $\underline{q}$, *with* $n = d + 3$, *or for* $n = 3$ *if* A *is commutative and* $d = 1$.

Proof. By (3.5) we have $SR_{d+2}(A)$, and $SR_{d+2}(A^o)$ also, by symmetry. By (3.3) (b), $SR_n \Rightarrow SR'_n$, and, trivially, $SR_n \Rightarrow SR_{n+1}$. Hence we have SR_{d+3} and SR'_{d+2} for A. Thus we've established the hypotheses of (4.1) and (4.2), respectively, in the indicated ranges. Moreover, if A is commutative, then $SR'_2(A, \underline{q})$ is always satisfied according to (3.4) (b). Hence, if further $d = 1$, then we have $SR_3(=SR_{d+2})$ and SR'_2, i.e. the hypotheses of (4.2) for $n = 3$.

Remark. I conjecture that Theorem (4.2) is valid with only the hypothesis SR_n. It will be seen from the proof that the hypothesis SR'_{n-1} intervenes only at the last stage (see (8.1)).

§5. PROOF OF THEOREM (4.1).

We keep the setting of (4.1) and fix an $m \geq n$.

Proof of (a). According to (3.3) (1), $SR_n(A, \underline{q})$ implies $GL_m(A, \underline{q}) = E_m(A, \underline{q}) \cdot GL_{n-1}(A, \underline{q})$. Now assume $SR_n(A)$. We propose to show that $E_m(A, \underline{q})$ is normal in $GL_m(A)$. By definition $E_m(A, \underline{q})$ is generated by elements of the form τ^ε where $\varepsilon \in E_m(A)$ and $\tau = I_m + \alpha e_{ij}$ ($\alpha \in \underline{q}$, $i \neq j$). Since $E_m(A)$ contains all permutation matrices of determinant 1 (cf (1.8) (c)) it suffices even to restrict i to the value m, in which case τ has the form $\tau = \begin{pmatrix} I_{m-1} & 0 \\ t & 1 \end{pmatrix}$. In order to show that $E_m(A, \underline{q})$ is normal in $GL_m(A)$ it suffices to show for each $\sigma \in GL_m(A)$, that $(\tau^\varepsilon)\sigma \in E_m(A, \underline{q})$. Since $(\tau^\varepsilon)^\sigma = (\tau^{\varepsilon\sigma\varepsilon^{-1}})\varepsilon$, and since $\varepsilon \in E_m(A)$ normalizes $E_m(A, \underline{q})$ (by definition), it suffices, after replacing σ by $\varepsilon\sigma\varepsilon^{-1}$, to show that $\tau^\sigma \in E_m(A, \underline{q})$. Write $\sigma^{-1} = \varepsilon_o{}^{-1} \sigma'^{-1}$ with $\varepsilon_o \in E_m(A)$ and $\sigma' = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_{m-1}(A)$; this uses the conclusion of the last paragraph. Then $\tau^\sigma = \sigma^{-1} \tau\sigma = (\tau^{\sigma'})^{\varepsilon_o}$, and, again, it suffices to show that $\tau^{\sigma'} \in E_m(A, \underline{q})$. We calculate:

$$(1) \qquad \tau^{\sigma'} = \begin{pmatrix} \sigma_1{}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ t\sigma_1 & 1 \end{pmatrix}$$

$\in E_m(A, \underline{q})$. q.e.d.

Proof of (b). Assuming $SR_n(A, \underline{q})$ we shall show that $[GE_m(A), GL_m(A, \underline{q})] \subset E_m(A, \underline{q})$. The equality when $m \geq 3$ follows from (1.5).

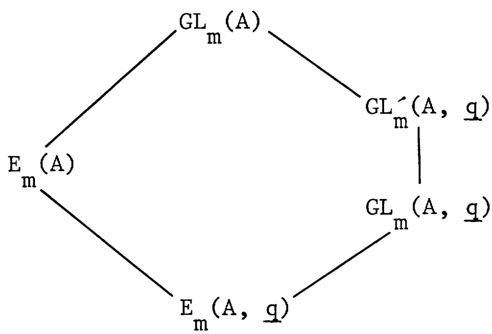
As above, $E_m(A)$ is generated by elements τ^ε, where $\varepsilon \in E_m(A)$ and $\tau = \begin{pmatrix} I & 0 \\ t & 1 \end{pmatrix}$. By (1.8) (b) $GE_m(A)$ is generated with $E_m(A)$ by elements $\delta = \text{diag}(1,\ldots,1, u)$ where $u \in U(A)$. Therefore, it suffices to show that, given $\sigma \in GL_m(A, \underline{q})$, $[\sigma, \delta\tau^{\varepsilon}] \in E_m(A, \underline{q})$. We have $[\sigma, \delta\tau^{\varepsilon}] = [\sigma, \tau^{\varepsilon}]\,[\sigma, \delta]^{\tau^{\varepsilon}}$ and $[\sigma, \tau^{\varepsilon}] = [\sigma^{\varepsilon^{-1}}, \tau]^{\varepsilon}$. Since $E_m(A)$ normalizes $E_m(A, \underline{q})$ (by definition) it suffices to show, for all $\sigma \in GL_m(A, \underline{q})$, that $[\sigma, \delta]$ and $[\sigma, \tau]$ are in $E_m(A, \underline{q})$. By part (a) (applied to σ^{-1}) we can write $\sigma = \sigma_1\varepsilon_1$ where $\sigma_1 = \begin{pmatrix} \sigma_1' & 0 \\ 0 & 1 \end{pmatrix} \in GL_{m-1}(A, \underline{q})$ and $\varepsilon_1 \in E_m(A, \underline{q})$. Then $[\delta, \sigma] = [\delta, \varepsilon_1]\,[\delta, \sigma_1]^{\varepsilon_1}$. Since δ and σ_1 commute and since δ normalizes $E_m(A, \underline{q})$ (see (1.8) (b)) this shows that $[\delta, \sigma] \in E_m(A, \underline{q})$. Next $[\tau, \sigma] = [\tau, \varepsilon_1]\,[\tau, \sigma_1]^{\varepsilon_1}$ so it remains to be shown that $[\tau, \sigma_1]$ $E_m(A, \underline{q})$. But (cf. formula (1) above) we have

$$[\tau, \sigma_1] = \left[\begin{pmatrix} I & 0 \\ t & 1 \end{pmatrix}, \begin{pmatrix} \sigma_1' & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} I & 0 \\ t(\sigma_1'-I) & 1 \end{pmatrix}$$

$\in E_m(A, \underline{q})$. (Recall $\sigma_1' \equiv I \bmod \underline{q}$).

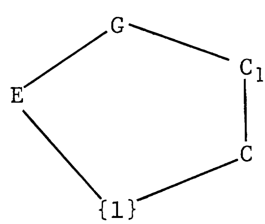
Now assume $SR_n(A)$, and let $\alpha \in GL_m(A)$ and $\beta \in GL_m(A, \underline{q})$. Then, if $m \geq 2(n-1)$, we claim α and β commute modulo (the normal subgroup) $E_m(A, \underline{q})$. For we have just seen that $E_m(A)$ commutes with β, mod $E_m(A, \underline{q})$, and mod $E_m(A)$ we can assume $\alpha = \begin{pmatrix} \alpha_0 & 0 \\ 0 & I \end{pmatrix}$ with $\alpha_0 \in GL_{n-1}(A)$. Moreover, mod $E_m(A, \underline{q})$ we can assume $\beta = \begin{pmatrix} I & 0 \\ 0 & \beta_0 \end{pmatrix}$. In each of these matrices $I = I_{m-(n-1)}$. Since $m \geq 2(n-1)$ it follows that α and β now actually commute. This shows that $[GL_m(A), GL_m(A, \underline{q})] \subset E_m(A, \underline{q})$ for $m \geq 2(n-1)$.

It remains to be shown that $[E_m(A), GL'_m(A, \underline{q})] = E_m(A, \underline{q})$ for $m \geq n$ and ≥ 3. The inclusion $\supset$ follows from (1.5). Consider the diagram of subgroups:



In view of what has been shown above we see that the opposite inclusion $\subset$ follows from the next lemma, applied to $G = GL_m(A)/E_m(A, \underline{q})$:

(5.1) LEMMA. Let



be a diagram of normal subgroups of a group G. Assume that $[E, C_1] \subset C$, $[E, C] = \{1\}$, and $[E, E] = E$. Then $[E, C_1] = \{1\}$.

Proof. Fix $\gamma \in C_1$ and define

$$h: E \longrightarrow E \cap C \subset \text{center } (E)$$

by $h(\varepsilon) = [\gamma, \varepsilon]$. Then $h(\varepsilon_1\varepsilon_2) = [\gamma, \varepsilon_1\varepsilon_2] = [\gamma, \varepsilon_2] [\gamma, \varepsilon_1]^{\varepsilon_2}$ (see (IV, 4.2)) $= [\gamma, \varepsilon_2] [\gamma, \varepsilon_1]$ (because [E, C]

$= \{1\}) = h(\varepsilon_1)\ h(\varepsilon_2)$ (because $C \cap E$ is commutative). Thus h is a homomorphism into an abelian group. Since $[E, E] = E$ it follows that $[\gamma, \varepsilon] = 1$ for all $\varepsilon \in E$. Thus $[E, C_1] = \{1\}$. q.e.d.

Proof of (c). If, for some two sided ideal $\underline{q}$, $E_m(A, \underline{q}) \subset H \subset GL'_m(A, \underline{q})$, and if $m \geq \max(n, 3)$, then it follows from part (b) that

$$E_m(A, \underline{q}) = [E_m(A), E_m(A, \underline{q})] \subset [E_m(A), H]$$

$$\subset [E_m(A), GL'_m(A, \underline{q})] = E_m(A, \underline{q}),$$

and hence $E_m(A)$ normalizes H.

Now suppose, conversely, that $H \subset GL_m(A)$ is a subgroup normalized by $E_m(A)$, and the $m \geq \max(n, 3)$. We must show that H has the above form.

(5.2) LEMMA. If H is not central then $E_m(A, \underline{q}) \subset H$ for some two sided ideal $\underline{q} \neq 0$.

We shall first conclude the proof assuming the lemma. Let $\underline{q}$ be the largest two sided ideal such that $E_m(A, \underline{q}) \subset H$; this clearly exists. We must show that the image, H', of H in $GL_m(A')$, $A' = A/\underline{q}$, lies in the center of $GL_m(A')$. Thanks to (3.2) (b) the hypothesis $SR_n(A)$ of Theorem (4.1) implies $SR_n(A')$. Moreover H' is normalized by the image of $E_n(A)$ which, according to (1.1), is $E_n(A')$. Therefore, if H' is not central, Lemma (5.2) implies H' contains $E_m(A', \underline{q}'/\underline{q})$ for some $\underline{q}' \neq \underline{q}$. Taking inverse images, we deduce that $E_m(A, \underline{q}') \subset GL_m(A, \underline{q}) \cdot H$. Suppose $\varepsilon \in E_m(A)$, $\alpha \in H$, and $\beta \in GL_m(A, \underline{q})$. Then $[\varepsilon, \beta\alpha] = [\varepsilon, \alpha]\ [\varepsilon, \beta]^{\alpha}$. Since $E_m(A)$ normalizes H, $[\varepsilon, \alpha] \in H$. Moreover part (b) of the theorem implies $[\varepsilon, \beta] \in E_m(A, \underline{q}) \subset H$. Thus $[\varepsilon, \beta\alpha] \subset H$. Hence we have

$$E_m(A, \underline{q}') = [E_m(A), E_m(A, \underline{q}')]$$
$$\subset [E_m(A), GL_m(A, \underline{q}) \cdot H] \subset H.$$

This contradicts the maximality of $\underline{q}$, and thus completes the proof of (c), modulo the:

Proof of (5.2)

Case 1. H contains a non central element $\sigma = u\alpha$ where $u \in U(\text{center }(A))$ and where

$$\alpha = \begin{pmatrix} 1 & 0 \\ x & \alpha_o \end{pmatrix} \in \text{Aff}_{m-1}(A) = \begin{pmatrix} 1 & 0 \\ A^{m-1} & GL_{m-1}(A) \end{pmatrix}.$$

Write A^{m-1} for the set of matrices $\tau(t) = \begin{pmatrix} 1 & 0 \\ t & I \end{pmatrix}$ $(t \in A^{m-1})$.

It suffices to show that $H \cap A^{m-1} \neq \{I\}$. For then (IV, 4.3 (c)) implies H contains all $\tau(t)$ such that t has coordinates in some left ideal $\underline{a} \neq 0$, and then (1.4) implies that H contains $E_m(A, \underline{a}A)$.

We have $[\tau(t), \sigma] = [\tau(t), \alpha] = \tau((\alpha_o - I)t)$ so we are done if $\alpha_o \neq I$. Otherwise, since σ is not central, we have $x \neq 0$, so there is an $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon_o \end{pmatrix} \in E_m(A)$ such that $\varepsilon_o(x) \neq x$. Therefore H contains $[\sigma, \varepsilon] = [\alpha, \varepsilon] = [\tau(x), \varepsilon]$ $= \tau((\varepsilon_o - I)(x)) \neq I$ in A^{m-1}.

Case 2. H contains a non central element σ with at least one off diagonal coordinate equal to zero.

After conjugation by an element of $E_m(A)$ we can assume σ has a zero in its first row, $\alpha = (a_1, \ldots, a_m)$, and even that $a_m = 0$. For $t \in A$ write $\tau(t) = I + te_{21}$. Then $\sigma^{-1}\tau(t)\sigma = I + \beta t\alpha$ where $\beta = {}^T(b_1, \ldots, b_m)$ is the second column of σ^{-1}. (The "T" denotes transpose). Suppose σ commutes with all $\tau(t)$. Setting $t = 1$ we find that $\alpha = (u,0, \ldots,0)$ and $\beta = {}^T(0, u^{-1},0,\ldots,0)$. Moreover, $u^{-1}tu = t$ for all t so $u \in U(\text{center }(A))$ and we are in case 1.

Therefore we can assume there is a $\tau = \tau(t)$ such that $\gamma = [\tau, \sigma] \neq I$. $\gamma = \tau^{-1} + \tau^{-1}\beta t\alpha$. Since $a_m = 0$ the last column of $\tau^{-1}\beta t\alpha$ is zero, so γ has the form $\gamma = \begin{pmatrix} \gamma_o & 0 \\ x & 1 \end{pmatrix}$. In particular γ is non central. Moreover ${}^T\gamma = \begin{pmatrix} {}^T\gamma_o & {}^Tx \\ 0 & 1 \end{pmatrix}$ lies in an $E_m(A)$-conjugate of $\mathrm{Aff}_{m-1}(A)$, so we can argue as in case 1 to conclude the proof.

<u>General Case</u>. Choose a non central σ in H, say with first column $\alpha = {}^T(a_1,\ldots,a_m)$. If $\varepsilon = I + \Sigma_{1\leq i<m} b_i e_{im}$ then $\varepsilon\sigma\varepsilon^{-1}$ has first column ${}^T(a_1 + b_1 a_m,\ldots,a_{m-1} + b_{m-1} a_m, a_m)$. Thanks to the hypothesis $SR_n(A)$, therefore, we can, after conjugating by an element of $E_m(A)$, assume that $(a_1, \ldots,a_{m-1})$ is unimodular. Then we can write $a_m = \Sigma s_i a_i$ $(1 \leq i < m)$. Setting $\varepsilon = I - \Sigma_{1\leq i<m} s_i e_{mi}$ now we have $\varepsilon\alpha = {}^T(a_1,\ldots,a_{m-1},0)$

Let $\tau = I + te_{12}$ and consider $\delta = \sigma\tau\sigma^{-1}\tau^{-1} = (I + \alpha t\beta)\tau^{-1}$, where β is the second row of σ^{-1}. As in case 2 we can choose t so that $\delta \neq I$, or else we are back in case 2. Now set $\sigma' = \varepsilon\delta\varepsilon^{-1} = \varepsilon\tau^{-1}\varepsilon^{-1} + \varepsilon\alpha t\beta\varepsilon^{-1}$. Since $\varepsilon\alpha$ has last coordinate 0 the bottom row of $(\varepsilon\alpha)(t\beta\varepsilon^{-1})$ is zero. Therefore σ' and $\varepsilon\tau^{-1}\varepsilon^{-1} = \tau^{-1} + \varepsilon te_{12}\varepsilon^{-1}$ have the same botton row. But $\varepsilon te_{12}\varepsilon^{-1} = (I + \Sigma_{i<m} s_i e_{mi}) te_{12} (I - \Sigma_{i<m} s_i e_{mi}) = (te_{12} + s_1 te_{m2})(I - \Sigma_{i<m} s_i e_{mi}) = te_{12} + s_1 te_{m2}$ (recall $m \geq 3$). Therefore σ' has last row $(0, s_1t, 0,\ldots,0,1)$. Since $\sigma' \neq I$ it cannot be central (it has a 1 on the diagonal) so we can apply case 2 to $\sigma' \in H$.

This finishes the proof of (5.2), and hence of Theorem (4.1).

§6. PROOF OF (4.2): I. THE CONSTRUCTION OF κ'.

For the next three sections we fix a ring A and a two sided ideal $\underline{q}$.

Recall from §3 that

$$\overline{E}_m(A, \underline{q})$$

is the group generated by $E_m(A, \underline{q})$ together with $[GE_m(A), GL_m(A, \underline{q})]$. Thus, for example, it follows from (1.5) that

(1) $\overline{E}_m(A, \underline{q}) = [GE_m(A), GL_m(A, \underline{q})]$ *if* $m \geq 3$.

On the other hand (4.1)' and (4.1) imply:

(2) *If* A *satisfies* $SR_n(A, \underline{q})$ *then, for all* $m \geq n$, $\overline{E}_m(A, \underline{q}) = E_m(A, \underline{q})$. *It is a normal subgroup of* $GL_m(A)$ *if* A *satisfies* $SR_n(A)$.

The proof of Theorem (4.2) will be organized around the following proposition, which we shall establish under suitable hypotheses:

$(6.1)_n$ *Given a homomorphism* $\kappa: GL_n(A, \underline{q}) \longrightarrow C$ *such that* $\overline{E}_n(A, \underline{q}) \subset Ker(\kappa)$, *there is a homomorphism* κ': $GL_{n+1}(A, \underline{q}) \longrightarrow C$ *extending* κ *and such that* $\overline{E}_{n+1}(A, \underline{q}) \subset Ker(\kappa')$.

We shall prove, in particular, the following:

(6.2) THEOREM. *If the conditions* $SR_n(A, \underline{q})$, $SR_n(A^o, \underline{q})$, and $SR'_{n-1}(A, \underline{q})$ *are satisfied then* $(6.1)_n$ holds.

Proof that (6.2) $\Rightarrow$ (4.2). For $m \geq n$ let κ_m: $GL_m(A, \underline{q}) \longrightarrow C_m(\underline{q}) = GL_m(A, \underline{q})/E_m(A, \underline{q})$ (cf. (2) above) be the natural projection. There are natural homomorphisms

$S_m: C_n(\underline{q}) \longrightarrow C_m(\underline{q})$ for $m \geq n$ and (4.1) (a) implies that they are surjective, and even that $GL_{n-1}(A, \underline{q}) \longrightarrow C_m(\underline{q})$ is surjective. To complete the proof of (4.2), therefore, we must show that each S_m is an isomorphism.

According to (6.2) we can apply $(6.1)_n$ to κ_n. The κ' so obtained clearly induces an inverse, $C_{n+1}(\underline{q}) \longrightarrow C_n(\underline{q})$, to S_{n+1}. Now we can finish, by induction on $m - n$, thanks to the fact that $SR_n(A) \Rightarrow SR_m(A)$ and $SR'_m(A)$ (see (3.3)) for all $m \geq n$, and similarly for A^o. q.e.d.

In the proof of $(6.1)_n$ all but the last stage of the argument will be carried out with hypotheses weaker than those of (6.2), and this added generality will be used in §9, as well as in Chapter VI.

Throughout §§6-8, $\kappa: GL_n(A, \underline{q}) \longrightarrow C$ <u>denotes a fixed homomorphism as in</u> $(6.1)_n$.

We shall say that an element of $GL_m(A)$ is of <u>type</u> L if its last row is $(0,\ldots,0,1)$, and of <u>type</u> R if its first column is ${}^T(1,0,\ldots,0)$. For example a type L looks like

$$\bar{\alpha} = \begin{pmatrix} \alpha & \gamma \\ 0 & 1 \end{pmatrix}$$

where $\alpha \in GL_{m-1}(A)$, ${}^T\gamma \in A^{m-1}$, etc. Similarly, we can write a type R in the form

$$\bar{\beta} = \begin{pmatrix} 1 & \rho \\ 0 & \beta \end{pmatrix}.$$

If $\sigma \in GL_m(A, \underline{q})$ we define a <u>standard form</u> for σ to be a factorization

$$\bar{\sigma} = \bar{\alpha}\, \varepsilon\, \bar{\beta}, \tag{3}$$

with all factors in $GL_m(A, \underline{q})$, where $\bar{\alpha}$ and $\bar{\beta}$ are of types L

R, respectively, as above, and where $\varepsilon = I + te_{m1}$ for some $t \in \underline{q}$. (In fact t must be the (m, 1) coordinate of σ).

These notions are, of course, only provisional. Their importance here is explained by the next proposition.

(6.3) PROPOSITION. (a) _Assuming_ $SR_{n+1}(A, \underline{q})$ _and_ $SR_n'(A, \underline{q})$, _every_ $\sigma \in GL_{n+1}(A, \underline{q})$ _has a standard form_ $\sigma = \overline{\alpha}\, \varepsilon\, \overline{\beta}$, _as above in_ (3).

Now further assume $SR_n''(A, \underline{q})$.

(b) _The map_

$$\kappa' : GL_{n+1}(A, \underline{q}) \longrightarrow C,$$

$\kappa'(\sigma) = \kappa(\alpha)\,\kappa(\beta)$ _if_ $\sigma = \overline{\alpha}\,\varepsilon\,\overline{\beta}$ _in standard form, is well defined and extends_ κ.

(c) _If_ $\overline{\alpha}_1$, $\overline{\beta}_1 \in GL_{n+1}(A, \underline{q})$ _are of types_ L _and_ R, _respectively, then_ $\kappa'(\overline{\alpha}_1\,\sigma\,\overline{\beta}_1) = \kappa'(\overline{\alpha}_1)\,\kappa'(\sigma)\,\kappa'(\overline{\beta}_1)$.

(d) _If there is a homomorphism_ $\kappa'' : GL_{n+1}(A, \underline{q}) \longrightarrow C$ _extending_ κ _and such that_ $E_{n+1}(A, \underline{q}) \subset \mathrm{Ker}(\kappa'')$ _then_ $\kappa'' = \kappa'$.

We shall prove $(6.1)_n$ by showing eventually (i.e. after strengthening the hypotheses of (6.3)) that the κ' above is a homomorphism killing $\overline{E}_{n+1}(A, \underline{q})$.

Proof. (a). Let $\sigma_1 = {}^T(a_1,\ldots,a_{n+1})$ be the first column of σ. Using $SR_{n+1}(A, \underline{q})$ we can find $\overline{\gamma} = \begin{pmatrix} I & -\gamma \\ 0 & 1 \end{pmatrix}$ $\in E_{n+1}(A, \underline{q})$ such that $\gamma\sigma_1 = {}^T(b_1,\ldots,b_n,a_{n+1})$ where $\sigma_1' = {}^T(b_1,\ldots,b_n)$ is $\underline{q}$-unimodular. By $SR_n'(A, \underline{q})$ there is an $\alpha \in GL_n(A, \underline{q})$ such that $\alpha^{-1}\sigma_1' = {}^T(1,0,\ldots,0)$. Set $\overline{\alpha}_1 =$

$\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}$. Then $\overline{\alpha}_1\ \overline{\gamma}\ \sigma_1 = {}^T(1,0,\ldots,0,a_{n+1})$; set $\varepsilon = I + a_{n+1}\, e_{n+1,1}$. Then $\overline{\beta} = \varepsilon^{-1}\ \overline{\alpha}_1\ \overline{\gamma}\ \sigma$ has first column ${}^T(1,0,\ldots,0)$, i.e. $\overline{\beta}$ is of type R. Then $\sigma = \overline{\alpha}\ \varepsilon\ \overline{\beta}$ where

$$\overline{\alpha} = \overline{\gamma}^{-1}\ \overline{\alpha}_1^{-1} = \begin{pmatrix} I & \gamma\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & \gamma\\ 0 & 1\end{pmatrix}.\quad \text{q.e.d.}$$

(b) Let $\sigma = \overline{\alpha}_1\ \varepsilon\ \overline{\beta}_1 = \overline{\alpha}_2\ \varepsilon\ \overline{\beta}_2$ be two standard forms for σ. (We have noted that ε is determined by (the $(n+1, 1)$ coordinate of) σ). Write

$$\overline{\alpha}_i = \begin{pmatrix}\alpha_i & \gamma_i\\ 0 & 1\end{pmatrix} \text{ and } \overline{\beta}_i = \begin{pmatrix}1 & \rho_i\\ 0 & \beta_i\end{pmatrix}\quad (i = 1, 2).$$

We must show that $\kappa(\alpha_1)\ \kappa(\beta_1) = \kappa(\alpha_2)\ \kappa(\beta_2)$. Since κ is a homomorphism this is equivalent to $\kappa(\alpha) = \kappa(\beta)$, where $\alpha = \alpha_1^{-1}\alpha_2$ and $\beta = \beta_1\beta_2^{-1}$. We shall deduce this from equation:

$$\overline{\alpha}\varepsilon = \varepsilon\overline{\beta};\ \overline{\alpha} = \overline{\alpha}_1^{-1}\overline{\alpha}_2 = \begin{pmatrix}\alpha & \gamma\\ 0 & 1\end{pmatrix},\ \overline{\beta} = \overline{\beta}_1\overline{\beta}_2^{-1} = \begin{pmatrix}1 & \rho\\ 0 & \beta\end{pmatrix}.$$

Write $\alpha = (a_{ij})$, $\beta = (b_{ij})$, $\gamma = {}^T(c_1,\ldots,c_n)$, and $\rho = (r_1,\ldots,r_n)$.

$$\overline{\alpha}\varepsilon = \begin{pmatrix} a_{11} + c_1 t & a_{12} & . & . & a_{1n} & c_1\\ . & . & & & . & .\\ . & . & & & . & .\\ a_{n1} + c_n t & a_{n2} & . & . & a_{nn} & c_n\\ t & 0 & . & . & 0 & 1\end{pmatrix}$$

and

$$\varepsilon\overline{\beta} = \begin{pmatrix} 1 & r_1 & \cdot & \cdot & \cdot & r_n \\ 0 & b_{11} & \cdot & \cdot & \cdot & b_{1n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & b_{n-1,1} & \cdot & \cdot & & b_{n-1,n} \\ t & b_{n1} + tr_1 & \cdot & \cdot & & b_{nn} + tr_n \end{pmatrix}.$$

Set $a = c_1$, which also equals r_n. Then we have

$$\alpha = \begin{pmatrix} 1-at & \alpha_{12} \\ -\alpha_{21}t & \alpha_{22} \end{pmatrix}$$

where $\alpha_{12} = (a_{12},\ldots,a_{1n}) = (r_1,\ldots,r_{n-1})$, $\alpha_{21} = {}^T(c_2,\ldots,c_n)$ and $a_{22} = (a_{ij})_{2\le i,j\le n} = (b_{ij})_{1\le i,j\le n-1}$. We therefore also have

$$\beta = \begin{pmatrix} \alpha_{22} & \alpha_{21} \\ -t\alpha_{12} & 1-ta \end{pmatrix}$$

Let $\pi = \begin{pmatrix} 0 & 1 \\ I_{n-1} & 0 \end{pmatrix} \varepsilon\ GE_n(A)$, the matrix of the permutation

$i \longmapsto i + 1 \pmod n$. Then

$$\beta^{\pi^{-1}} = \pi\ \beta\ \pi^{-1} = \begin{pmatrix} 1-ta & -t\alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

Thus α above is of type $(\underline{q}, t)$ (see (3.1)") and $\beta^{\pi^{-1}}$ is $(\underline{q}, t)$-related to α. The hypothesis $SR_n''(A, \underline{q})$ says that if α' is $(\underline{q}, t)$ related to α then $\alpha'\alpha^{-1} \varepsilon\ \overline{E}_n(A, \underline{q})$. Since $\overline{E}_n(A, \underline{q}) \subset \operatorname{Ker}(\kappa)$ (by hypothesis) this implies that $\kappa(\alpha)$ $\kappa(\alpha')$. In particular we have $\kappa(\beta^{\pi^{-1}}) = \kappa(\alpha)$. Since $\beta^{-1}\ \beta^{\pi^{-1}}$

$= [\beta, \pi^{-1}] \in \overline{E}_n(A, \underline{q})$ we further have $\kappa(\beta) = \kappa(\beta^{\pi^{-1}})$ and so $\kappa(\alpha) = \kappa(\beta)$. This shows that κ is well defined. If $\overline{\alpha} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ $\in GL_n(A, \underline{q})$ then $\overline{\alpha}$ is already of type L, so $\kappa'(\overline{\alpha}) = \kappa(\alpha)$, i.e. κ' extends κ. q.e.d.

(6.4) Remark. The last stage of the proof above is the only place in our arguments where the hypothesis $SR''_n(A, \underline{q})$ is used. For future reference we record the following observation, which is evident from the argument above: Proposition (6.3) remains valid if we replace the hypothesis $SR''_n(A, \underline{q})$ by the assumption that $\kappa(\alpha) = \kappa(\alpha')$ whenever, for some $t \in \underline{q}$, $\alpha \in GL_n(A, \underline{q})$ is of type $(\underline{q}, t)$ and α' is $(\underline{q}, t)$-related to α.

(c) Let $\sigma = \overline{\alpha} \in \overline{\beta}$ be a standard form for $\sigma \in GL_{n+1}(A, \underline{q})$, as above, and let

$$\overline{\alpha}_1 = \begin{pmatrix} \alpha_1 & \gamma_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{\beta}_1 = \begin{pmatrix} 1 & \rho_1 \\ 0 & \beta_1 \end{pmatrix}$$

be elements of $GL_{n+1}(A, \underline{q})$. Then $\overline{\alpha}_1 \sigma \overline{\beta}_1 = (\overline{\alpha}_1\overline{\alpha}) \in (\overline{\beta}\,\overline{\beta}_1)$ is a standard form, where

$$\overline{\alpha}_1\overline{\alpha} = \begin{pmatrix} \alpha_1\alpha & * \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{\beta\beta_1} = \begin{pmatrix} 1 & * \\ 0 & \beta\beta_1 \end{pmatrix}.$$

Hence $\kappa'(\overline{\alpha}_1 \sigma \overline{\beta}_1) = \kappa(\alpha_1\alpha)\, \kappa(\beta\beta_1) = \kappa(\alpha_1)\, \kappa(\alpha)\, \kappa(\beta)\, \kappa(\beta_1) = \kappa'(\overline{\alpha}_1)\, \kappa'(\sigma)\, \kappa'(\overline{\beta}_1)$. q.e.d.

(d) Let $\kappa'': GL_{n+1}(A, \underline{q}) \longrightarrow C$ be a homomorphism extending κ and killing $E_{n+1}(A, \underline{q})$. According to (4.1)' and our hypothesis $SR_{n+1}(A, \underline{q})$ we have $GL_{n+1}(A, \underline{q}) = E_{n+1}(A, \underline{q}) \cdot GL_n(A, \underline{q})$. Hence κ'' is uniquely determined by the conditions above. If $\sigma = \overline{\alpha} \in \overline{\beta}$ is a standard form, as above, then

$\kappa''(\varepsilon) = 1$. Since $\bar{\alpha} = \begin{pmatrix} I & \gamma \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ we also have $\kappa''(\bar{\alpha}) = \kappa(\alpha)$.

Finally, $\bar{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 1 & \rho \\ 0 & I \end{pmatrix}$, and the first factor is conjugate, by a permutation matrix, to $\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$. Therefore

$\kappa''(\bar{\beta}) = \kappa(\beta)$ because $[GE_{n+1}(A), GL_{n+1}(A, \underline{q})] \subset E_{n+1}(A, \underline{q})$ (cf (2) above)). This shows that $\kappa'' = \kappa'$, thus proving (d) and concluding the proof of (6.3).

§7. PROOF OF (4.2): II. THE NORMALIZER OF κ'

Let A^o be the opposite ring of A. Then transposition is a self inverse pair of antiisomorphisms

$$GL_m(A, \underline{q}) \underset{T}{\rightleftarrows} GL_m(A^o, \underline{q}).$$

Let C^o denote the opposite group of C (in $(6.1)_n$). Then we have a homomorphism κ^o: $GL_n(A^o, \underline{q}) \longrightarrow C^o$ defined by the commutativity of

$$\begin{array}{ccc} GL_n(A, \underline{q}) & \xrightarrow{\ T\ } & GL_m(A^o, \underline{q}) \\ \kappa \downarrow & & \downarrow \kappa^o \\ C & "=" & C^o . \end{array}$$

I.e. $\kappa^o(\alpha) = \kappa({}^T\alpha)$, as a set map. To avoid confusion we shall use a dot when writing products in $GL_m(A^o)$ or in C^o. E.g. if $x, y \in C$ then $x \cdot y = yx$.

Throughout this section we shall work with the following:

(7.1) HYPOTHESES. <u>The conditions</u> SR_{n+1}, SR_n', <u>and</u>

SR''_n _hold for both_ $(A, \underline{q})$ _and_ $(A^o, \underline{q})$.

These hypotheses make available all the conclusions of Proposition (6.3) for both κ and κ^o. Thus we have the κ' of (6.3) which extends κ, and the analogously defined (with the aid of standard forms in $GL_{n+1}(A^o, \underline{q})$ κ'^o extending κ^o. By virtue of the symmetry in our hypotheses all definitions and propositions concerning κ' have analogues for κ'^o.

It is important to note that: _The hypotheses of_ (6.2) _imply those of_ (7.1). For the hypotheses of (6.2) are $SR_n(A, \underline{q})$, $SR_n(A^o, \underline{q})$, and $SR_{n-1}(A, \underline{q})$. But, for all $m \geq n$, $SR_n \Rightarrow SR'_m$ (see (3.3) (1)) and $SR_n \Rightarrow SR''_m$ (see (3.3) (2)). In view of this observation all of the arguments of this section are legitimate contributions to the proof of (6.2). The stronger hypotheses of (6.2) will intervene only in §8 (see (8.1) (b) and (8.2)).

Finally we remark, for use in Chapter VI, that the hypothesis SR''_n above is present only to make the conclusions of (6.3) available. Therefore one can substitute for SR''_n the condition on κ described in (6.4), and then all the results of this section remain valid.

Consider the groups

$$H = \{\sigma \in GL_{n+1}(A, \underline{q}) \mid \kappa'(\sigma\sigma') = \kappa'(\sigma)\,\kappa'(\sigma') \text{ for all } \sigma' \in GL_{n+1}(A, \underline{q})\}$$

and

$$N = \{\tau \in GL_{n+1}(A) \mid \kappa'(\sigma^\tau) = \kappa'(\sigma) \text{ for all } \sigma \in GL_{n+1}(A, \underline{q})\}.$$

That they are groups follows from:

(7.2) LEMMA. (a) H _is a subgroup, containing all matrices of type_ L, _of_ $GL_{n+1}(A, \underline{q})$.

(b) N _is a subgroup of_ $GL_{n+1}(A)$ _and_ N _normalizes_ H.

(c) *Let* K *be a subgroup of* $GL_{n+1}(A, \underline{q})$ *containing all matrices of type* L *and normalized by* $E_{n+1}(A)$. *Then* K = $GL_{n+1}(A, \underline{q})$.

Proof. (a). If $\sigma \in H$ then $1 = \kappa'(I) = \kappa'(\sigma\sigma^{-1}) = \kappa'(\sigma)\, \kappa'(\sigma^{-1})$, so $\kappa'(\sigma^{-1}) = \kappa'(\sigma)^{-1}$. Now if $\sigma' \in GL_{n+1}(A, \underline{q})$ then $\kappa'(\sigma') = \kappa'(\sigma\sigma^{-1}\sigma') = \kappa'(\sigma)\, \kappa'(\sigma^{-1}\sigma')$, so $\kappa'(\sigma^{-1}\sigma') = \kappa'(\sigma)^{-1}\, \kappa'(\sigma') = \kappa'(\sigma^{-1})\, \kappa'(\sigma')$. This shows that $\sigma^{-1} \in H$. If $\sigma_1, \sigma_2 \in H$ then, for any σ' as above, $\kappa'(\sigma_1\sigma_2\sigma') = \kappa'(\sigma_1)\, \kappa'(\sigma_2\sigma') = \kappa'(\sigma_1)\, \kappa'(\sigma_2)\, \kappa'(\sigma) = \kappa'(\sigma_1\sigma_2)\, \kappa'(\sigma)$, so $\sigma_1\sigma_2 \in H$. Thus H is a group. That H contains all type L's follows from (6.3) (c).

(b). Let $\sigma \in GL_{n+1}(A, \underline{q})$. If $\tau \in N$ then $\kappa'(\sigma^{\tau^{-1}}) = \kappa'((\sigma^{\tau^{-1}})\tau) = \kappa'(\sigma)$, so $\tau^{-1} \in N$. If $\tau_1, \tau_2 \in N$ then $\kappa'(\sigma^{\tau_1\tau_2}) = \kappa'(\sigma^{\tau_1}) = \kappa'(\sigma)$, so $\tau_1\tau_2 \in N$. Thus N is a group. Suppose $\tau \in N$, and $\sigma_1 \in H$. Then $\kappa'(\sigma_1{}^{\tau}\,\sigma) = \kappa'((\sigma_1\,\sigma^{\tau^{-1}})^{\tau}) = \kappa'(\sigma_1\,\sigma^{\tau-1}) = \kappa'(\sigma_1)\, \kappa'(\sigma^{\tau^{-1}}) = \kappa'(\sigma_1{}^{\tau})\, \kappa'(\sigma)$, so $\sigma_1{}^{\tau} \in H$. Thus N normalizes H.

(c) Let $E = \{I + te_{12} \mid t \in \underline{q}\}$; clearly $E \subset K$ since all $I + te_{12}$ are of type L. As a subgroup of $GL_{n+1}(A, \underline{q})$ the group $GL_n(A, \underline{q})$ consists of matrices of type L. Moreover the normal subgroup of $E_{n+1}(A)$ generated by E is $E_{n+1}(A, \underline{q})$. Hence the hypotheses on K imply K contains $E_{n+1}(A, \underline{q}) \cdot GL_n(A, \underline{q})$. But, thanks to $SR_{n+1}(A, \underline{q})$, (4.1)´ implies the latter is all of $GL_{n+1}(A, \underline{q})$. q.e.d.

(7.3) COROLLARY. *If* $E_{n+1}(A) \subset N$ *then* κ' *is a homomorphism whose kernel contains* $E_{n+1}(A, \underline{q})$, *and hence* $(6.1)_n$ *is established*.

Proof. If $E_{n+1}(A) \subset N$ then (7.2) implies $H = GL_{n+1}(A, \underline{q})$, i.e. that κ' is a homomorphism. Moreover $\operatorname{Ker}(\kappa') \supset$

$\mathrm{Ker}(\kappa) \supset E_n(A, \underset{\sim}{q})$, and $\mathrm{Ker}(\kappa')$ is normalized by $E_{n+1}(A)$, so $\mathrm{Ker}(\kappa') \supset E_{n+1}(A, \underset{\sim}{q})$. q.e.d.

Because of this corollary the rest of our efforts will be spent trying to show that $E_{n+1}(A) \subset N$.

(7.4) LEMMA. N <u>contains all matrices of the form</u>

$$\tau = \begin{pmatrix} u & * & * \\ 0 & \gamma & * \\ 0 & 0 & v \end{pmatrix}$$

<u>with</u> $u, v \in U(A)$ <u>and</u> $\gamma \in GE_{n-1}(A)$, <u>provided</u> $n \geq 2$. <u>If</u> $n = 1$ <u>then</u> N <u>still contains</u> $D_2(A)$.

<u>Proof</u>. These matrices form a group, generated by those of the following types: $\tau_o = \mathrm{diag}(u_1,\ldots,u_{n+1}) \in D_{n+1}(A)$: $\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\gamma \in E_{n-1}(A)$; and $\tau_2 = I + te_{ij}$ with $t \in A$ and $(i, j) = (1, 2)$ or $(n, n + 1)$.

Let $\sigma \in GL_{n+1}(A, \underset{\sim}{q})$ have a standard form $\sigma = \overline{\alpha}\,\varepsilon\,\overline{\beta}$. Then, if $\tau = \tau_0$ or τ_1, it is easy to see that $\sigma^\tau = \overline{\alpha}^\tau\,\varepsilon^\tau\,\overline{\beta}^\tau$ is still a standard form. Moreover since $GE_n(A, \underset{\sim}{q})$ normalizes κ (this is one of the hypotheses on κ in $(6.1)_n$) it follows easily that $\kappa'(\sigma^\tau) = \kappa'(\sigma)$. This is a simple calculation which we leave to the reader.

Suppose next that $\tau = I + te_{12}$, say. Then τ is simultaneously of type L and of type R. Therefore $\overline{\alpha}^\tau$ and $\overline{\beta}^\tau$ are still of types L and R, respectively, and $\kappa'(\overline{\alpha}^\tau) = \kappa'(\overline{\alpha})$ and $\kappa'(\overline{\beta}^\tau) = \kappa'(\overline{\beta})$.

Now we invoke the assumption $n \geq 2$. If $\varepsilon = I + se_{n+1,1}$ then $\varepsilon^\tau = I + se_{n+1,1} + st\, e_{n+1,2} = \varepsilon\overline{\beta}_1$ where $\overline{\beta}_1 = I +$

st $e_{n+1,2}$ is of type R, and $\kappa'(\overline{\beta}_1) = 1$. Therefore $\sigma^\tau = \overline{\alpha}^\tau\ \varepsilon\ (\overline{\beta}_1\ \overline{\beta}^\tau)$ is a standard form for σ^τ, and we deduce from (6.3) (c) that $\kappa'(\sigma^\tau) = \kappa'(\overline{\alpha}^\tau)\ \kappa'(\overline{\beta}_1)\ \kappa'(\overline{\beta}^\tau) = \kappa'(\overline{\alpha})\ \kappa'(\overline{\beta}) = \kappa'(\sigma)$.

In case $\tau = I + te_{n,n+1}$ the argument is similar, except that this time we have $\varepsilon^\tau = \overline{\alpha}_1\ \varepsilon$ where $\overline{\alpha}_1$ is of type L and $\kappa'(\overline{\alpha}_1) = 1$. We omit the details.

Now we shall use the map $\kappa'^{o}: GL_{n+1}(A^o, \underline{q}) \longrightarrow C^o$ described at the beginning of this section. There is also the analogue of N,

$$N^o = \{\tau \in GL_{n+1}(A^o) \mid \kappa'^{o}(\sigma^\tau) = \kappa'^{o}(\sigma) \text{ for all } \sigma \in GL_{n+1}(A^o, \underline{q})\}.$$

Since our hypotheses on A and A^o are symmetric we can apply conclusions, proved for N, to N^o also.

Let $\phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and note that $\phi = {}^T\phi = \phi^{-1}$ $\in GE_{n+1}(A)$ (or $GL_{n+1}(A^o)$, as the case may be). For $\sigma \in GL_{n+1}(A)$ define

$$\tilde{\sigma} = \phi \cdot {}^T\sigma \cdot \phi = ({}^T\sigma)^\phi = {}^T(\sigma^\phi) \in GL_{n+1}(A^o).$$

Then $\sigma \longmapsto \tilde{\sigma}$ is an antiisomorphism. It exchanges the first and last rows and columns, and then transposes.

(7.5) LEMMA. <u>If</u> $\tau \in GL_{n+1}(A)$ <u>and</u> $\tilde{\tau} \in N^o$ <u>then</u> $\tau \in N$.

<u>Proof</u>. Let $\sigma = \overline{\alpha}\ \varepsilon\ \overline{\beta}$ be a standard form for $\sigma \in GL_{n+1}(A, \underline{q})$. Then we claim $\sigma = \tilde{\tilde{\beta}} \cdot \varepsilon \cdot \tilde{\tilde{\alpha}}$ is a standard form for σ in $GL_{n+1}(A^o, \underline{q})$. For if $\overline{\alpha} = \begin{pmatrix} \alpha & \gamma \\ 0 & 1 \end{pmatrix}$ then $\tilde{\tilde{\alpha}} = \begin{pmatrix} 1 & \gamma' \\ 0 & \alpha' \end{pmatrix}$ where α' is obtained from ${}^T\alpha$ by putting the first row and

column last. I.e. $\alpha' = ({}^T\alpha)^\pi$ where $\pi = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix} \in GE_n$.

Similarly $\tilde{\tilde{\beta}} = \begin{pmatrix} \beta' & \rho' \\ 0 & 1 \end{pmatrix}$, where $\beta' = ({}^T\beta)^{\pi^{-1}}$. Finally $\tilde{\varepsilon} = \varepsilon$ (recall ε is of the form $I + e_{n+1,1}$). Therefore we can compute: $\kappa'^o(\tilde{\sigma}) = \kappa^o(\beta') \cdot \kappa^o(\alpha')$. Since κ^o is normalized by ${}^TGE_n(A) = GE_n(A^o)$ we have $\kappa^o(\beta') = \kappa^o({}^T\beta)$ and $\kappa^o(\alpha') = \kappa^o({}^T\alpha)$. Therefore $\kappa'^o(\tilde{\sigma}) = \kappa^o({}^T\beta) \cdot \kappa^o({}^T\alpha) = \kappa(\alpha)\ \kappa(\beta) = \kappa'(\sigma)$.

Now if $\tilde{\tau} \in N^o$ then $\kappa'(\sigma^\tau) = \kappa'^o((\sigma^\tau)^\sim) = \kappa'^o(\tilde{\sigma}^{\tilde{\tau}}) = \kappa'^o(\tilde{\sigma}) = \kappa'(\sigma)$. Thus $\tau \in N$. q.e.d.

(7.6) LEMMA. <u>Assume</u> $n \geq 2$, <u>and set</u> $\pi = \begin{pmatrix} I_{n-1} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}$

<u>If</u> $\pi \in N$ <u>then</u> $E_{n+1}(A) \subset N$.

<u>Proof</u>. According to (7.4) N contains all $I + te_{ij}$ ($t \in A$, $i \neq j$, $i \neq n + 1$, $j \neq 1$). By symmetry, N^o contains all $I + te_{ij}$ ($t \in A^o$, $i \neq j$, $i \neq n + 1$, $j \neq 1$), so (7.5) implies N contains all $\tau = I + te_{ij}$ such that $\tilde{\tau}$ is of the above type.

Let $\tau_o = I + e_{n,n+1} \in N$. By assumption $\pi \in N$ so $\tau_o^{\ \pi} = I + e_{n+1,n} \in N$. If $\tau = I + te_{nj}$ ($j \neq 1, n, n + 1$) then

$$[\tau_o^{\ \pi}, \tau] = [I + e_{n+1,n}, I + te_{nj}] = I + te_{n+1,j} \in N.$$

Therefore we lack only the elementary matrices with off diagonal entry in the first column to generate $E_{n+1}(A)$. For $1 < j < n + 1$ we have $(I + te_{j1})^\sim = I + te_{n+1,j} \in N^o$, so

$I + te_{j1} \in N$ by the first paragraph above. Now we lack only the generators $I + te_{n+1,1}$. But we obtain these from the ones already obtained with the formula

$$[I + e_{n+1,n}, I + te_{n,1}] = I + te_{n+1,1}. \text{ q.e.d.}$$

Now we can summarize the conclusions of this section as follows:

(7.7) PROPOSITION. *Keep the hypotheses* (7.1) *and assume further that* $n \geq 2$ *and that*

(1) $\kappa'(\sigma^{\pi}) = \kappa'(\sigma)$ *for all* $\sigma \in GL_{n+1}(A, \underline{q})$,

where $\pi = \begin{pmatrix} I_{n-1} & & 0 \\ & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. *Then* κ' *is a homomorphism extending* κ *and* $E_{n+1}(A, \underline{q}) \subset \mathrm{Ker}(\kappa')$, *thus establishing* $(6.1)_n$.

This follows from (7.3) and (7.6). In the next section we shall conclude the proof of (4.2) by establishing (1) above under hypotheses somewhat stronger than those of (7.1). In Chapter VI we shall use (7.7) again, but in a setting where the stronger hypotheses are not available. It is for this reason that we have kept such careful track of our assumptions.

§8. PROOF OF (4.2): III. CONCLUSION

We assume $n \geq 2$, with π as in (7.7) we set

$$S = \{\sigma \in GL_{n+1}(A, \underline{q}) \mid \kappa'(\sigma^{\pi}) = \kappa'(\sigma)\}.$$

The task that remains for us is, according to (7.7), to show that $S = GL_{n+1}(A, \underline{q})$.

(8.1) LEMMA. *Let* $\sigma \in GL_{n+1}(A, \underline{q})$.

(a) *If* $\overline{\beta}_1 \in GL_{n+1}(A, \underline{q})$ *is of type* R *and if* $\overline{\alpha}_1 \in$

$GL_{n+1}(A, \underline{q})$ *is of the form*

$$\overline{\alpha_1} = \begin{pmatrix} \alpha_1' & \gamma_1' \\ 0 & I_2 \end{pmatrix}$$

(*where* $\alpha_1' \in GL_{n-1}(A, \underline{q})$) *then* $\sigma \in S \iff \overline{\alpha_1}\ \sigma\ \overline{\beta_1} \in S$.

(b) *We can choose* $\overline{\alpha_1}$ *and* $\overline{\beta_1}$ *as above so that* $\overline{\alpha_1}\ \sigma\ \overline{\beta_1}$ $= \overline{\alpha}\varepsilon$ *where* $\varepsilon = I + ae_{n+1,1}$ and where $\overline{\alpha} = \begin{pmatrix} \alpha & \gamma \\ 0 & 1 \end{pmatrix}$ *is of type* L, *with* $\gamma = {}^T(0,..,0,c)$. *Assuming* $SR_n(A, \underline{q})$ *we can arrange that the first column,* ${}^T(a_1,...,a_n)$, *of* α *is such that* $(a_1,...,a_{n-1})$ *is unimodular. If, further, we assume* $SR_{n-1}(A, \underline{q})$ *we can arrange that* $(a_1,...,a_{n-1}) = (1,0,..0)$.

Proof. (a). With $\overline{\alpha_1}$ and $\overline{\beta_1} = \begin{pmatrix} 1 & \rho_1 \\ 0 & \beta_1 \end{pmatrix}$ as above,

$\overline{\alpha_1}^{\pi} = \begin{pmatrix} \alpha_1' & \gamma_1'' \\ 0 & I_2 \end{pmatrix}$ is still of type L, and $\overline{\beta_1}^{\pi} = \begin{pmatrix} 1 & \rho_1' \\ 0 & \beta_1^{\pi_1} \end{pmatrix}$ is still of type R. Hence, by (6.3) (c), $\kappa'((\overline{\alpha_1}\ \sigma\ \overline{\beta_1})^{\pi}) = \kappa'(\overline{\alpha_1}^{\pi})\ \kappa'(\sigma^{\pi})\ \kappa'(\overline{\beta_1}^{\pi})$, and it is clear that $\kappa'(\overline{\alpha_1}^{\pi}) = \kappa'(\overline{\alpha_1})$ and $\kappa'(\overline{\beta_1}^{\pi}) = \kappa'(\overline{\beta_1})$. This proves (a).

(b) If $\sigma = \overline{\alpha}\ \varepsilon\ \overline{\beta}$ in standard form we first take $\overline{\beta_1} = \overline{\beta}^{-1}$. It remains to be seen how we can modify $\overline{\alpha} = \begin{pmatrix} \alpha & \gamma \\ 0 & 1 \end{pmatrix}$ with left multiplication by an $\overline{\alpha_1}$ as above. The matrices of the latter type are clearly a group, so we are at liberty to perform a succession of such left multiplications.

Say $\gamma = {}^T(c_1,...,c_n)$. Then left multiplication by

$\begin{pmatrix} I & -\gamma' \\ 0 & 1 \end{pmatrix}$, where $\gamma' = {}^T(c_1,\ldots,c_{n-1},0)$, is admissible. The result is to replace γ by ${}^T(0,\ldots,0,c)$, where $c = c_n$, and to leave α unaltered. Therefore we can achieve the required form for γ, and even if it is upset by the operations to follow on α, we can restore it without harm to the work done on α.

Let $\beta = {}^T(a_1,\ldots,a_n)$ be the first column of α. Assuming $SR_n(A, \underline{q})$ we can find an $\alpha_1 = \begin{pmatrix} I_{n-1} & \gamma_1 \\ 0 & 1 \end{pmatrix}$ $E_n(A, \underline{q})$ such that $\alpha_1\beta = {}^T(a_1',\ldots,a_{n-1}',a_n)$ with $(a_1',\ldots,a_{n-1}')$ unimodular. After left multiplication by $\overline{\alpha}_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 1 \end{pmatrix}$, which is admissible, we can therefore assume $(a_1,\ldots,a_{n-1})$ is unimodular. Assuming $SR_{n-1}'(A, \underline{q})$ we can now further find α_2 $GL_{n-1}(A, \underline{q})$ such that $\alpha_2{}^T(a_1,\ldots,a_{n-1}) = {}^T(1,0,\ldots,0)$. Therefore left multiplication by $\overline{\alpha}_2 = \begin{pmatrix} \alpha_2 & 0 \\ 0 & I_2 \end{pmatrix}$ achieves the last condition indicated in part (b) for α, thus completing the proof of (b). q.e.d.

(8.2) LEMMA. Suppose σ $GL_{n+1}(A, \underline{q})$ has the form $\sigma = \overline{\alpha}\varepsilon$ where $\varepsilon = I + ae_{n+1,n}$ $(a \in \underline{q})$ and

$$\overline{\alpha} = \begin{pmatrix} \alpha & \gamma \\ 0 & 1 \end{pmatrix}$$

with $\gamma = {}^T(0,\ldots,0,c)$ *and* $\alpha = \begin{pmatrix} 1 & a_{12} & \cdot & \cdot & a_{1n} \\ 0 & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & & & & \\ b & a_{n2} & & & a_{nn} \end{pmatrix}$. *Then* $\sigma \in S$, *i.e.* $\kappa'(\sigma^\pi) = \kappa'(\sigma)$.

Let us first note that *(8.2) completes the proof of (6.2), and hence also of (4.2)*. For we have already noted that (6.2) $\Rightarrow$ (4.2) and that the hypotheses of (6.2) imply the hypotheses (7.1). The hypothesis $SR_n(A, \underline{q})$ can be fulfilled only for $n \geq 2$, so this restriction on n above is innocent. Moreover, all of the hypotheses in (8.1) (b) above are among those of (6.2). Hence, by (8.1), to prove $S = GL_{n+1}(A, \underline{q})$, if suffices to show that $\sigma \in S$ if σ is of the form presented in (8.2). Therefore (8.2) will, indeed, complete the proof of (6.2). q.e.d.

Proof of (8.2). By definition of κ', $\kappa'(\sigma) = \kappa(\alpha)$. To save some writing we shall put these matrices in block form:

$$\alpha = \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ b & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad \text{where } \alpha_{22} = (a_{ij})_{2 \leq i,j \leq n-1},$$

and the rest of the notation is clear. Set

$$\alpha' = \begin{pmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} - b\alpha_{12} & \alpha_{33} - b\alpha_{13} \end{pmatrix}.$$

Then $\alpha \equiv \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & \multicolumn{2}{c}{\alpha'} \\ 0 & & \end{pmatrix} \equiv \begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix}$ mod $\overline{E}_n(A, \underline{q})$, so

$$(*) \qquad \kappa'(\sigma) = \kappa(\alpha) = \kappa\begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix}.$$

Next we display $\sigma = \overline{\alpha}\varepsilon =$

$$\begin{pmatrix} \alpha & \gamma \\ 0 & 1 \end{pmatrix} (I + ae_{n+1,1}) = \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} & 0 \\ 0 & \alpha_{22} & \alpha_{23} & 0 \\ b+ca & \alpha_{32} & \alpha_{33} & c \\ a & 0 & 0 & 1 \end{pmatrix}.$$

Since $\sigma \longmapsto \sigma^{\pi}$ just exchanges the last two rows and the last two columns we have

$$\sigma^{\pi} = \begin{pmatrix} 1 & \alpha_{12} & 0 & \alpha_{13} \\ 0 & \alpha_{22} & 0 & \alpha_{23} \\ a & 0 & 1 & 0 \\ d & \alpha_{32} & c & \alpha_{33} \end{pmatrix} \qquad (d = b + ca).$$

We can write $\sigma^{\pi} = (I + ae_{n,1})(I + de_{n+1,1})\overline{\beta}$, in standard form, where

$$\overline{\beta} = \begin{pmatrix} 1 & \rho \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & \alpha_{12} & 0 & \alpha_{13} \\ 0 & \alpha_{22} & 0 & \alpha_{23} \\ 0 & -a\alpha_{12} & 1 & -a\alpha_{13} \\ 0 & \alpha_{32}-d\alpha_{12} & c & \alpha_{33}-d\alpha_{13} \end{pmatrix}$$

Therefore we have

$$\kappa'(\sigma^{\pi}) = \kappa(I_n + ae_{n,1})\,\kappa(\beta) = \kappa(\beta).$$

To compute $\kappa(\beta)$ we can replace β by anything to which it is congruent modulo $E_n(A, \underline{q})$. The congruences which follow are all modulo $\overline{E}_n(A, \underline{q})$.

$$\beta = \begin{pmatrix} \alpha_{22} & 0 & \alpha_{23} \\ -a\alpha_{12} & 1 & -a\alpha_{13} \\ \alpha_{32}-d\alpha_{12} & c & \alpha_{33}-d\alpha_{13} \end{pmatrix}$$

$$\equiv \begin{pmatrix} \alpha_{22} & 0 & \alpha_{23} \\ -a\alpha_{12} & 1 & -a\alpha_{13} \\ \alpha_{32}+(ca-d)\alpha_{12} & 0 & \alpha_{33}+(ca-d)\alpha_{13} \end{pmatrix}$$

Recall that $d = b + ca$ so $ca - d = -b$. Therefore

$$\beta \equiv \begin{pmatrix} \alpha_{22} & 0 & \alpha_{23} \\ -a\alpha_{12} & 1 & -a\alpha_{13} \\ \alpha_{32}-b\alpha_{12} & 0 & \alpha_{33}-b\alpha_{13} \end{pmatrix}$$

$$\equiv \begin{pmatrix} \alpha_{22} & 0 & \alpha_{23} \\ 0 & 1 & 0 \\ \alpha_{32}-b\alpha_{12} & 0 & \alpha_{33}-b\alpha_{13} \end{pmatrix} \equiv \begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix},$$

where $\alpha' = \begin{pmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32}-b\alpha_{12} & \alpha_{33}-b\alpha_{13} \end{pmatrix}$ is the same α' that appears in (*) above. It follows from (*) therefore that $\kappa'(\sigma^{\pi}) =$

$$\kappa(\beta) = \kappa \begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix} = \kappa'(\sigma). \text{ q.e.d.}$$

§9. SEMI-LOCAL RINGS

(9.1) THEOREM. Let $\underline{q}$ be a two sided ideal in a ring A. Assume either that A is semi-local or that $\underline{q} \subset$ rad A. Then

(1) $\qquad U(A, \underline{q}) \longrightarrow K_1(A, \underline{q})$

is surjective, and, for all $m \geq 2$,

$$GL_m(A, \underline{q})/E_m(A, \underline{q}) \longrightarrow K_1(A, \underline{q})$$

is an isomorphism. Moreover $[GL_m(A), GL_m(A, \underline{q})] \subset E_m(A, \underline{q})$, *with equality for* $m \geq 3$.

Proof. The conclusions above are just the conclusions of (4.2) and (4.1) (b) in the case $n = 2$. Therefore we need only verify the relevant hypotheses: $(A, \underline{q})$ and $(A^o, \underline{q})$ satisfy SR_2 and SR_1'. These both follow from (3.4). q.e.d.

(9.2) COROLLARY. *Suppose that* A *above is commutative. Then* (1) *is an isomorphism*,

$$E_n(A, \underline{q}) = SL_n(A, \underline{q}) \quad \text{for all } n \geq 1,$$

and $SK_1(A, \underline{q}) = 0$.

Proof. The determinant induces the inverse, det: $K_1(A, \underline{q}) \longrightarrow U(A, \underline{q})$, to (1). In particular, if $\alpha \in GL_n(A, \underline{q})$ and $\det(\alpha) = 1$ then $\alpha \in E_n(A, \underline{q})$, i.e. $SL_n(A, \underline{q}) \subset E_n(A, \underline{q})$. The opposite inclusion is trivial. Finally, $SK_1(A, \underline{q}) = SL(A, \underline{q})/E(A, \underline{q}) = 0$. q.e.d.

(9.3) COROLLARY. *Let* A *be a commutative ring and let* $\underline{q}$ *and* $\underline{q}'$ *be ideals such that* $A/\underline{q}$ *is semi-local. Then, for* all $n \geq 1$,

$$SL_n(A, \underline{q} + \underline{q}') = E_n(A, \underline{q}') \cdot SL_n(A, \underline{q}),$$

and

$$E_n(A, \underline{q}') \longrightarrow SL_n(A/\underline{q}, \underline{q} + \underline{q}'/\underline{q})$$

is surjective. In particular,

$$E_n(A) \longrightarrow SL_n(A/\underline{q})$$

is surjective. Moreover $SK_1(A, \underline{q}) \longrightarrow SK_1(A, \underline{q} + \underline{q}')$ *is surjective*.

Proof. It follows from (1.1) that $E_n(A, \underline{q}') \longrightarrow$

$E_n(A/\underline{q}, \underline{q} + \underline{q}'/\underline{q})$ is surjective. Since $A/\underline{q}$ is semi-local (9.2) implies the latter group equals $SL_n(A/\underline{q}, \underline{q} + \underline{q}'/\underline{q})$. Taking inverse images modulo $SL_n(A, \underline{q})$ this implies $SL_n(A, \underline{q} + \underline{q}') = E_n(A, \underline{q}') \cdot SL_n(A, \underline{q})$, and hence $SK_1(A, \underline{q}) \longrightarrow SK_1(A, \underline{q} + \underline{q}')$ is surjective. In case $\underline{q}' = A$ we see also that $E_n(A) \longrightarrow SL_n(A/\underline{q})$ is surjective. q.e.d.

In terms of the general theorems of this chapter the results above represent, in some sense, their most effective case. Nevertheless there remain, even here (i.e. in the setting of (9.1)) a few loose ends:

(i) When is the inclusion $[GL_2(A), GL_2(A, \underline{q})] \subset E_2(A, \underline{q})$ an equality?

(ii) What is the kernel of the epimorphism $U(A, \underline{q}) \longrightarrow K_1(A, \underline{q})$ in (1) (when A is not commutative)?

(iii) What are the normal subgroups of $GL_n(A, \underline{q})$ for $n = 1$ and $n = 2$?

In connection with (i) we can deduce certain information from the commutator formula,

$$(2) \qquad \left[\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}\right] = \begin{pmatrix} 1 & v^{-1}tu-t \\ 0 & 1 \end{pmatrix}.$$

(9.4) PROPOSITION. _Let_ A _be a commutative ring and let_ $\underline{q}$ _and_ $\underline{q}'$ _be ideals in_ A. _Let_ $\underline{q}_1$ _and_ $\underline{q}_2$ _denote the ideals generated by_ $\{1 - u\}$, _resp._ $\{1 - u^2\}$, _where_ u _ranges over_ $U(A, \underline{q})$. _Then_

$$[GE_2(A, \underline{q}), E_2(A, \underline{q}')] \supset E_2(A, \underline{q}_1\underline{q}')$$

and

$$[E_2(A, \underline{q}), E_2(A, \underline{q}')] \supset E_2(A, \underline{q}_2\underline{q}').$$

Proof. Formula (2) (with $v = 1$), and its transpose, shows that $[GE_2(A, \underline{q}), E_2(A, \underline{q}')]$ contains all elementary matrices of the form $I + t(1 - u)\, e_{ij}$ with $t \in \underline{q}'$ and $u \in$

$U(A, \underline{q})$. The $E_2(A)$-normalized subgroup genererated by these is clearly $E_2(A, \underline{q}_1\underline{q}')$, so the latter is contained in $[GL_2(A, \underline{q}), E_2(A, \underline{q}')]$. The second inclusion follows from (2) similarly, in the case $v = u^{-1}$. It follows from the Whitehead Lemma (1.7) that $\operatorname{diag}(u^{-1}, u) \in E_2(A, \underline{q})$. q.e.d.

Of course one can obtain similar conclusions when A is not commutative, but we will not pursue the matter.

Of much greater interest are questions (ii) and (iii). In case A is a division ring there are essentially complete results, due to Dieudonné. We quote the answer to (ii) (see Artin [1], Chapter V)).

(9.5) THEOREM (Dieudonné). _Let_ A _be a division ring. Then the kernel of_ $U(A) \longrightarrow K_1(A)$ _is the commutator subgroup_ $[U(A), U(A)]$. _The commutator factor group of_ $GL_n(A)$ _is isomorphic to_ $K_1(A)$ _for all_ $n \geq 1$ _except, for_ $n = 2$, _when_ A _is the field of two elements_.

More generally, if A is a local ring, then results of this type have been proved by Klingenberg [1]. We shall treat only the following case which, unfortunately, does not cover Dieudonné's Theorem.

(9.6) THEOREM. _Let_ $f: U(A, \underline{q}) \longrightarrow K_1(A, \underline{q})$ _be the epimorphism_ (1) _in Theorem_ (9.1). _Let_ E _denote the subgroup of_ $U(A, \underline{q})$ _generated by_ $[U(A), U(A, \underline{q})]$ _together with all elements of the form_ $(1 + ts)(1 + st)^{-1}$, _where_ $s, t \in \underline{q}$ _and_ $1 + st \in U(A)$. _Then_ $E \subset \operatorname{Ker}(f)$. _Assume that_ A _is generated by_ U(A) _as an algebra over_ R = center(A). _Assume further that_ $\underline{q} \subset \operatorname{rad} A$. _Then_ $E = \operatorname{Ker}(f)$.

Remark. The hypothesis that $A = R[U(A)]$ is quite innocent; for example any local ring satisfies it. The undesirable hypothesis is that $\underline{q} \subset \operatorname{rad} A$. In fact the proof is arranged so that this hypothesis is invoked only at the very last step. I lacked the patience to work out the details in the general case. Let me indicate, at least, that Dieudonné's theorem would follow from (9.6) if the restriction on $\underline{q}$ were dropped. For suppose A is a division ring. First the group E equals $[U(A), U(A)]$. For if $1 + st \in U(A)$ (we may assume $t \neq 0$) then $1 + ts = t(1 + st)t^{-1}$. Finally, if A has more than two elements then formula (2) above shows that $[GL_2(A), GL_2(A)] \supset E_2(A)$; hence (9.1) implies $E_n(A) = [GL_n(A), GL_n(A)]$ for all $n \geq 2$.

Proof of (9.6). Since $K_1(A, \underline{q}) = GL(A, \underline{q})/[GL(A), GL(A, \underline{q})]$ we evidently have $[U(A), U(A, \underline{q})] \subset Ker(f)$. If $s, t \in \underline{q}$ and if $1 + st \in U(A)$ then $1 + st$ is of type $(\underline{q}, t)$ (see (3.1)") and $1 + ts$ is $(\underline{q}, t)$-related to $(1 + st)$. Since we have condition $SR_2(A, \underline{q})$ it follows from (3.3) (b) that $(1 + ts)(1 + st)^{-1}$ belongs to $E_2(A, \underline{q})$, and hence also to Ker(f). Thus $E \subset Ker(f)$.

Let $\kappa: U(A, \underline{q}) \longrightarrow C = U(A, \underline{q})/E$ be the natural projection. If we can show that κ extends to a homomorphism $\kappa': GL_2(A, \underline{q}) \longrightarrow C$ such that $E(A, \underline{q}) \subset Ker(\kappa')$ then κ' will induce an inverse to the obvious homomorphism $C \longrightarrow GL_2(A, \underline{q})/E_2(A, \underline{q})$. Thus the theorem will be proved by virtue of (9.1).

We saw in the proof of (9.1) that $(A, \underline{q})$ satisfies SR_2 and SR_1''. Moreover the definiton of E shows that $\kappa(\alpha) = \kappa(\alpha')$ whenever, for some $t \in \underline{q}$, $\alpha \in U(A, \underline{q})$ if of type $(\underline{q}, t)$, and α' is $(\underline{q}, t)$-related to α. Therefore (see Remark (6.4)) we can apply (6.3) to obtain a well defined extension, $\kappa': GL_2(A, \underline{q}) \longrightarrow C$, of κ, defined with the aid of "standard forms" in $GL_2(A, \underline{q})$. Precisely, every $\sigma \in GL_2(A, \underline{q})$ can be factored in $GL_2(A, \underline{q})$ in the form

$$(*) \qquad \sigma = \begin{pmatrix} \alpha & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & b \end{pmatrix},$$

and then $\kappa'(\sigma) = \kappa(a)\,\kappa(b)$.

Let $N = \{\tau \in GL_2(A) \mid \kappa'(\sigma^\tau) = \kappa'(\sigma)$ for all $\sigma \in GL_2(A, \underline{q})\}$. It follows from (7.3) that, if the group N contains $E_2(A)$, then κ' is a homomorphism whose kernel contains $E_2(A, \underline{q})$. Thus the proof will be complete if we show that $E_2(A) \subset N$. We further note that (7.4) implies $D_2(A) \subset N$.

Now we claim, under the assumption that $A = R[U(A)]$, that the group H generated by $D_2(A)$ together with $\{\varepsilon(s) = I + se_{12} \mid s \in R\}$ and $\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is all of $GE(A)$. For if formula (2) above shows that $M = \{t \mid \varepsilon(t) \in H\}$ contains all us with $u \in U(A)$, $s \in R$. But these additively generate A so $M = A$. Since $\pi\,\varepsilon(t)\,\pi^{-1} = I + te_{21}$ we see that $H \supset E_2(A)$, as well as $D_2(A)$, so $H = GE_2(A)$ as claimed.

In view of what has been said the theorem will be proved if we show that $\pi \in N$ and that $\varepsilon(s) \in N$ for all $s \in R$.

If σ has the standard form $\sigma = \overline{\alpha}\,\varepsilon\,\overline{\beta}$ as in (*) above then, since $\varepsilon(s)$ is both of type L and of type R, it follows easily from (6.3) (c) that

$$\begin{aligned}\kappa'(\sigma^{\varepsilon(s)}) &= \kappa'(\overline{\alpha}^{\varepsilon(s)}\ \varepsilon^{\varepsilon(s)}\ \overline{\beta}^{\varepsilon(s)}) \\ &= \kappa'(\overline{\alpha}^{\varepsilon(s)})\ \kappa'(\varepsilon^{\varepsilon(s)})\ \kappa'(\overline{\beta}^{\varepsilon(s)}) \\ &= \kappa(a)\kappa'(\varepsilon^{\varepsilon(s)})\ \kappa(b).\end{aligned}$$

Therefore it suffices to show, in this case, that $\kappa'(\varepsilon^{\varepsilon(s)}) = 1$. Let B be a commutative subring of A containing $R[t]$ and such that $U(A) \cap B = U(B)$; e.g. any maximal commutative subring has this property. Let $\underline{q}_o = \underline{q} \cap B$, and suppose we know that $(B, \underline{q}_o)$ satisfies the same hypotheses that we have made on $(A, \underline{q})$. Then we have a map $\kappa'': GL_2(B, \underline{q}_0) \longrightarrow U(B, \underline{q}_0)$ analogous to κ', and evidently $\kappa'(\varepsilon^{\varepsilon(s)}) = h\kappa''(\varepsilon^{\varepsilon(s)})$, where $h: U(B, \underline{q}_0) \longrightarrow C$ is the inclusion $U(B, \underline{q}_0) \subset U(A, \underline{q})$ followed by κ. Since B is commutative it follows from (9.2) that $\kappa'' = \det$, and manifestly $\det(\varepsilon^{\varepsilon(s)}) = \det(\varepsilon) = 1$.

To check the hypotheses on $(B, \underline{q}_0)$ first note that $\text{rad } A \cap B \subset \text{rad } B$. For if $b \in B$ and $b \equiv 1 \text{ mod } (\text{rad } A \cap B)$ then $b \in U(A) \cap B = U(B)$. Therefore if $\underline{q} \subset \text{rad } A$ we have $\underline{q}_0 \subset \text{rad } B$, as required. (Note also that if R is semi-local and if A is a finite R-algebra, then it is easy to show that B is also semi-local). Again, if A is local then B must be also. Hence the proof so far works under any of these hypotheses.

Finally, we complete the proof by showing that $\pi \in N$. We will show that $\kappa'(\sigma^\pi) = \kappa'(\sigma)$ if $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A, \underline{q})$ is such that $a, d \in U(A, \underline{q})$. This last condition is automatic if $\underline{q} \subset \text{rad } A$, of course. With it we have the standard forms,

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & d-ca^{-1}b \end{pmatrix}$$

$$= \begin{pmatrix} a-bd^{-1}c & bd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix},$$

and so

$$\kappa'(\sigma) = \kappa(ad - aca^{-1}b) = \kappa(ad - bd^{-1}cd).$$

Similarly, $\sigma^\pi = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$, so

$$\kappa'(\sigma^\pi) = \kappa(d \quad - dbd^{-1}c).$$

But $d^{-1}(d\alpha - dbd^{-1}c)d = \alpha d - bd^{-1}cd$, and thus $\kappa'(\sigma^{\pi}) = \kappa'(\sigma)$. q.e.d.

Let A be a semi-simple finite algebra over a field, L, and let C denote the center of A. Then C is the product the centers of the simple factors of A, and these factors of C are finite field extensions of L. Recall from (III, §8, discussion preceding (8.5)) that there is a reduced norm homomorphism

$$\mathrm{Nrd} = \mathrm{Nrd}_{A/C}\colon U(A) \longrightarrow U(C).$$

It is defined as the product of the reduced norms in the simple factors, it is stable under an extension of the base field (which preserves semi-simplicity), and it is the ordinary determinant when $A = M_n(C)$ for some $n > 0$. These properties characterize it. The last one implies that it has the same stability properties as det. Explicitly, suppose $n > 0$. Then $M_n(A)$ is semi-simple with center C, so we have

$$\text{"det"} = \mathrm{Nrd}_{M_n(A)/C}\colon GL_n(A) \longrightarrow U(C),$$

whose kernel we shall denote by $SL_n(A)$, the elements of reduced norm one. If $\alpha \in GL_n(A)$ and $\beta \in GL_m(A)$ then

$$\det \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \det\alpha \ \det\beta.$$

In particular $\det(\alpha \oplus I_m) = \det(\alpha)$, so we obtain

$$\det\colon GL(A) \longrightarrow U(C).$$

This is a homomorphism into an abelian group, so its kernel, which we shall denote SL(A) contains E(A). Thus we have an exact sequence

$$0 \longrightarrow SK_1(A) \longrightarrow K_1(A) \xrightarrow{\det} U(C)$$

where $SK_1(A) = SL(A)/E(A)$.

<u>Problem</u>. Is $SK_1(A) = 0$?

The answer is yes if A is commutative and, more generally, if A is a product of full matrix algebras over fields (not just division rings). If A is simple then it is known that there is a field extension C′ of C such that $A \otimes_C C' \simeq M_n(C')$, where $[A: C] = n^2$. We have just noted that $SK_1(A \otimes_C C') = 0$. From this one can deduce easily (cf (IX, 4.7)).that $SK_1(A)$ is a torsion group of exponent $[C': C]$. While no examples are known for which the answer to the question above is negative the only positive result of any generality is the following theorem of Wang [1]. Wang's proof, which we omit, uses rather deep theorems from number theory.

(9.7) THEOREM (Wang). *Let* A *be a semi-simple finite algebra over a number field. Then, for all* $n \geq 1$, *the group* $SL_n(A)$, *of elements of reduced norm one in* $GL_n(A)$, *coincides with the commutator subgroup of* $GL_n(A)$. *In particular* $SK_1(A) = 0$.

Remark. By virtue of Dieudonné's theorem the last assertion is equivalent to the first.

§10. CRITERIA FOR FINITE GENERATION

In Chapter X we will prove theorems stating, in some circumstances, that $K_1(A, \underline{q})$ is finitely generated. With the aid of some purely group theoretic facts this can sometimes be deduced from the finite generation of $K_1(A)$. This section records some of these propositions from group theory.

(10.1) PROPOSITION. *Let* G *be a group and let* H *be a subgroup of finite index*.

(a) H *has only a finite number of conjugates in* G, *and their intersection is a normal subgroup of finite index*.

(b) G *is finitely generated if and only if* H *is*.

Proof. (a). The number of conjugates of H is [G: N], where N is the normalizer of H, and hence if finite. For the rest it suffices to show that $H \cap H'$ has finite index in G if H and H′ do. G acts as permutations of the cosets, G/H, as well as G/H′. Therefore we have a homomorphism $G \longrightarrow$ Permutations of $((G/H) \times (G/H'))$ whose kernel is clearly in $H \cap H'$, and has finite index in G.

(b) Let X be a set of generators for G and let C be a set of coset representatives for G/H containing 1. If $\alpha \in G$ there is a unique factorization $\alpha = c(\alpha)\, h(\alpha)$ with $c(\alpha) \in C$ and $h(\alpha) \in H$. Let H_0 be the subgroup of H generated by

$$Y = \{h(x^{\pm 1}c) \mid x \in X,\ c \in C\}.$$

We claim $H_0 = H$. If X is finite then so is Y (because C is) so this will imply H is finitely generated if G is. The converse is trivial because H and C generate G.

If $\alpha \in G$ then α is a product of elements $x^{\pm 1}$ $(x \in X)$. To show that $h(\alpha) \in H_0$ we can use induction on the number n of such factors. If $n = 1$ then $h(\alpha) \in Y \subset H_0$. It suffices now to show that, if $h(\alpha) \in H_0$, and if $y = x^{\pm 1}$ for some $x \in X$, then $h(y\alpha) \in H_0$. But $y\alpha = yc(\alpha)\, h(\alpha) = c(yc(\alpha))\, h(yc(\alpha))\, h(\alpha)$, so $h(y\alpha) = h(yc(\alpha))\, h(\alpha)$. We have $h(\alpha) \in H_0$, by assumption, and $h(yc(\alpha)) \in Y$, so $h(y\alpha) \in H_0$. q.e.d.

(10.2) PROPOSITION. _Let_

(1) $$1 \longrightarrow G' \longrightarrow G \xrightarrow{\ p\ } G'' \longrightarrow 1$$

be a group extension (i.e. exact sequence of groups) and assume G″ _is finite. Then_

(2) $$G'/[G, G'] \longrightarrow G/[G, G]$$

has finite kernel and cokernel.

We will not prove this here, but simply indicate that it follows from an exact sequence, due to Schur, which occurs as the "exact sequence of low order terms" in the Hochschild-Serre spectral sequence of (1). The sequence in question is

$$(3) \qquad H_2(G) \longrightarrow H_2(G'') \xrightarrow{t} H_o(G'', H_1(G')) \xrightarrow{j} H_1(G) \longrightarrow H_1(G'') \longrightarrow 0.$$

Here $H_i(G) = G_i(G, \underline{\underline{Z}})$, the i^{th} homology group of G with integer coefficients. The homomorphism (2) above can be identified with j in the sequence (3). The proposition now follows from the fact that, since G'' is finite, $H_i(G'')$ is a finite group for all $i > 0$.

The same reasoning shows that _if_ G'' _is finitely presented then the kernel and cokernel of_ (2) _are finitely generated_. For it is trivial that $H_1(G'') = G''/[G'', G'']$ is finitely generated if G'' is, and it is well known that $H_2(G'')$ is finitely generated if G'' has a presentation with a finite number of defining relations.

In case the projection p is split by a homomorphism $s: G'' \longrightarrow G$ then the maps $H_i(G) \longrightarrow H_i(G'')$ in (3) are split epimorphisms, so the exactness implies that (2) is a split monomorphism. The existence of s just means that G is a semi-direct product (see (IV, §4)). We shall record this conclusion for future reference; it is a simple exercise to prive it directly.

(10.3) PROPOSITION. _Suppose_ $G = G' \times_{s-d} G''$ _is a semi-direct product. Then_

$$G/[G, G] = (G'/[G, G']) \oplus (G''/[G'', G'']).$$

Let A be a ring and suppose that, for each $n \geq 1$, we are given a group $S_n(A)$ such that

$$E_n(A) \subset S_n(A) \subset GL_n(A)$$

and

$$S_{n+m}(A) \cap GL_n(A) = S_n(A) \qquad \text{for all } m \geq 0.$$

If $\underline{q}$ is a two sided ideal we put

$$S_n(A, \underline{q}) = S_n(A) \cap GL_n(A, \underline{q})$$

and

$$S(A, \underline{q}) = \bigcup_n S_n(A, \underline{q}).$$

(10.4) PROPOSITION. (cf. (4.4)). *Suppose* A *satisfies the hypotheses of* (4.2) (*with* $\underline{q}$ = A) for some $n \geq 2$. *Then if* $S(A)/E(A)$ ($\subset K_1(A)$) *is finite (resp. finitely generated) the same is true of* $S(A, \underline{q})/E(A, \underline{q})$ *for all ideals* $\underline{q}$ *such that* $A/\underline{q}$ *is finite. If, further*, A *is a finitely generated* $\mathbb{Z}$*-algebra, then for all* $m \geq \max(n, 3)$, $S_m(A, \underline{q})$ *is a finitely generated group*.

Conversely, if $S_m(A)$ *is finitely generated for any* $m \geq n - 1$ *then* $S(A)/E(A)$ *is finitely generated.*

Proof. If $m \geq n$ and if $\alpha \in GL_m(A, \underline{q})$ then, by (4.2), $\alpha = \varepsilon\beta$ with $\varepsilon \in E_m(A, \underline{q})$ and $\beta \in GL_{n-1}(A, \underline{q})$. If $\alpha \in S_m(A,\underline{q})$ it follows that $\beta \in S_m(A, \underline{q}) \cap GL_{n-1}(A) = S_{n-1}(A, \underline{q})$. Thus we see from (4.2) that

$$S_{n-1}(A, \underline{q}) \longrightarrow S(A, \underline{q})/E(A, \underline{q})$$

is surjective, and

$$S_m(A, \underline{q})/E_m(A, \underline{q}) \longrightarrow S(A, \underline{q})/E(A, \underline{q})$$

is bijective for all $m \geq n$. This establishes the last assertion of the proposition. It further follows from (4.3) that, for $m \geq n$,

$$[E_m(A), S_m(A, \underline{q})] \subset [GL_m(A), S_m(A, \underline{q})] \subset E_m(A, \underline{q}),$$

with equality if $m \geq 3$.

If $A/\underline{q}$ is finite then $S_m(A, \underline{q})$ is a normal subgroup of finite index in $S_m(A)$, so it follows from (10.2) above that

$$S_m(A, \underline{q})/[S_m(A), S_m(A, \underline{q})] \longrightarrow S_m(A)/[S_m(A), S_m(A)]$$

has finite kernel and cokernel. For large m this map is isomorphic to

$$S(A, \underline{q})/E(A, \underline{q}) \longrightarrow S(A)/E(A),$$

and the first assertion of the proposition now follows.

If A is a finitely generated $\underline{\underline{Z}}$-algebra then (see (1.3)) $E_m(A)$ is finitely generated for all $m \geq 3$. Therefore if S(A)/E(A) is finitely generated the same is true of $S_m(A)$ for all $m \geq (n, 3)$. If $A/\underline{q}$ is finite then $S_m(A, \underline{q})$ has finite index in $S_m(A)$ so (10.1) (b) implies $S_m(A, \underline{q})$ is also finitely generated. q.e.d.

HISTORICAL NOTES

As mentioned in the introduction, the material above is taken primarily from Bass [1] and from Bass-Milnor-Serre [1]. Some improvements in the exposition were supplied by Hervé Jacquet, to whom I am grateful.

The questions treated here fall within the tradition of the work of Dickson, Dieudonné, Artin,... on the classical groups (over a field). Klingenberg, in a series of papers (cf. Klingenberg [1] and [2]) has extended much of that theory to the classical groups over local rings, and it is now reasonable to seek a "globalization" of his results, such as we have obtained here for GL.

Presumably the most natural setting for such a theory would be the theory of semi-simple algebraic groups, or rather group schemes, over a commutative ring A. Stability conjectures could be formulated in terms of dim(max(A)) and the ranks of a split tori in the group.

Chapter VI
MENNICKE SYMBOLS AND RECIPROCITY LAWS

In this chapter something quite remarkable happens. We start with a Dedekind ring A, with the intention of refining the results of Chapter V on the groups $SK_1(A, \underline{q})$. The latter imply that

$$SL_n(A, \underline{q})/E_n(A, \underline{q}) \longrightarrow SK_1(A, \underline{q})$$

is an isomorphism for $n \geq 3$, and that

(1) $$\kappa_{\underline{q}}: SL_2(A, \underline{q}) \longrightarrow SK_1(A, \underline{q})$$

is surjective. It is natural to ask for the kernel of $\kappa_{\underline{q}}$.

We note first that if $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(A, \underline{q})$ then $\kappa_{\underline{q}}(\alpha)$

depends only on (a, b), so we can denote this by

$$\kappa_{\underline{q}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}.$$

Here (a, b) varies over a set we denote by $W_{\underline{q}}$. The first theorem, due to Mennicke, states that the function [ ]: $W_{\underline{q}} \longrightarrow SK(A, \underline{q})$ has some pleasant algebraic properties, the most striking of which is that it is bimultiplicative in (a, b). The main result then is that it is the <u>universal</u>

function from $W_{\underline{q}}$ into a group having these properties; such functions are called "Mennicke symbols". The proof consists in showing first that a Mennicke symbol induces a homomorphism from $SL_2(A, \underline{q})$ with certain properties - this step is a theorem of Kubota. Next we must extend Kubota's homomorphism from SL_2 to SL_3, and thence to SL_n $(n \geq 3)$. This is accomplished by the methods of Chapter V.

The remarkable fact now is that Mennicke symbols are intimately related to "reciprocity laws", of a type that includes, for example, the classical quadratic and higher reciprocity laws in number fields, as well as certain "geometric reciprocity laws" on algebraic curves.

This connection was already apparent in the paper Bass-Milnor-Serre [1], from which the present material is adapted. The classical reciprocity laws are most naturally expressed as "product formulas" for certain local symbols. In the case of quadratic reciprocity these are the Hilbert symbols. The Mennicke symbols in the present context are then analogous to the Legendre symbols in the quadratic reciprocity law.

In §§5-6 we show how, over an arbitrary Dedekind ring A, with a non zero ideal $\underline{q}$, one can construct certain local symbols, one for each $\underline{p} \in \max(A)$. Then we formally define a "$\underline{q}$-reciprocity" to be a certain collection of data satisfying a product formula relative to these local symbols (definition (6.1)). The definition poses a universal mapping problem, and hence there is a universal $\underline{q}$-reciprocity. In §6 we establish an equivalence between Mennicke symbols on $W_{\underline{q}}$ and $\underline{q}$-reciprocities. The upshot is that $SK_1(A, \underline{q})$, which we originally investigated in order to determine the normal subgroups of $SL_n(A)$, is now characterized as the group defined by the universal $\underline{q}$-reciprocity.

So far A has been any Dedekind ring. In §7 we take A to be the ring of integers in a number field L. The main theorem of Bass-Milnor-Serre [1] is then quoted without proof. It states that SK $(A, \underline{q}) = 0$ for all $\underline{q}$ if L has a real embedding. If, on the other hand, L is totally imaginary, then the power reciprocity laws in L give rise to non trivial reciprocity laws in A. Moreover, there are no others. From this it follows that $SK_1(A, \underline{q}) \simeq \mu_r$, the r^{th} roots of unity, where $r = r(\underline{q})$ is a divisor of the number of roots of unity in L. An exact formula is given for r.

Finally, in §8, we take A to be the coordinate ring of an absolutely non singular and irreducible algebraic curve X over a field k. Following Serre [3], we give the proof of a reciprocity law on $\overline{X}$, the complete non singular curve determined by X. This reciprocity law, which has been attributed to Weil, is sometimes formulated as "f((g)) = g((f))" where f and g are non zero rational functions on X. We show how to obtain from this a sometimes non trivial induced reciprocity law on the affine curve X. No non trivial examples can occur this way when k is finite, and, indeed, it is proved in Bass-Milnor-Serre [1] that $SK_1(A, \underline{q}) = 0$ for all $\underline{q}$ when k is finite.

On the other hand we show that there is a non trivial reciprocity law, with values in $\mu_2 = \{\pm 1\}$, defined on the coordinate ring of the real circle: $\underline{\underline{R}}[x, y]$, $x^2 + y^2 = 1$. We give a direct proof of the reciprocity law in this case, using elementary arguments, and we also give a topological interpretation of the corresponding homomorphism $SK_1(A) \longrightarrow \mu_2$.

It is natural to ask whether there are any reciprocity laws on algebraic curves other than those which can be deduced, by the method of §8, from that of Weil. The answer is "yes", as we shall see in Chapter XIII. The reason is that $SK_1(A)$ can be much larger than Weil's reciprocity law can account for. The proof of this relies heavily on the machinery developed in subsequent chapters which we use to compute $SK_1(A)$. This is an illustration of the double edged nature of the theory. In Bass-Milnor-Serre the classical recprocity laws were used to settle the problem of congruence subgroups, i.e., essentially, the computation of the groups $SK_1(A, \underline{q})$. Since the "K-theory" methods developed below give a direct means for computing $SK_1(A, \underline{q})$ in certain cases, we can then go back and use the K-theory to discover new reciprocity laws in those cases.

§1. MENNICKE SYMBOLS $\begin{bmatrix} b \\ a \end{bmatrix}$

We fix a commutative ring A _and an ideal_ $\underline{q}$ _in_ A. _We_ shall write $W_{\underline{q}}$ for the set of $\underline{q}$-unimodular elements in A^2.

Explicitly,

$$W_{\underline{q}} = \{(a, b) \in A^2 \mid (a, b) \equiv (1, 0) \bmod \underline{q};\ aA + bA = A\}.$$

The object of study in this section is described by:

(1.1) DEFINITION. Let C be a group. A function

$$[\ ]: W_{\underline{q}} \longrightarrow C, \quad (\alpha, b) \longmapsto \begin{bmatrix} b \\ \alpha \end{bmatrix},$$

is called a <u>Mennicke symbol</u> if it satisfies MS1 and MS2 below.

<u>MS1</u>

<u>MS1a.</u> $\begin{bmatrix} b+ta \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$ whenever $(a, b) \in W_{\underline{q}}$ and $t \in \underline{q}$

<u>MS1b.</u> $\begin{bmatrix} b \\ a+tb \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$ whenever $(a, b) \in W_{\underline{q}}$ and $t \in A$

(Note the assymetry).

<u>MS2</u>

<u>MS2a.</u> $\begin{bmatrix} b_1 \\ a \end{bmatrix} \begin{bmatrix} b_2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 b_2 \\ a \end{bmatrix}$ whenever (a, b_1), $(a, b_2) \in W_{\underline{q}}$.

<u>MS2b.</u> $\begin{bmatrix} b \\ a_1 \end{bmatrix} \begin{bmatrix} b \\ a_2 \end{bmatrix} = \begin{bmatrix} b \\ a_1 a_2 \end{bmatrix}$ whenever (a_1, b), $(a_2, b) \in W_{\underline{q}}$.

It is clear from the definition that there is a <u>universal Mennicke symbol</u>, $[\ ]_{\underline{q}}: W_{\underline{q}} \longrightarrow C_{\underline{q}}$, characterized by the fact that any other Mennicke symbol [ ], as above, is of the form $h \circ [\ ]_{\underline{q}}$ for a unique homomorphism $h: C_{\underline{q}} \longrightarrow C$. Moreover this defines $C_{\underline{q}}$ up to a unique isomorphism. It can be constructed, for example, as the group with generators $W_{\underline{q}}$ and relations MS1 and MS2. (We shall see below that the axioms are not independent, so that this presentation of $C_{\underline{q}}$ is redundant). The main theorems of this chapter will show that *if* A *is a Dedekind ring then* $C_{\underline{q}} \approx SK_1(A, \underline{q})$. This explains our interest in Mennicke symbols. The effect of this result is that to give a homomorphism $SK_1(A, \underline{q}) \longrightarrow C$ is equivalent to giving a Mennicke symbol $W_{\underline{q}} \longrightarrow C$, for any group C. In the later sections we will exhibit examples where $SK_1(A, \underline{q})$ can be computed with the aid of Mennicke symbols. We begin here by establishing some of their elementary properties.

(1.2) PROPOSITION. (a) *If* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A, \underline{q})$ *then* $(a, b) \in W_{\underline{q}}$. *The resulting map* $GL_2(A, \underline{q}) \xrightarrow{1^{st}\text{ row}} W_{\underline{q}}$ *induces bijections*

$$SN/SL_2(A, \underline{q}) \longrightarrow N/GL_2(A, \underline{q}) \longrightarrow W_{\underline{q}},$$

where the left and middle terms denote coset spaces modulo the subgroups $SN = \{I + te_{21} \mid t \in \underline{q}\}$, *and*

$N = \{\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \in GL_2(A, \underline{q})\}$, *respectively.*

(b) *Let* $\kappa: SL_2(A, \underline{q}) \longrightarrow C$ *be a homomorphism such that* $\text{Ker}(\kappa)$ *contains both* $E_2(A, \underline{q})$ *and* $[E_2(A), SL_2(A, \underline{q})]$. *Then* κ *admits a factoriation*

(1)
$$\begin{array}{ccc} SL_2(A, \underline{q}) & \xrightarrow{\quad\kappa\quad} & C \\ & {\scriptstyle 1^{st}\text{ row}} \searrow \quad \nearrow {\scriptstyle [\ ]} & \\ & W_{\underline{q}} & \end{array}$$

and [ ] *satisfies* MS1.

In §2 we shall see what further conditions on κ are required to make [ ] a Mennicke symbol.

Proof. (a). Clearly $(a, b) \equiv (1, 0) \bmod \underline{q}$; moreover $ad - bc \in U(A)$. Thus $(a, b) \in W_{\underline{q}}$, and we have a map $GL_2(A, \underline{q})$

$\xrightarrow{1^{st}\ row} W_{\underline{q}}$. Suppose $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\alpha' = \begin{pmatrix} a & b \\ c' & d' \end{pmatrix}$ have the same first row, i.e. $\varepsilon\alpha = \varepsilon\alpha'$ where $\varepsilon = (1, 0)$. Then $\varepsilon\alpha'\alpha^{-1} = \varepsilon$, i.e. $\alpha'\alpha^{-1} \in N$. Since $N \cap SL_2(A, \underline{q}) = SN$ the maps of coset spaces above are well defined and injective. To establish bijectivity we must show that every (a, b) $W_{\underline{q}}$ is the first row of some $\alpha \in SL_2(A, \underline{q})$. This follows from (V, 3.4 (b)), but we shall recall the proof. Write $1 = ax + by$; then set $c = -by^2 \in \underline{q}$ and $d = x + bxy$. We have $ad - bc = a(x + bxy) + b^2y^2 = ax + by(ax + by) = 1$. Reading mod $\underline{q}$ shows that $d \equiv 1 \bmod \underline{q}$, and thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(A, \underline{q})$. q.e.d.

(b) Since $SN \subset E_2(A, \underline{q}) \subset \mathrm{Ker}(\kappa)$, it follows from part (a) that κ factors through $W_{\underline{q}}$ $(= SN/SL_2(A, \underline{q}))$, thus giving us diagram (1). Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(A, \underline{q})$. If $\varepsilon = I + te_{12}$ $(t \in \underline{q})$ then $\varepsilon \in E_2(A, \underline{q}) \subset \mathrm{Ker}(\kappa)$ so $K(\alpha\varepsilon) = \kappa(\alpha)$. But $\alpha\varepsilon = \begin{pmatrix} a & b+ta \\ * & * \end{pmatrix}$, so we have $\begin{bmatrix} b+ta \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$ for $t \in \underline{q}$. If $\varepsilon = I + te_{21}$ $(t \in A)$ then $[\alpha, \varepsilon] = \alpha^{-1}\alpha^{\varepsilon} \in \mathrm{Ker}(\kappa)$ also so $\kappa(\alpha^{\varepsilon}) = \kappa(\alpha)$. But $\alpha^{\varepsilon} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} a+tb & b \\ * & * \end{pmatrix}$, so $\begin{bmatrix} b \\ a+tb \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$ for $t \in A$. Thus we have verified MS1a and MS1b, respectively, for [ ]. q.e.d.

Let $H \subset SL_2(A)$ denote the group generated by all $\tau_{12}(t) = I + te_{12}$ $(t \in A)$ and all $\tau_{21}(t) = I + t_{21}$ $(t \in \underline{q})$. If $(a, b) \in A^2$ then $(a, b)\,\tau_{12}(t) = (a, b + ta)$ while (a, b)

$\tau_{21}(t) = (a + tb, b)$. If $(a_1, b_1), (a_2, b_2) \in A^2$ we shall write

$$(a_1, b_1) \sim_{\underline{q}} (a_2, b_2)$$

if there is a $\tau \in H$ such that $(a_2, b_2) = (a_1, b_1)\tau$. This is the equivalence relation generated by $(a, b + t) \sim_{\underline{q}} (a, b)$ for all $t \in \underline{q}$, and $(a + tb, b) \sim_{\underline{q}} (a, b)$ for all $t \in A$. Note that if $(a_1, b_1) \in W_{\underline{q}}$ above then $(a_2, b_2) \in W_{\underline{q}}$ also. Moreover, we can restate axiom MS1 for a Mennicke symbol with this notation, as follows:

$$\underline{\text{MS1}} \quad \begin{bmatrix} b' \\ a' \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix} \text{ if } (a, b) \in W_{\underline{q}} \text{ and } (a', b') \sim_{\underline{q}} (a, b).$$

(1.3) PROPOSITION. _Let_ $(a, b) \in W_{\underline{q}}$ _and let_ $u \in U(A)$. _Then_ $(a, b) \sim_{\underline{q}} (a, (1 - a) b)$, _and_ $(a, b) \sim_{\underline{q}} (1, 0)$ _if either_ $a \equiv u \bmod b$ _or_ $b \equiv u \bmod a$.

Proof. $(a, b) \sim_{\underline{q}} (a, b - ab) = (a, (1 - a) b)$. If $a = u - tb$ then $(a, b) \sim_{\underline{q}} (a + tb, b) = (u, b) \sim_{\underline{q}} (u, b + u \cdot u^{-1}(1 - u - b)) = (u, 1 - u) \sim_{\underline{q}} (1, 1 - u) \sim_{\underline{q}} (1, 0)$.

Suppose, finally, that $b = u + ta$, and set $q = 1 - a$. Then $(a, b) \sim_{\underline{q}} (a, qb) \sim_{\underline{q}} (a, qb - qta) = (a, qu) \sim_{\underline{q}} (a - u^{-1} \cdot uq, uq) = (1, uq) \sim_{\underline{q}} (1, 0)$. q.e.d.

(1.4) PROPOSITION. _Let_ $\underline{q}'$ _be an ideal containing_ $\underline{q}$ _and assume that_ $A/\underline{q}$ _is semi-local. Then given_ $(a', b') \in W_{\underline{q}'}$, _there is an_ $(a, b) \in W_{\underline{q}}$ _such that_ $(a, b) \sim_{\underline{q}'} (a', b')$.

Proof. After passing to $A/\underline{q}$ and $\underline{q}'/\underline{q}$ we can assume $\underline{q} = 0$, so A is semi-local. Then, by (III, 2.8), we can choose $t \in A$ so that $a' + tb' \in U(A)$. Applying (1.3) then we have $(a', b') \sim_{\underline{q}'} (a' + tb', b') \sim_{\underline{q}'} (1, 0)$. q.e.d.

This proposition is especially useful when A is a noetherian integral domain of dimension ≤ 1. For then $A/\underline{q}$ is semi-local (in fact, an Artin ring) for every ideal $\underline{q} \neq 0$.

(1.5) PROPOSITION. Let A be a noetherian integral domain of dimension ≤ 1 (e.g. a Dedekind ring).

(a) Let $\underline{q}$ be a non zero ideal in A and let $t \neq 0$ be an element of $\underline{q}$. Given $(a_1, b_1), \ldots, (a_n, b_n) \in W_{\underline{q}}$, there exist $(a, c_1 t), \ldots, (a, c_n t) \in W_t$ $(= W_{tA})$ such that $(a_i, b_i) \sim_{\underline{q}} (a, c_i t)$ $(1 \leq i \leq n)$.

(b) Let S be a multiplicative set in A, let $\underline{q}'$ be an ideal of $A' = S^{-1} A$, and let $\underline{q} = \underline{q}' \cap A$. Given $(a', b') \in W_{\underline{q}'}$, there is an $(a, b) \in W_{\underline{q}}$ such that $(a, b) \sim_{\underline{q}'} (a', b')$.

Remark. It follows easily from (1.5) (b) that, for all $n \geq 1$, $SL_n(A', \underline{q}')$ is generated by $E_n(A', \underline{q}')$ together with $SL_n(A, \underline{q})$. In particular, $SK_1(A, \underline{q}) \longrightarrow SK_1(A', \underline{q}')$ is surjective.

Proof. (a). Since A/tA is semi-local we can use (1.4) to find $(a_i', b_i' t) \in W_t$ such that $(a_i', b_i' t) \sim_{\underline{q}} (a_i, b_i)$ $(1 \leq i \leq n)$. Assume, by induction on n, that we have found $(a', c_i t) \in W_t$ such that $(a', c_i t) \sim_t (a_i, c_i t)$ $(1 \leq i < n)$. We can, of course, arrange that each $c_i \neq 0$. Let $c = c_1 \ldots c_{n-1} \neq 0$. Since b_n' is comaximal with $a_n A$ we can, by (III, 2.8), find $c_n \equiv b_n' \bmod a_n$ such that c_n maps to a unit in (the semi-local ring) A/cA. Then we have $(a_n', b_n' t) \sim_t (a_n', c_n t)$, clearly. Moreover, writing $a' - a_n = dt$, we can solve $d = rc_n - sc$. Then $a' - a_n = rc_n t - sct$ so $a_n' + rc_n t = a' + sct$; call this element a. Then $(a_n', c_n t) \sim_t (a, c_n t)$, and, since $c = c_1 \ldots c_{n-1}$, $(a', c_i t) \sim_t (a, c_i t)$ $(1 \leq i < n)$. q.e.d.

(b) Let $\underline{a} \neq 0$ be an ideal in A. Then $A/\underline{a}$ is an Artin ring, so $A/\underline{a} = \Pi A/\underline{q}_i$ where each $A/\underline{q}_i$ is a local ring with maximal ideal $\underline{p}_i/\underline{q}_i$. Moreover $\underline{p}_i^{n_i} \subset \underline{q}_i$ for some $n_i > 0$ so the $\underline{q}_i$ are comaximal, and we have $\underline{a} = \cap \underline{q}_i = \Pi \underline{q}_i$ (see Chinese Remainder Theorem (III, 2.4)). Moreover $S^{-1}\underline{a} = \Pi S^{-1}\underline{q}_i$ ($\underline{p}_i \cap S = \phi$), and $A'/S^{-1}\underline{a} = \Pi A/\underline{q}_i$ ($\underline{p}_i \cap S = \phi$). These facts follow from standard properties of localization (see (III, §4)) and the fact that the $\underline{p}_i$ are maximal ideals. Since any ideal $\underline{a}'$ in A' is of the form $\underline{a}' = S^{-1}\underline{a}$ ($\underline{a} = \underline{a}' \cap A$) it follows, in particular, that the composite $A \subset A' \longrightarrow A'/\underline{a}'$ is surjective for $\underline{a}' \neq 0$.

We are given $(a', b') \in W_{\underline{q}'}$ and we seek $(a, b) \in W_{\underline{q}}$ such that $(a, b) \sim_{\underline{q}'} (a', b')$. If a' or b' is zero then the other is a unit, and (1.3) implies $(a', b') \sim_{\underline{q}'} (1, 0)$. If not we can find $b \neq 0$ in A such that $b \equiv b' \bmod a'\,\underline{q}'$, by the paragraph above. Moreover the same paragraph shows that $bA = \underline{b}_1\underline{b}_2$ where $\underline{b}_1 = bA' \cap A$ (so that $bA' = \underline{b}_1A'$) and where $\underline{b}_2$ is comaximal with $\underline{b}_1$. Choose $a_1 \in A$ such that $a_1 \equiv a' \bmod bA'$, and then choose $a \in A$ to solve

$$a \equiv a_1 \bmod \underline{b}_1$$

$$a \equiv 1 \bmod \underline{b}_2.$$

The first congruence implies $a \equiv a_1 \equiv a' \bmod bA'$, so $(a', b') \sim_{\underline{q}'} (a', b) \sim_{\underline{q}'} (a, b)$. Hence $(a, b) \equiv (1, 0) \bmod \underline{q}' \cap A$ $(=\underline{q})$. To show that $(a, b) \in W_{\underline{q}}$ it remains to be shown that $\underline{a} = aA + bA$ equals A. Of course $\underline{a}A' = A'$ and $\underline{a} \supset bA = \underline{b}_1\underline{b}_2$. If $s \in S \cap \underline{a}$ then s belongs to no maximal ideal containing $\underline{b}_1$. Moreover $a \in \underline{a}$ and $a \equiv 1 \bmod \underline{b}_2$. Therefore $A = sA + aA + \underline{b}_1\underline{b}_2 \subset \underline{a}$. q.e.d.

(1.6) PROPOSITION. Let $\underline{q}$ be an ideal in a commutative ring A, let C be a group, and let $[\ ]: W_{\underline{q}} \longrightarrow C$ be a function satisfying MS1, and such that $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$.

(a) If $(a, b) \in W_{\underline{q}}$, and if there is a $u \in U(A)$ such that $a \equiv u \bmod b$ or $b \equiv u \bmod a$ then $\begin{bmatrix} b \\ a \end{bmatrix} = 1$.

(b) Suppose $t \in \underline{q}$, and $a \equiv 1 \bmod t$. Then the map $b \longmapsto \begin{bmatrix} bt \\ a \end{bmatrix}$ for $b \in A$, $bA + aA = A$, induces a map

(2) $U(A/aA) \longrightarrow C$

whose composite with $U(A) \longrightarrow U(A/aA)$ is the constant map 1.

(c) Suppose A is a noetherian integral domain of dimension ≤ 1. Then given $(a_1, b_1), \ldots, (a_n, b_n) \in W_{\underline{q}}$, there exist t and a as above such that $\begin{bmatrix} b_i \\ a_i \end{bmatrix}$ $(1 \leq i \leq n)$ all lie in the image of (2).

Now suppose that [ ] satisfies MS2a also. (This implies $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$).

(d) The map (2) is a homomorphism. If A is as in (c) then $[W_{\underline{q}}]$ is an abelian subgroup of C. Moreover, if $0 \neq \underline{q}' \subset \underline{q}$ then $[W_{\underline{q}'}] = [W_{\underline{q}}]$.

(e) (Kervaire). Let $t \in \underline{q}$ and suppose a, $d \in A$ are that $a \equiv 1 \equiv d \bmod t$ and $aA + dA = A$. Then

$$\begin{bmatrix} dt \\ a \end{bmatrix} = \begin{bmatrix} at \\ d \end{bmatrix}.$$

Proof. (a) follows from (1.3).

(b) Let $(a, b_1 t), (a, b_2 t) \in W_{\underline{q}}$. If $b_2 = b_1 - xa$

then $\begin{bmatrix} b_2t \\ \alpha \end{bmatrix} = \begin{bmatrix} b_2t + xt\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} b_1t \\ \alpha \end{bmatrix}$. Thus $\begin{bmatrix} bt \\ \alpha \end{bmatrix}$ depends only on the class of b in $U(A/\alpha A)$. If $b \in U(A)$ then $\alpha \equiv 1 \bmod tA$ $(= btA)$ so (a) implies $\begin{bmatrix} bt \\ \alpha \end{bmatrix} = 1$.

(c) follows from (1.5) (a).

(d) We have $\begin{bmatrix} t \\ \alpha \end{bmatrix} = 1$, by part (a), so $\begin{bmatrix} b_1b_2t \\ \alpha \end{bmatrix} = \begin{bmatrix} b_1b_2t \\ \alpha \end{bmatrix} \begin{bmatrix} t \\ \alpha \end{bmatrix} = \begin{bmatrix} b_1t\ b_2t \\ \alpha \end{bmatrix} = \begin{bmatrix} b_1t \\ \alpha \end{bmatrix} \begin{bmatrix} b_2t \\ \alpha \end{bmatrix}$, using MS2a. Hence (2) is a homomorphism. In particular, the image of (2) is an abelian subgroup of C. If A is as in part (c) then the latter implies $[W_{\underline{q}}]$ is the direct union of the images of (2), for variable t and α. Therefore $[W_{\underline{q}}]$ is an abelian subgroup of C. If $0 \neq \underline{q}' \subset \underline{q}$ then $A/\underline{q}'$ is semi-local so it follows from (1.4) that $[W_{\underline{q}'}] = [W_{\underline{q}}]$.

(e) Write $d - \alpha = xt$. Then $\begin{bmatrix} dt \\ \alpha \end{bmatrix} = \begin{bmatrix} dt - \alpha t \\ \alpha \end{bmatrix} = \begin{bmatrix} xt^2 \\ \alpha \end{bmatrix}$

$$= \begin{bmatrix} xt \\ \alpha \end{bmatrix} = \begin{bmatrix} xt \\ \alpha + xt \end{bmatrix} = \begin{bmatrix} xt \\ d \end{bmatrix} = \begin{bmatrix} -x^2t \\ d \end{bmatrix} = \begin{bmatrix} \alpha t - dt \\ d \end{bmatrix} = \begin{bmatrix} \alpha t \\ d \end{bmatrix}. \quad \text{q.e.d.}$$

(1.7) PROPOSITION. (Lam) Let A, $\underline{q}$, and C be as in (1.6), and let $[\ ]: W_{\underline{q}} \longrightarrow C$ be a function satisfying MS1.

(a) MS2b $\Rightarrow$ MS2a.

(b) If A is a noetherian integral domain of dimension ≤ 1 and if $\underline{q}$ is an invertible ideal then MS2a $\Rightarrow$ MS2b.

Proof (a). We assume MS2b. Suppose $t \in \underline{q}$, $\alpha \equiv 1 \bmod t$, and $(\alpha, bt) \in W_{\underline{q}}$; say $\alpha = 1 + st$. Then

$$(*) \qquad \begin{bmatrix} bt^2 \\ a \end{bmatrix} = \begin{bmatrix} -at \\ a + bt \end{bmatrix}.$$

For we have $\begin{bmatrix} bt^2 \\ 1 + bt \end{bmatrix} = \begin{bmatrix} bt^2 - t(1 + bt) \\ 1 + bt \end{bmatrix} = \begin{bmatrix} -t \\ 1 + bt \end{bmatrix} = 1,$

and hence $\begin{bmatrix} bt^2 \\ a \end{bmatrix} = \begin{bmatrix} bt^2 \\ a \end{bmatrix} \begin{bmatrix} bt^2 \\ 1 + bt \end{bmatrix} = \begin{bmatrix} bt^2 \\ (1 + st)(1 + bt) \end{bmatrix}$

$$= \begin{bmatrix} bt^2 \\ 1 + st + bt + sbt^2 \end{bmatrix} = \begin{bmatrix} bt^2 \\ a + bt \end{bmatrix}$$

$$= \begin{bmatrix} bt^2 - t(a + bt) \\ a + bt \end{bmatrix} = \begin{bmatrix} -at \\ a + bt \end{bmatrix}.$$

Next suppose above that $b = b_1 b_2$. Then

$$\begin{bmatrix} b_1 t^2 \\ a \end{bmatrix} \begin{bmatrix} b_2 t^2 \\ a \end{bmatrix} = \begin{bmatrix} -at \\ a + b_1 t \end{bmatrix} \begin{bmatrix} -at \\ a + b_2 t \end{bmatrix} \quad \text{(using (*))}$$

$$= \begin{bmatrix} -ta \\ a^2 + at(b_1 + b_2) + b_1 b_2 \ t^2 \end{bmatrix}$$

$$= \begin{bmatrix} -ta \\ a(1 + st) + b_1 b_2 \ t^2 \end{bmatrix}$$

$$= \begin{bmatrix} -ta \\ a + b_1 b_2 \ t^2 \end{bmatrix}$$

$$= \begin{bmatrix} (b_1 b_2 t) t^2 \\ a \end{bmatrix} \qquad \text{(using (*))}.$$

Finally, suppose (a, b_1), $(a, b_2) \in W_{\underline{q}}$. We claim

$\begin{bmatrix} b_1 b_2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 \\ a \end{bmatrix} \begin{bmatrix} b_2 \\ a \end{bmatrix}$. Set $t = 1 - a$. Then if $(a, b) \in W_{\underline{q}}$ we have $\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} bt \\ a \end{bmatrix} = \begin{bmatrix} bt^n \\ a \end{bmatrix}$ for all $n \geq 0$. Hence, using the calculation above, we have

$$\begin{bmatrix} b_1 \\ a \end{bmatrix} \begin{bmatrix} b_2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 t^2 \\ a \end{bmatrix} \begin{bmatrix} b_2 t^2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 b_2 t^3 \\ a \end{bmatrix} = \begin{bmatrix} b_1 b_2 \\ a \end{bmatrix}. \quad \text{q.e.d.}$$

(b) We now assume A is a noetherian integral domain of dimension ≤ 1, that $\underline{q}$ is invertible, and MS2a. We claim that if (a_1, b), $(a_2, b) \in W_{\underline{q}}$ then

$$(**) \qquad \begin{bmatrix} b \\ a_1 a_2 \end{bmatrix} = \begin{bmatrix} b \\ a_1 \end{bmatrix} \begin{bmatrix} b \\ a_2 \end{bmatrix}.$$

<u>Case 1.</u> There is a $t \in \underline{q}$ such that $a_1 \equiv 1 \equiv a_2 \bmod t$.

Then $\begin{bmatrix} t \\ a_1 a_2 \end{bmatrix} = 1 = \begin{bmatrix} t \\ a_i \end{bmatrix}$ $(i = 1, 2)$, so it suffices to show

that $\begin{bmatrix} bt \\ a_1 a_2 \end{bmatrix} = \begin{bmatrix} bt \\ a_1 \end{bmatrix} \begin{bmatrix} bt \\ a_2 \end{bmatrix}$. Neither side of this equation is altered if we vary b modulo $a_1 a_2$. If $a_1 a_2 = 0$ then $b \in U(A)$ and (1.6) (b) implies all these symbols equal 1. Otherwise we can, after changing b modulo $a_1 a_2$, arrange that b is comaximal with t. (We can assume $t \neq 0$ for the problem here is otherwise trivial). Then we can find $b' \in A$ so that $b' \equiv 1 \bmod a_1 a_2$ and $b_1 \equiv 1 \bmod t$, where $b_1 = b'b$. Now using (1.6) (d), we have

$$\begin{bmatrix} b_1 t \\ a_1 a_2 \end{bmatrix} = \begin{bmatrix} b' t \\ a_1 a_2 \end{bmatrix} \begin{bmatrix} bt \\ a_1 a_2 \end{bmatrix} = \begin{bmatrix} bt \\ a_1 a_2 \end{bmatrix}$$

and

$$\begin{bmatrix} b_1 t \\ a_i \end{bmatrix} = \begin{bmatrix} b' t \\ a_i \end{bmatrix} \begin{bmatrix} bt \\ a_i \end{bmatrix} = \begin{bmatrix} bt \\ a_i \end{bmatrix} \qquad (i = 1, 2).$$

Finally, with the aid of (1.6) (c), we obtain

$$\begin{bmatrix} b_1 t \\ \alpha_1\alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 t \\ b_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 t \\ b_1 \end{bmatrix}\begin{bmatrix} \alpha_2 t \\ b_1 \end{bmatrix} = \begin{bmatrix} b_1 t \\ \alpha_1 \end{bmatrix}\begin{bmatrix} b_1 t \\ \alpha_2 \end{bmatrix},$$

and this concludes the proof in case 1.

<u>General case</u>. Write $\alpha_1 = 1 - t$; if $t = 0$ we are in case 1 so assume $t \neq 0$. If we replace b by $b_1 = b + s\alpha_1\alpha_2$ for some $s \in \underline{q}$ then neither side of (**) is altered. We claim s can be chosen so that t and b_1 generate $\underline{q}$. For it suffices to choose s so that this is so at each of the (finite number of) maximal ideals containing t. By the Chinese Remainder Theorem it suffices to do this locally. But then the invertible ideal $\underline{q}$ is principal and either $\alpha_1\alpha_2$ or b is a unit, so such an s clearly exists.

Since $\underline{q} = At + Ab_1$ we can write $\alpha_2 = 1 + xt + yb_1$ Then neither side of the alleged equation,

$$\begin{bmatrix} b_1 \\ \alpha_1\alpha_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ \alpha_1 \end{bmatrix}\begin{bmatrix} b_1 \\ \alpha_2 \end{bmatrix},$$

is altered if we replace α_2 by $\alpha_2' = \alpha_2 - yb_1$. But (α_1, b_1), (α_2', b_1) satisfy the conditions of case 1, so this concludes the proof.

§2. THE MAIN THEOREMS

Let $\underline{q}$ be an ideal in a commutative ring A, and let κ: $SL_2(A, \underline{q}) \longrightarrow C$ be a group homomorphism such that $\mathrm{Ker}(\kappa)$ contains both $E_2(A, \underline{q})$ and $[E_2(A), SL_2(A, \underline{q})]$. Then, according to (1.2) (b), κ admits a factorization

(1)
$$\begin{array}{ccc} SL_2(A, \underline{q}) & \xrightarrow{\quad\kappa\quad} & C \\ & {\scriptstyle 1^{st}\ \mathrm{row}} \searrow \quad \nearrow {\scriptstyle [\ ]} & \\ & W_{\underline{q}} & \end{array}$$

and [ ] satisfies MS1. Clearly also $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$.

(2.1) THEOREM. (a) [ ] satisfies MS2a if and only if κ satisfies the following condition:

(2) If $\alpha, \alpha' \in SL_2(A, \underline{q})$ are of the form $\alpha = \begin{pmatrix} 1 + ta & b \\ tc & d \end{pmatrix}$ and $\alpha' = \begin{pmatrix} 1 + ta & tb \\ c & d \end{pmatrix}$ for some $a, t \in \underline{q}$, then $\kappa(\alpha) = \kappa(\alpha')$.

(b) (Mennicke) If κ is the restriction of a homomorphism $\kappa': SL_3(A, \underline{q}) \longrightarrow C$ such that $[E_3(A), SL_3(A, \underline{q})] \subset \mathrm{Ker}(\kappa')$ then [ ] is a Mennicke symbol.

Let $MS(A, \underline{q})$ denote the normal subgroup of $SL_2(A, \underline{q})$ generated by $E_2(A, \underline{q})$, $[E_2(A), SL_2(A, \underline{q})]$, and all $\alpha^{-1}\alpha'$, where α, α' are as in (2) above.

(2.2) COROLLARY. In the setting of (2.1) assume that A is a noetherian integral domain of dimension ≤ 1, and that $\underline{q}$ is an invertible ideal. Then [ ] is a Mennicke symbol $\Longleftrightarrow MS(A, \underline{q}) \subset \mathrm{Ker}(\kappa)$.

Proof. This is an immediate consequence of (2.1) (a) and (1.7) (b).

Proof of (2.1) (a). If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(A, \underline{q})$ then $\begin{bmatrix} b \\ a \end{bmatrix} = \kappa(\alpha)$.

MS2a $\Rightarrow$ (2). Given α and α' as in (2) we have

$$\kappa(\alpha') = \kappa\begin{pmatrix} 1 + at & bt \\ c & c \end{pmatrix} = \begin{bmatrix} bt \\ 1 + at \end{bmatrix} = \begin{bmatrix} b \\ 1 + at \end{bmatrix}\begin{bmatrix} t \\ 1 + at \end{bmatrix}$$

$$= \begin{bmatrix} b \\ 1 + at \end{bmatrix} = \kappa(\alpha). \text{ q.e.d.}$$

<u>(2) ⟹ MS2a</u>. We first note the following immediate consequence of (2):

(*) If $(a, b) \in W_{\underline{q}}$ and $a = 1 + xt$ with $x, t \in \underline{q}$ then

$$\begin{bmatrix} bt \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}.$$

We must prove now that if $(a, b_o), (a, b_1) \in W_{\underline{q}}$ then

$\begin{bmatrix} b_o b_1 \\ a \end{bmatrix} = \begin{bmatrix} b_o \\ a \end{bmatrix} \begin{bmatrix} b_1 \\ a \end{bmatrix}$. Write $a = 1 + q$ (so $q \in \underline{q}$) and choose $c \in A$ such that

$$c \equiv 1 \mod q; \quad \text{and}$$

$$b_o b_1 c \equiv -1 \mod a; \quad \text{say } 1 + b_o b_1 c = ad.$$

Then $\alpha_i = \begin{pmatrix} a & b_i \\ b_{1-i}c & d \end{pmatrix} \in SL_2(A, \underline{q})$ and $\begin{bmatrix} b_i \\ a \end{bmatrix} = \kappa(\alpha_i)$ $(i = 0, 1)$.

Let $\pi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in E_2(A)$. Then, by hypothesis, $\kappa(\alpha^\pi) = \kappa(\alpha)$ for $\alpha \in SL_2(A, \underline{q})$. Hence we have

$$\begin{bmatrix} b_o \\ a \end{bmatrix} \begin{bmatrix} b_1 \\ a \end{bmatrix} = \kappa(\alpha_o)\, \kappa(\alpha_1{}^\pi) = \kappa(\alpha_o\, \alpha_1{}^\pi),$$

where

$$\alpha_o\, \alpha_1{}^\pi = \begin{pmatrix} a & b_o \\ b_1 c & d \end{pmatrix} \begin{pmatrix} d & -b_o c \\ -b_1 & a \end{pmatrix} = \begin{pmatrix} ad - b_o b_1 & ab_o - ab_o c \\ * & * \end{pmatrix}$$

$$= \begin{pmatrix} 1 - b_0 b_1 (1 - c) & a b_0 (1 - c) \\ * & * \end{pmatrix}.$$

Since $c \equiv 1 \bmod q$ (recall $q = a - 1$) we can write $1 - c = tq$, and we have

$$\begin{bmatrix} b_0 \\ a \end{bmatrix} \begin{bmatrix} b_1 \\ a \end{bmatrix} = \begin{bmatrix} a b_0 tq \\ 1 - b_0 b_1 tq \end{bmatrix}$$

$$= \begin{bmatrix} aq \\ 1 - b_0 b_1 tq \end{bmatrix} \quad \text{((*) with t there} = b_0 t \text{ here)}$$

$$= \begin{bmatrix} aq \\ a - q(1 + b_0 b_1 t) \end{bmatrix} \quad \text{(because } a = 1 + q\text{)}$$

$$= \begin{bmatrix} aq - q(a - q(1 + b_0 b_1 t)) \\ a - q(1 + b_0 b_1 t) \end{bmatrix}$$

$$= \begin{bmatrix} q^2 (1 + b_0 b_1 t) \\ a - q(1 + b_0 b_1 t) \end{bmatrix}$$

$$= \begin{bmatrix} q(1 + b_0 b_1 t) \\ a - q(1 + b_0 b_1 t) \end{bmatrix} \quad \text{((*) with } t = q \; (= a - 1))$$

$$= \begin{bmatrix} q(1 + b_0 b_1 t) \\ a \end{bmatrix}$$

$$= \begin{bmatrix} q + b_0 b_1 (1 - c) \\ a \end{bmatrix} \quad (1 - c = qt)$$

$$= \begin{bmatrix} b_0 b_1 + q - b_0 b_1 c \\ a \end{bmatrix}$$

$$= \begin{bmatrix} b_0 b_1 + q + 1 - ad \\ a \end{bmatrix} \quad (ad - b_0 b_1 c = 1)$$

$$= \begin{bmatrix} b_0 b_1 + a(1-d) \\ a \end{bmatrix} = \begin{bmatrix} b_0 b_1 \\ a \end{bmatrix}. \text{ q.e.d.}$$

<u>Proof of (2.1) (b)</u>. Thanks to (1.7) (a) it suffices, in order to show that [ ] is a Mennicke symbol, to verify MS2b. Thus, given (a_1, b), $(a_2, b) \in W_{\underline{q}}$, it suffices to show that $\begin{bmatrix} b \\ a_1 a_2 \end{bmatrix} = \begin{bmatrix} b \\ a_1 \end{bmatrix} \begin{bmatrix} b \\ a_2 \end{bmatrix}$, under the assumption that κ extends to a homomorphism κ' on $SL_3(A, \underline{q})$ that kills $E_3(A, \underline{q})$.

Choose $\alpha_i = \begin{pmatrix} a_i & b \\ c_i & d_i \end{pmatrix} \in SL_2(A, \underline{q})$, and write $\overline{\alpha}_i = \begin{pmatrix} \alpha_i & 0 \\ 0 & 1 \end{pmatrix}$ $(i = 1, 2)$. Let $\overline{\pi} = \begin{pmatrix} -1 & 0 \\ 0 & \pi \end{pmatrix} \in E_3(A)$, where $\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we have

$$\begin{bmatrix} b \\ a_1 \end{bmatrix} \begin{bmatrix} b \\ a_2 \end{bmatrix} = \kappa(\alpha_1)\ \kappa(\alpha_2) = \kappa'(\overline{\alpha}_1)\ \kappa'(\overline{\alpha}_2^{\overline{\pi}}) = \kappa'(\alpha),$$

where

$$\alpha = \overline{\alpha}_1\ \overline{\alpha}_2^{\overline{\pi}} = \begin{pmatrix} a_1 & b & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 & -b \\ 0 & 1 & 0 \\ -c_2 & 0 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_2 & b & -a_1 b \\ c_1 a_2 & d_1 & -c_1 b \\ -c_2 & 0 & d_2 \end{pmatrix}.$$

Let $\varepsilon_1 = I + a_1 e_{23} \in E_3(A)$. Then

$$\alpha^{\varepsilon_1} = \begin{pmatrix} a_1a_2 & b & 0 \\ c_1a_2 - c_2a_1 & d_1 & a_1d_1 - a_1d_2 - c_1b \\ -c_2 & 0 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1a_2 & b & 0 \\ x & d_1 & 1 - a_1d_2 \\ -c_2 & 0 & d_2 \end{pmatrix}.$$

Let $\varepsilon_2 = I + (a_1 - 1)e_{23}$; then

$$\varepsilon_2\ \alpha^{\varepsilon_1} = \begin{pmatrix} a_1a_2 & b & 0 \\ y & d_1 & 1 - d_2 \\ -c_2 & 0 & d_2 \end{pmatrix}.$$

Let $\varepsilon_3 = I - e_{32} \in E_3(A)$. Then

$$(\varepsilon_2\ \alpha^{\varepsilon_1})^{\varepsilon_3} = \begin{pmatrix} a_1a_2 & b & 0 \\ y & d_1 + d_2 - 1 & 1 - d_2 \\ y - c_2 & d_1 - 1 & 1 \end{pmatrix}.$$

Let $\varepsilon_4 = I + (d_2 - 1)e_{23} \in E_3(A, \underline{q})$; then

$$\varepsilon_4(\varepsilon_2\ \alpha^{\varepsilon_1})^{\varepsilon_3} = \begin{pmatrix} a_1a_2 & b & 0 \\ z & d_1d_2 & 0 \\ y - c_2 & d_1 - 1 & 1 \end{pmatrix}$$

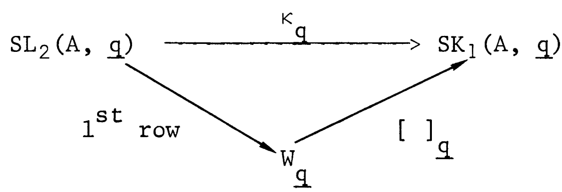
Let $\beta = \begin{pmatrix} a_1a_2 & b \\ z & d_1d_2 \end{pmatrix}$, and $\overline{\beta} = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$. Then $\varepsilon_4(\varepsilon_2\ \alpha^{\varepsilon_1})^{\varepsilon_3} = \varepsilon_5\overline{\beta}$ with $\varepsilon_5 \in E_3(A, \underline{q})$, clearly. Since $E_3(A, \underline{q}) \subset [E_3(A), SL_3(A, \underline{q})] \subset \mathrm{Ker}(\kappa')$ we conclude that

$$\begin{bmatrix} b \\ a_1 \end{bmatrix} \begin{bmatrix} b \\ a_2 \end{bmatrix} = \kappa'(\alpha) = \kappa'(\varepsilon_4(\varepsilon_2 \alpha^{\varepsilon_1})^{\varepsilon_3}) = \kappa'(\varepsilon_5\bar{\beta})$$

$$= \kappa'(\bar{\beta}) = \kappa(\beta) = \begin{bmatrix} b \\ a_1 a_2 \end{bmatrix}. \text{ q.e.d.}$$

The main theorem of this chapter is the following result, which contains a partial converse to (2.1). In the later sections we shall explore some of its consequences.

(2.3) THEOREM. *Let* A *be a noetherian integral domain of dimension* ≤ 1, *and let* $\mathfrak{q}$ *be an ideal in* A. *The natural homomorphism* $\kappa_{\mathfrak{q}}: SL_2(A, \mathfrak{q}) \longrightarrow SK_1(A, \mathfrak{q})$ *admits a factorization*



as in (1) *above, and* $[\ ]_{\mathfrak{q}}$ *is a universal Mennicke symbol.*

This theorem will be deduced from the following, more explicit, statements (which are themselves consequences of the theorem).

(2.4) I. (Kubota). *Let* $[\ ]: W_{\mathfrak{q}} \longrightarrow C$ *be a Mennicke symbol, and let* $\kappa: GL_2(A, \mathfrak{q}) \longrightarrow C$ *be the composite* $GL_2(A,\mathfrak{q}) \xrightarrow{1^{st}\text{ row}} W_{\mathfrak{q}} \xrightarrow{[\ ]} C$. *Then* κ *is a homomorphism whose kernel contains* $GE_2(A, \mathfrak{q})$, $[GE_2(A), GL_2(A, \mathfrak{q})]$, *and all elements* $\alpha^{-1}\alpha'$ *where* $\alpha, \alpha' \in GL_2(A, \mathfrak{q})$ *are of the form*

$$\alpha = \begin{pmatrix} 1 + at & b \\ ct & d \end{pmatrix} \text{ and } \alpha' = \begin{pmatrix} 1 + at & tb \\ c & d \end{pmatrix} \text{ with } a, t \in \mathfrak{q}.$$

(2.5) II. *The homomorphism* κ *of* I *can be extended to* a homomorphism κ': $GL_3(A, \underline{q}) \longrightarrow C$ such that $E_3(A, \underline{q}) \subset \mathrm{Ker}(\kappa')$.

Proof that I and II $\Rightarrow$ (2.3). First it follows from (2.1) (b) that $[\ ]_{\underline{q}}$ is a Mennicke symbol (without any assumptions on A). To prove that it is universal let $[\ ]$: $W_{\underline{q}} \longrightarrow C$ be a universal Mennicke symbol. Then $[\ ]_{\underline{q}} = h \circ [\ ]$ for a unique homomorphism h: $C \longrightarrow SK_1(A, \underline{q})$, and we must show that h is an isomorphism. Let κ and κ' be the homomorphisms whose existences are guaranteed by I and II, respectively, and let f': $SL_3(A, \underline{q})/E_3(A, \underline{q}) \longrightarrow C$ be the homomorphism induced by κ'. Since $\dim A \leq 1$ it follows from (IV, 4.5) that the natural homomorphism $SL_3(A, \underline{q})/E_3(A, \underline{q}) \longrightarrow SK_1(A, \underline{q})$ is an isomorphism. Thus f' induces f: $SK_1(A, \underline{q}) \longrightarrow C$. If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(A, \underline{q})$ then $\begin{bmatrix} b \\ a \end{bmatrix} = f$ (class of $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ in SK (A, $\underline{q}$)) $= f \begin{bmatrix} b \\ a \end{bmatrix}_{\underline{q}} = fh \begin{bmatrix} b \\ a \end{bmatrix}$. Since, again by (V, 4.5), $SL_2(A, \underline{q}) \longrightarrow SK_1(A, \underline{q})$ is surjective, it follows that h is an epimorphism. We have just seen that $fh = 1_C$, and hence h is an isomorphism with inverse f. q.e.d.

§3. PROOF OF THEOREM (2.3): I. KUBOTA'S THEOREM

A and $\underline{q}$ are as in (2.3), and κ: $GL_2(A, \underline{q}) \longrightarrow C$ is defined by

$$\kappa(\alpha) = \begin{bmatrix} b \\ a \end{bmatrix}, \quad \text{if } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $[\ ]$ is a Mennicke symbol. We shall prove, in several steps, that κ is a homomorphism having the properties described in (2.4).

(i) If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A, \underline{q})$ then $\kappa(\alpha) = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}^{-1} = \begin{bmatrix} c \\ a \end{bmatrix}^{-1}$.

For $u = ad - bc \in U(A)$ so, with the aid of (1.6) (a), we have

$$\kappa(\alpha) = \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} b \\ ad \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix}^{-1} = \begin{bmatrix} b \\ d \end{bmatrix}^{-1} \begin{bmatrix} bc \\ d \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} bc \\ a \end{bmatrix} \begin{bmatrix} c \\ a \end{bmatrix}^{-1} = \begin{bmatrix} c \\ a \end{bmatrix}^{-1}.$$

Let $H = \{\alpha \in GL_2(A, \underline{q}) \mid \kappa(\alpha'\alpha) = \kappa(\alpha')\,\kappa(\alpha)$ for all $\alpha' \in GL_2(A, \underline{q})\}$ and let $N = \{\tau \quad GL_2(A) \mid \kappa(\alpha^{\tau}) = \kappa(\alpha)$ for all $\alpha \in GL_2(A, \underline{q})\}$. Then, just as in (V, 7.2), H and N are groups and N normalizes H.

(ii) $GE_2(A) \subset N$ and $GE_2(A, \underline{q}) \subset H$. In fact $\kappa(\alpha\varepsilon) = \kappa(\alpha)$ for $\varepsilon \in GE_2(A, \underline{q})$.

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A, \underline{q})$, and let $\varepsilon_{ij}(t) = I + te_{ij}$.

If $t \in \underline{q}$ then $\alpha\varepsilon_{12}(t) = \begin{pmatrix} a & b+ta \\ * & * \end{pmatrix}$ and $\alpha\varepsilon_{21}(t) = \begin{pmatrix} a+tb & b \\ * & * \end{pmatrix}$

so $\kappa(\alpha\varepsilon_{12}(t)) = \begin{bmatrix} b+ta \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix} = \kappa(\alpha)$, and, similarly, $\varepsilon_{21}(t) \in H$.

For any $t \in A$ we have $\alpha^{\varepsilon_{12}(t)} = \begin{pmatrix} a - ct & * \\ c & * \end{pmatrix}$ and

$$\alpha^{\varepsilon_{21}(t)} = \begin{pmatrix} a+tb & b \\ * & * \end{pmatrix}. \text{ Hence } \kappa(\alpha^{\varepsilon_{12}(t)}) = \begin{bmatrix} c \\ a - ct \end{bmatrix}^{-1} = \begin{bmatrix} c \\ a \end{bmatrix}^{-1}$$

$= \kappa(\alpha)$, and, similarly, $\varepsilon_{21}(t) \in N$.

If $\delta = \operatorname{diag}(1, u)$ then $\alpha^{\delta} = \begin{pmatrix} a & bu \\ * & * \end{pmatrix}$ and $\alpha\delta = \begin{pmatrix} a & bu \\ * & * \end{pmatrix}$

so $\kappa(\alpha^{\delta}) = \begin{bmatrix} bu \\ a \end{bmatrix} = \begin{bmatrix} bu(1-a) \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}\begin{bmatrix} u(1-a) \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix} =$

$\kappa(\alpha)$, and also $\kappa(\alpha\delta) = \kappa(\alpha)$ if $u \quad U(A, \underline{q})$.

Since the $\varepsilon_{12}(t)$ and $\varepsilon_{21}(t)$ $(t \in A)$, and δ as above, generate $GE_2(A)$, we conclude that $GE_2(A) \subset N$. Since N normalizes H and all $\varepsilon_{12}(t)$, $\varepsilon_{21}(t)$ $(t \in \underline{q})$ belong to H we have $E_2(A, \underline{q}) \subset H$ as well. Finally $GE_2(A, \underline{q}) \subset H$ since $GE_2(A, \underline{q})$ is generated by $E_2(A, \underline{q})$ and those δ as above such that $u \in U(A, \underline{q})$.

(iii) Suppose $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\alpha' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in $GL_2(A, \underline{q})$ are such that $d \equiv 1 \equiv a' \bmod t$ for some $t \in \underline{q}$, and assume further that $a'A + dA = A$. Then $\kappa(\alpha'\alpha) = \kappa(\alpha')\,\kappa(\alpha)$.

Suppose $(a', xy) \in W_{\underline{q}}$ with $y \in \underline{q}$. Then $\begin{bmatrix} xy \\ a' \end{bmatrix} = \begin{bmatrix} xy \\ a' \end{bmatrix}$

$\begin{bmatrix} t \\ a' \end{bmatrix} = \begin{bmatrix} xyt \\ a' \end{bmatrix} = \begin{bmatrix} y \\ a' \end{bmatrix}\begin{bmatrix} xt \\ a' \end{bmatrix}$. An analogous remark applies to d.

Now, for the proof, we have

$$\alpha'\alpha = \begin{pmatrix} a'a + b'c & a'b + b'd \\ * & * \end{pmatrix},$$

and hence

$$\kappa(\alpha'\alpha) = \begin{bmatrix} a'b + b'd \\ a'a + b'c \end{bmatrix}$$

$$= \begin{bmatrix} a'b + b'd \\ (a'a + b'c)d \end{bmatrix} \begin{bmatrix} a'b + b'd \\ d \end{bmatrix}^{-1}$$

$$(a'A + dA = A, \text{ so } (d, a'b) \in W_{\underline{q}})$$

$$= \begin{bmatrix} a'b + b'd \\ a'u + c(a'b + b'd) \end{bmatrix} \begin{bmatrix} a'b \\ d \end{bmatrix}^{-1}$$

$$(u = ad - bc \in U(A, \underline{q}))$$

$$= \begin{bmatrix} a'b + b'd \\ a'u \end{bmatrix} \begin{bmatrix} a't \\ d \end{bmatrix}^{-1} \begin{bmatrix} b \\ d \end{bmatrix}^{-1}$$

(remark above)

$$= \begin{bmatrix} b'd \\ a'u \end{bmatrix} \begin{bmatrix} a't \\ d \end{bmatrix}^{-1} \kappa(\alpha)$$

(see (i))

$$= \begin{bmatrix} b'd \\ u \end{bmatrix} \begin{bmatrix} b' \\ a' \end{bmatrix} \begin{bmatrix} dt \\ a' \end{bmatrix} \begin{bmatrix} a't \\ d \end{bmatrix}^{-1} \kappa(\alpha)$$

(remark above)

$$= \kappa(\alpha') \begin{bmatrix} dt \\ a' \end{bmatrix} \begin{bmatrix} a't \\ d \end{bmatrix}^{-1} \kappa(\alpha)$$

((i) and (1.6)(a))

$$= \kappa(\alpha')\ \kappa(\alpha) \qquad ((1.6)(e)).$$

(iv) κ <u>is a homomorphism, i.e.</u> $H = GL_2(A, \underline{q})$

Let $\alpha, \alpha' \in GL_2(A, \underline{q})$; we claim $\kappa(\alpha'\alpha) = \kappa(\alpha')\ \kappa(\alpha)$.

Suppose $\alpha = \alpha_1\alpha_2$ with $\alpha_2 \in GE_2(A, \underline{q}) \subset H$ (see (ii)). Then $\kappa(\alpha'\alpha) = \kappa(\alpha'\alpha_1)$ while $\kappa(\alpha) = \kappa(\alpha_1)$ also, by (ii). Therefore we are free to replace α by $\alpha\ \alpha_2^{-1}$ for any $\alpha_2 \in GE_2(A, \underline{q})$. We first arrange that $\det(\alpha) = 1$.

Write $a' = 1 + t$. If $t = 0$ we can apply (iii) to finish the proof. If not then, by (V, 9.3), we can choose $\varepsilon_1 \in E_2(A, \underline{q})$ such that $\alpha\varepsilon_1 \in SL_2(A, tA)$; say $\alpha\varepsilon_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. Since $d_1A + c_1A = A = d_1A + c_1{}^2A$ we can find a $d_2 = d_1 + s\ c_1{}^2$ $(s \in A)$ which is a unit modulo a'. Set $\varepsilon_2 = I + c_1 s\ e_{12} \in E_2(A, tA)$. Then we have $\alpha\varepsilon_1\varepsilon_2 = \begin{pmatrix} a_1 & b_1 + c_1 s a_1 \\ c_1 & d_2 \end{pmatrix}$, so we have achieved the hypotheses of part (iii) for $\alpha\varepsilon_1\varepsilon_2$ and α'. Therefore, by (iii) and (ii), we have

$$\kappa(\alpha'\alpha) = \kappa(\alpha'\alpha\varepsilon_1\varepsilon_2) = \kappa(\alpha')\ \kappa(\alpha\varepsilon_1\varepsilon_2) = \kappa(\alpha')\ \kappa(\alpha).$$

(v) Ker (κ) <u>contains</u> $GE_2(A, \underline{q})$ <u>and</u> $[GE_2(A), GL_2(A,\underline{q})]$.

This follows immediately from (ii).

(vi) <u>If</u> $\alpha, \alpha' \in GL_2(A, \underline{q})$ <u>are of the form</u> $\alpha = \begin{pmatrix} 1+at & b \\ ct & d \end{pmatrix}$ <u>and</u> $\alpha' = \begin{pmatrix} 1+at & bt \\ c & d \end{pmatrix}$ <u>with</u> $a, t \in \underline{q}$ <u>then</u> $\kappa(\alpha) = \kappa(\alpha')$.

$$\kappa(\alpha') = \begin{bmatrix} bt \\ 1+at \end{bmatrix} = \begin{bmatrix} b \\ 1+at \end{bmatrix} \begin{bmatrix} t \\ 1+at \end{bmatrix} = \begin{bmatrix} b \\ 1+at \end{bmatrix} = \kappa(\alpha).$$

(cf. (2.1) (a)).

The assertions of Kubota's Theorem (2.4) are contained in (iv), (v), and (vi) above. q.e.d.

§4. PROOF OF THEOREM (2.3): II. CONCLUSION

The homomorphism $\kappa: GL_2(A, \underline{q}) \longrightarrow C$ constructed in Kubota's Theorem ((2.4); see §3) is to be extended to a κ': $GL_3(A, \underline{q}) \longrightarrow C$ so that $E_3(A, \underline{q}) \subset \text{Ker}(\kappa')$. We have seen in §2 that this will complete the proof of (2.3). We can assume, of course, that $\underline{q} \neq 0$.

Since dim $A \leq 1$ and A is commutative we have the stable range conditions $SR_3(A, \underline{q})$ (see (IV, 3.5)) and $SR_2'(A, \underline{q})$ (see (V, 3.4 (b))). Furthermore κ satisfies the condition in (V, 6.4) (see part (vi) of the proof of Kubota's Theorem). It therefore follows from (V, 6.4) that there is a map $\kappa': GL_3(A, \underline{q}) \longrightarrow C$ extending κ, defined with the aid of "standard forms". Explicitly, if

$$\sigma = \begin{pmatrix} \alpha & \gamma \\ 0 & 1 \end{pmatrix} (I + te_{31}) \begin{pmatrix} 1 & \rho \\ 0 & \beta \end{pmatrix}$$

with all factors in $GL_3(A, \underline{q})$, then $\kappa'(\sigma) = \kappa(\alpha)\,\kappa(\beta)$. Moreover all of the results of (V, §7) apply to κ', in particular (7.7). The upshot is that in order to show that κ' is a homomorphism whose kernel contains $E_3(A, \underline{q})$ it will suffice to show that

(1) $$\kappa(\sigma^{\pi}) = \kappa(\sigma)$$

for all $\sigma \in GL_3(A, \underline{q})$, where

$$\pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It follows further from (V, 8.1) that it suffices to verify (1) for $\sigma = \overline{\alpha}\varepsilon$ where $\overline{\alpha}, \varepsilon \in GL_3(A, \underline{q})$ are of the form

$$\overline{\alpha} = \begin{pmatrix} \alpha & \gamma \\ 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 \\ c \end{pmatrix}, \quad \varepsilon = I + te_{31}.$$

We are further allowed (by (V, 8.1)) to replace σ by $\tau\sigma$ for any $\tau = I + qe_{12}$ $(q \in \underline{q})$ and we can thereby arrange that $a_{11} \neq 0$.

Now

$$\sigma = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} + tc & a_{22} & c \\ t & 0 & 1 \end{pmatrix} \text{ and}$$

$$\sigma^{\pi} = \begin{pmatrix} a_{11} & 0 & a_{12} \\ t & 1 & 0 \\ a_{21} + tc & c & a_{22} \end{pmatrix}.$$

Since $A/a_{11}A$ is semi-local there is an $s \in \underline{q}$ such that $t + s(a_{21} + tc)$ is comaximal with a_{11}. Since s is determined only modulo $a_{11}\underline{q}$ we can further choose s so that $d = 1 + sc \neq 0$; note that $t + s(a_{21} + tc) = sa_{21} + td$.

Put $\delta = I + se_{23}$, so that

$$\delta\sigma^{\pi} = \begin{pmatrix} a_{11} & 0 & a_{12} \\ sa_{21} + td & d & sa_{22} \\ a_{21} + tc & c & a_{22} \end{pmatrix}.$$

Since $(a_{11}, sa_{21} + dt) \in W_{\underline{q}}$ (by construction) there is an

$\omega = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \in SL_2(A, \underline{q})$ such that

$$(2) \qquad \omega \begin{pmatrix} a_{11} & 0 \\ sa_{21} + td & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$$

for some $x, y \in A$. Writing $\overline{\omega} = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}$ we have

$$\bar{\omega}\delta\sigma^{\pi} = \begin{pmatrix} 1 & x & w_{11}\,a_{12} + w_{12}\,sa_{22} \\ 0 & y & w_{21}\,a_{12} + w_{22}\,sa_{22} \\ a_{21} + tc & c & a_{22} \end{pmatrix}$$

Let $u = a_{21} + tc$ and put $\varepsilon_1 = I + ue_{31}$, so that $\bar{\beta} = \varepsilon_1^{-1}\,\bar{\omega}\cdot$

$$\delta\sigma^{\pi} = \begin{pmatrix} 1 & \rho \\ 0 & \beta \end{pmatrix}, \text{ where } \beta = \begin{pmatrix} y & w_{21}\,a_{12} + w_{22}\,sa_{22} \\ c-ux & a_{22} - u(w_{11}\,a_{12} + w_{12}\,sa_{22}) \end{pmatrix}.$$

Now $\sigma^{\pi} = (\delta^{-1}\,\bar{\omega}^{-1})\varepsilon_1\bar{\beta}$ is a standard form for σ, since

$$\delta^{-1}\,\bar{\omega}^{-1} = \begin{pmatrix} \omega^{-1} & \begin{matrix} 0 \\ -s \end{matrix} \\ 0\ \ 0 & 1 \end{pmatrix}$$ is of "type L" (see (V, §6)). Therefore

$$\kappa'(\sigma^{\pi}) = \kappa(\omega)^{-1}\,\kappa(\beta),$$

and we must show that this equals $\kappa'(\sigma) = \kappa(\alpha) = \begin{bmatrix} a_{12} \\ a_{11} \end{bmatrix}$

To solve for ω we make equation (2) explicit,

$$\begin{pmatrix} w_{11}a_{11} + w_{12}(sa_{21} + td) & w_{12}d \\ w_{21}a_{11} + w_{22}(sa_{21} + td) & w_{22}d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}.$$

Since $\det(\omega) = 1$ the left side of (2) has determinant $a_{11}d$, so $y = a_{11}d$. Therefore $w_{22}d = a_{11}d$, and since $d \neq 0$ (by construction above) we conclude that $w_{22} = a_{11}$. Making this substitution in the equation of (2,1) coordinates, we can again cancel a_{11} to conclude that $w_{21} = -sa_{21} + td$. Therefore β has (1,2) coordinate $-(sa_{21} + td)a_{12} + a_{11}sa_{22} = s(a_{11}a_{22} - a_{12}a_{21}) - tda_{12} = s - tda_{12}$, so β has the form

$$\beta = \begin{pmatrix} a_{11}d & s - tda_{12} \\ * & * \end{pmatrix}.$$

On the other hand

$$\omega = \begin{pmatrix} w_{11} & w_{12} \\ (sa_{21} + td) & a_{22} \end{pmatrix}$$

where, recall, $d = 1 + sc$. We can now compute $\kappa(\omega)^{-1}\,\kappa(\beta)$.

$$\kappa(\omega)^{-1} = \begin{bmatrix} -(sa_{21} + td) \\ a_{11} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -(sa_{21} + td) \\ a_{11} \end{bmatrix}^{-1} \begin{bmatrix} a_{12} \\ a_{11} \end{bmatrix}^{-1} \kappa(\alpha)$$

$$= \kappa(\alpha) \begin{bmatrix} -sa_{12}a_{21} - tda_{12} \\ a_{11} \end{bmatrix}^{-1}$$

$$= \kappa(\alpha) \begin{bmatrix} s(1 - a_{11}a_{22} - tda_{12} \\ a_{11} \end{bmatrix}^{-1}$$

$$= \kappa(\alpha) \begin{bmatrix} s - tda_{12} \\ a_{11} \end{bmatrix}^{-1}.$$

Next

$$\kappa(\beta) = \begin{bmatrix} s - tda_{12} \\ a_{11}d \end{bmatrix}$$

$$= \begin{bmatrix} s - tda_{12} \\ a_{11} \end{bmatrix} \begin{bmatrix} s - tda_{12} \\ d \end{bmatrix}$$

$$= \begin{bmatrix} s - tda_{12} \\ a_{11} \end{bmatrix} \begin{bmatrix} s \\ d \end{bmatrix}$$

$$= \begin{bmatrix} s - t\alpha_{12} \\ \alpha_{11} \end{bmatrix},$$

because $d = 1 + sc$. Thus, indeed, $\kappa(\omega)^{-1}\,\kappa(\beta) = \kappa(\alpha)$. q.e.d.

§5. MENNICKE SYMBOLS $\left[\frac{b}{\alpha}\right]$.

In this section *we fix a noetherian integral domain* A *of dimension* ≤ 1, *and an ideal* $\underline{q} \neq 0$ in A. Starting from a Mennicke symbol $[\ ]: W_{\underline{q}} \longrightarrow C$, we propose to construct a symbol $(\alpha, \underline{b}) \longmapsto \left[\frac{\underline{b}}{\alpha}\right]$, $\overline{W}_{\underline{q}} \longrightarrow C$, where

$$\overline{W}_{\underline{q}} = \{(\alpha, \underline{b}) \mid \alpha \equiv \text{mod } \underline{q};\ \underline{b} \text{ is an invertible ideal } \subset \underline{q};\ \alpha A + \underline{b} = A\}.$$

(5.1) PROPOSITION. *There exists a function*

$$[\ ]: \overline{W}_{\underline{q}} \longrightarrow C$$

satisfying:

M0. *If* $(\alpha, b) \in W_{\underline{q}}$ *and* $b \neq 0$ then $\begin{bmatrix} bA \\ \alpha \end{bmatrix} = \begin{bmatrix} b \\ \alpha \end{bmatrix}$;

M1 (a). $\left[\frac{\underline{b}}{1}\right] = 1$ *for all* $(1, \underline{b}) \in \overline{W}_{\underline{q}}$;

M1 (b). $\left[\frac{\underline{b}}{\alpha + b}\right] = \left[\frac{\underline{b}}{\alpha}\right]$ *for all* $(\alpha, \underline{b}) \in \overline{W}_{\underline{q}}$ *and* $b \in \underline{b}$;

M2 (a). $\left[\frac{\underline{b}_2\,\underline{b}_2}{\alpha}\right] = \left[\frac{\underline{b}_1}{\alpha}\right]\left[\frac{\underline{b}_2}{\alpha}\right]$ *for all* $(\alpha, \underline{b}_1)$, $(\alpha, \underline{b}_2) \in \overline{W}_{\underline{q}}$; and

M2 (b). $\left[\frac{\underline{b}}{\mathfrak{a}_1\ \mathfrak{a}_2}\right] = \left[\frac{\underline{b}}{\mathfrak{a}_1}\right]\left[\frac{\underline{b}}{\mathfrak{a}_2}\right]$ for all $(\mathfrak{a}_1, \underline{b})$,

$(\mathfrak{a}_2, \underline{b}) \in \overline{W}_{\underline{q}}$.

Moreover this function is uniquely determined by conditions M0, MS1 (a) and (b), and MS2 (a).

For the proof we shall require:

(5.2) LEMMA. Let $\underline{b}$ be an invertible ideal and let $\mathfrak{a}$ be any non zero ideal in A. Then there is an invertible ideal $\underline{c}$, comaximal with $\mathfrak{a}$, such that $\underline{b}\ \underline{c}$ is principal.

Proof. Let $P = \underline{b}^{-1}$, and let $S^{-1}A$ be the (semi-) localization of A at the (finite set of) maximal ideals containing $\mathfrak{a}$. Then, since $S^{-1}A$ is semi-local, we have $S^{-1}P \simeq S^{-1}A$. It follows that there is a homomorphism $h: P \longrightarrow A$ such that $S^{-1}h$ is an isomorphism. Then $\underline{c} = \mathrm{Im}(h) \simeq P \simeq \underline{b}^{-1}$ is contained in no maximal ideal containing $\mathfrak{a}$. q.e.d.

Proof of (5.1).

Uniqueness. Given $(\mathfrak{a}, \underline{b}) \in \overline{W}_{\underline{q}}$ put $t = 1 - \mathfrak{a} \in \underline{q}$. If $t = 0$ then $\left[\frac{\underline{b}}{\mathfrak{a}}\right] = 1$ by M1 (a). If not use (5.2) to find a $\underline{c}$ comaximal with $t\underline{b}$ such that $\underline{b}\ \underline{c} = dA$ for some $d \in A$. By the Chinese Remainder Theorem we can find $\mathfrak{a}'$ so that

$$\mathfrak{a}' \equiv \mathfrak{a} \quad \mathrm{mod}(\underline{b} \cap tA)$$

$$\mathfrak{a}' \equiv 1 \quad \mathrm{mod}\ \underline{c}.$$

Since $\mathfrak{a}' \equiv \mathfrak{a} \equiv 1 \bmod tA$ we have $\mathfrak{a}' \equiv 1 \bmod (tA \cap \underline{c})$ $(= t\underline{c})$. Therefore

$$\left[\frac{\underline{b}}{\mathfrak{a}}\right] = \left[\frac{\underline{b}}{\mathfrak{a}'}\right]\left[\frac{t\underline{c}}{\mathfrak{a}'}\right] \qquad \text{(M1 (b) and M1 (a))}$$

$$= \begin{bmatrix} t\underline{bc} \\ a' \end{bmatrix} \qquad \text{(MS2 (a))}$$

$$= \begin{bmatrix} dt \\ a' \end{bmatrix} \qquad \text{(M0)}$$

$$= \begin{bmatrix} d \\ a' \end{bmatrix} \begin{bmatrix} t \\ a' \end{bmatrix}$$

$$= \begin{bmatrix} d \\ a' \end{bmatrix} .$$

<u>Existence</u>. Define $\begin{bmatrix} \underline{b} \\ a \end{bmatrix}$ as above. There is no ambiguity if $t = 0$ so assume $t \neq 0$. Since the congruences determine a' uniquely modulo $(\underline{b} \cap tA)\underline{c}$, and hence modulo $dA = \underline{bc}$, it follows that $\begin{bmatrix} d \\ a' \end{bmatrix}$ does not depend on the choice of a', for a given choice of $\underline{c}$. Moreover d is determined up to a unit factor, and $\begin{bmatrix} d \\ a' \end{bmatrix}$ is unaffected by such a change.

Suppose $\underline{c}_1$ and $\underline{c}_2$ both satisfy the conditions on $\underline{c}$ above, say $\underline{c}_i\underline{b} = d_iA$ $(i = 1, 2)$. Choose $\underline{b}'$ comaximal with $\underline{c}_1 \underline{c}_2 \underline{b}t$ in the same ideal class as $\underline{b}$, i.e. so that $\underline{b}'\underline{c}_i = e_iA$ for some $e_i \in A$ $(i = 1, 2)$; this is possible by (5.2). Choose a' now so that

$$a' \equiv a \mod (\underline{b} \cap tA)$$

$$a' \equiv 1 \mod \underline{c}_1\ c_2\ \underline{b}'.$$

Note that $\underline{c}_1 \underline{c}_2 \underline{b}' = \underline{c}_1 e_2 = \underline{c}_2 e_1$ is prime to $\underline{b}t$. Since $a' \equiv a \equiv 1 \mod tA$ we have $a' \equiv 1 \mod e_it$ $(i = 1, 2)$. Therefore

$$\begin{bmatrix} d_1 \\ a' \end{bmatrix} = \begin{bmatrix} d_1 \\ a' \end{bmatrix} \begin{bmatrix} e_2t \\ a' \end{bmatrix} = \begin{bmatrix} d_1e_2t \\ a' \end{bmatrix}, \text{ and similarly, } \begin{bmatrix} d_2 \\ a' \end{bmatrix} = \begin{bmatrix} d_2e_1t \\ a' \end{bmatrix}.$$

But $d_1e_2A = (\underline{c}_1\ \underline{b})\ (\underline{c}_2\ \underline{b}') = (\underline{c}_2\ \underline{b})\ (\underline{c}_1\ \underline{b}') = d_2c_1A$, so $d_1e_2 = ud_2e_1$ for some $u \in U(A)$. Therefore $\begin{bmatrix} d_1 \\ a' \end{bmatrix} = \begin{bmatrix} d_1e_2t \\ a' \end{bmatrix} = \begin{bmatrix} ud_2e_1t \\ a' \end{bmatrix} = \begin{bmatrix} d_2e_1t \\ a' \end{bmatrix}\begin{bmatrix} ut \\ a' \end{bmatrix} = \begin{bmatrix} d_2 \\ a' \end{bmatrix}$, because $a' \equiv 1 \bmod tA$ $(= utA)$. This shows that $[\ ]: \overline{W}_{\underline{q}} \longrightarrow C$ is well defined, as above. We must now check the axioms.

M0. If $\underline{b} = bA$ we can choose $\underline{c} = A$, $d = b$, and $a' = a$ in the construction above. Then we have $\begin{bmatrix} bA \\ a \end{bmatrix} = \begin{bmatrix} d \\ a' \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$.

M1 (a). This is part of the definition.

M1 (b). Given $(a, \underline{b}) \in \overline{W}_{\underline{q}}$ and $b \in \underline{b}$ we must show that $\begin{bmatrix} \underline{b} \\ a + b \end{bmatrix} = \begin{bmatrix} \underline{b} \\ a \end{bmatrix}$. Write $a = 1 + t$ $(t \in \underline{q})$. Choose $\underline{c}$ comaximal with $\underline{b}t$ if $t \neq 0$ and also with $\underline{b}(t + b)$ if $t + b \neq 0$ (the assertion is trivial if t or t + b equals zero) and such that $\underline{bc} = dA$ for some $d \in A$. Now choose a_1' and a_2' such that

$$a_1' \equiv a \bmod (\underline{b} \cap tA), \quad a_2' \equiv (a + b) \bmod (\underline{b} \cap (t + b)A)$$

$$a_1' \equiv 1 \bmod \underline{c}, \quad a_2' \equiv 1 \bmod \underline{c}.$$

Then, by definition, $\begin{bmatrix} \underline{b} \\ a \end{bmatrix} = \begin{bmatrix} d \\ a_1' \end{bmatrix}$ and $\begin{bmatrix} \underline{b} \\ a + b \end{bmatrix} = \begin{bmatrix} d \\ a_2' \end{bmatrix}$.

But $a_1' \equiv a \equiv a_2' \bmod \underline{b}$ since $b \in \underline{b}$, and $a_1' \equiv 1 \equiv a_2' \bmod \underline{c}$. Hence $a_1' \equiv a_2' \bmod \underline{bc}$ $(= dA)$, so $\begin{bmatrix} d \\ a_1' \end{bmatrix} = \begin{bmatrix} d \\ a_2' \end{bmatrix}$.

M2 (a). Given $(a, \underline{b}_1), (a, \underline{b}_2) \in \overline{W}_{\underline{q}}$ we claim that

$\begin{bmatrix} \underline{b}_1 \\ a \end{bmatrix} \begin{bmatrix} \underline{b}_2 \\ a \end{bmatrix} = \begin{bmatrix} \underline{b}_1\ \underline{b}_2 \\ a \end{bmatrix}$. This follows from M1 (a) if $a = 1$, so assume $t = 1 - a \neq 0$. Choose $\underline{c}_i$ comaximal with $\underline{b}_1\ \underline{b}_2 t$ such that $\underline{c}_i\ \underline{b}_i = d_i A$ $(i = 1, 2)$. Now choose a' so that

$$a' \equiv a \bmod (\underline{b}_1\ \underline{b}_2 \cap tA)$$

$$a' \equiv 1 \bmod \underline{c}_1\ \underline{c}_2 .$$

Since $\underline{c}_1\ \underline{c}_2\ \underline{b}_1\ \underline{b}_2 = d_1\ d_2\ A$ we have, by definition,

$$\begin{bmatrix} \underline{b}_1\ \underline{b}_2 \\ a' \end{bmatrix} = \begin{bmatrix} d_1\ d_2 \\ a' \end{bmatrix} = \begin{bmatrix} d_1 \\ a' \end{bmatrix} \begin{bmatrix} d_2 \\ a' \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ a \end{bmatrix} \begin{bmatrix} \underline{b}_2 \\ a \end{bmatrix} .$$

<u>M2 (b)</u>. Given $(a_1, \underline{b}), (a_2, \underline{b}) \in \overline{W}_{\underline{q}}$ we claim that $\begin{bmatrix} \underline{b} \\ a_1\ a_2 \end{bmatrix} = \begin{bmatrix} \underline{b} \\ a_1 \end{bmatrix} \begin{bmatrix} \underline{b} \\ a_2 \end{bmatrix}$. Write $t_i = 1 - a_i$ $(i = 1, 2)$ and $t = 1 - a_1\ a_2$. If $t_1\ t_2 = 0$ our assertion follows from M1 (a), so assume $t_1\ t_2 \neq 0$. Choose $\underline{c}$ comaximal with $\underline{b} t_1\ t_2$, and with t if $t \neq 0$, so that $\underline{cb} = dA$ for some $d \in A$, as in (5.2). Now choose a_1' and a_2' so that

$$a_i' \equiv a_i \bmod \underline{b} t_i \text{ (and mod } \underline{b} t_i t, \text{ if } t \neq 0)$$

$$a_i \equiv 1 \bmod \underline{c} \qquad (i = 1, 2).$$

Then we have $\begin{bmatrix} \underline{b} \\ a_1 \end{bmatrix} \begin{bmatrix} \underline{b} \\ a_2 \end{bmatrix} = \begin{bmatrix} d \\ a_1' \end{bmatrix} \begin{bmatrix} d \\ a_2' \end{bmatrix} = \begin{bmatrix} d \\ a_1\ a_2' \end{bmatrix}$. Observe that $a_1'\ a_2' \equiv a_1\ a_2 \bmod \underline{b}$ and $a_1'\ a_2' \equiv 1 \bmod \underline{c}$. Therefore, if $a_1\ a_2 = 1$, then $a_1'\ a_2' \equiv 1 \bmod \underline{bc}$ $(= dA)$ and we have $\begin{bmatrix} d \\ a_1'\ a_2' \end{bmatrix} = 1 = \begin{bmatrix} \underline{b} \\ a_1\ a_2 \end{bmatrix}$. If not, i.e. if $t \neq 0$, then we choose a' so that

$$a' \equiv a_1'\ a_2' \bmod (\underline{b} \cap tA)$$

$$a' \equiv 1 \qquad \bmod \underline{c} .$$

Then $\alpha' \equiv \alpha_1' \alpha_2' \equiv \alpha_1 \alpha_2 \bmod \underline{tb}$ so $\left[\frac{b}{\alpha_1 \alpha_2}\right] = \left[\frac{d}{\alpha'}\right]$. Moreover

$\alpha' \equiv \alpha_1' \alpha_2' \bmod \underline{bc}$ (= dA) so $\left[\frac{d}{\alpha'}\right] = \left[\frac{d}{\alpha_1' \alpha_2'}\right] = \left[\frac{b}{\alpha_1}\right]\left[\frac{b}{\alpha_2}\right]$

q.e.d.

Remark. The usual Mennicke symbol $\left[\frac{b}{\alpha}\right]$ equals 1 if $\alpha \in$ U(A). However it can happen that $\left[\frac{\underline{b}}{\alpha}\right] \neq 1$, even if $\alpha \in$ U(A), if $\underline{b}$ is not principal. We shall see examples of this in §8.

§6. RECIPROCITY LAWS OVER DEDEKIND RINGS, AND THEIR EQUIVALENCE WITH MENNICKE SYMBOLS.

Throughout this section $\underline{q}$ denotes a non zero ideal in a Dedekind ring A, and we shall write X = max(A). For $\underline{p} \in X$ we introduct the group

$$U_{\underline{p}}(\underline{q}) = U(A/\underline{pq}, \underline{q}/\underline{pq})$$
$$= \{\text{units in } A/\underline{pq} \text{ which are} \equiv 1 \bmod \underline{q}/\underline{pq}\}.$$

Its description depends on whether or not $\underline{p}$ divides $\underline{q}$.

Case $\underline{p} \nmid \underline{q}$. Then, by the Chinese Remainder Theorem, $A/\underline{pq} = (A/\underline{p}) \times (A/\underline{q})$, and the corresponding product decomposition of $U(A/\underline{pq})$ yields a canonical isomorphism

(1) $\quad U_{\underline{p}}(\underline{q}) \simeq U(A/\underline{p})$.

Case $\underline{p} \mid \underline{q}$. We can write $\underline{q} = \underline{p}^h \underline{q}'$ where $\underline{q}'$ is prime to $\underline{p}$ and $h = v_{\underline{p}}(\underline{q}) > 0$. In this case we can write $A/\underline{pq} = (A/\underline{p}^{h+1}) \times (A/\underline{q}')$, and we deduce a canonical isomorphism

$$U_{\underline{p}}(\underline{q}) \simeq U(A/\underline{p}^{h+1}, \underline{p}^h/\underline{p}^{h+1}) \quad (h = v_{\underline{p}}(\underline{q}) > 0).$$

Since $\underline{a} = \underline{p}^h/\underline{p}^{h+1}$ has square zero it follows that $a \mapsto 1 + a$ is an isomorphism from the additive group of $\underline{a}$ to the multiplicative group, $1 + \underline{a}$. Thus we can further write

(2) $$U_{\underline{p}}(\underline{q}) \simeq 1 + (\underline{p}^h/\underline{p}^{h+1}) \simeq \underline{p}^h/\underline{p}^{h+1} \quad (h = v_{\underline{p}}(\underline{q}) > 0).$$

This module is unchanged by localization at $A_{\underline{p}}$, which is a discrete valuation ring, and hence we see that

(3) $$\underline{p}^h/\underline{p}^{h+1} \simeq A/\underline{p}$$

(non canonically).

We conclude therefore that $U_{\underline{p}}(\underline{q})$ is isomorphic to the multiplicative group of $A/\underline{p}$ if $\underline{p} \nmid \underline{q}$, and to the additive group of $A/\underline{p}$ if $\underline{p} | \underline{q}$.

Let $U'_{\underline{p}}(\underline{q})$ denote the inverse image in A of $U_{\underline{p}}(\underline{q})$. Thus $U'_{\underline{p}}(\underline{q})$ is the set of $a \in A$ such that $a \notin \underline{p}$ and $a \equiv 1$ mod $\underline{q}$. If $\chi : U_{\underline{p}}(\underline{q}) \longrightarrow C$ is a homomorphism and is $a \in U'_{\underline{p}}(\underline{q})$, we allow ourselves to write $\chi(a)$ for the value of χ at the residue class of a in $U_{\underline{p}}(\underline{q})$.

We are going to show below that Mennicke symbols $[\ ]: W_{\underline{q}} \longrightarrow C$ are equivalent with the following objects.

(6.1) DEFINITION. A $\underline{q}$-reciprocity with values in an abelian group C is a collection $\{\chi_{\underline{p}} \mid \underline{p} \in X\}$ of homomorphisms

$$\chi_{\underline{p}}: U_{\underline{p}}(\underline{q}) \longrightarrow C$$

satisfying $\underline{q}$-R0 and $\underline{q}$-R1 below.

$\underline{q}$-R0. If $a \in U'_{\underline{p}}(\underline{q})$ then $\chi_{\underline{p}}(a)^{v_{\underline{p}}(1-a)} = 1$.

$\underline{q}$-R1. If $a \equiv 1$ mod $\underline{q}$, if $aA + bA = A$, and if $a \neq 0 \neq b$, then

(4) $$\Pi_{\underline{p}|b}\ \chi_{\underline{p}}(a)^{v_{\underline{p}}(b)} = \Pi_{\underline{p}|a}\ \chi_{\underline{p}}(b)^{v_{\underline{p}}(a)}.$$

The last axiom requires some comment. If $\underline{p}|b$ then $a \notin \underline{p}$ so $a \in U'_{\underline{p}}(\underline{q})$, and the left side makes sense. On the other hand, if $\underline{p}|a$ then, since $a \equiv 1 \bmod \underline{q}$, $\underline{p} \nmid \underline{q}$. In this case therefore we have a canonical isomorphism $U_{\underline{p}}(\underline{q}) \simeq U(A/\underline{p})$ (see (1)), and $b(\notin \underline{p})$ represents an element of this group. It is in the this sense that we interpret the right side of (4). In case a or b equals 0, the other is a unit. Then one side of (4) is the empty product (hence = 1) and all exponents in the other are zero (hence it is 1 also).

Concerning $\underline{q}$-RO it is automatically satisfied for $\underline{p} \nmid \underline{q}$, as the following result shows.

(6.2) PROPOSITION. Let $\{\chi_{\underline{p}}\}$ be a collection of homomorphisms as in (6.1). Then $\underline{q}$-RO is equivalent to each of the conditions:

$\underline{q}$-RO′. If $a \in U'_{\underline{p}}(\underline{q})$ then $\chi_{\underline{p}}(a)^{v_{\underline{p}}(\underline{q})} = 1$; and

$\underline{q}$-RO″. If $v_{\underline{p}}(\underline{q})$ is not a multiple of char$(A/\underline{p})$ then $\chi_{\underline{p}}$ is trivial.

Proof. $\underline{q}$-RO $\Rightarrow$ $\underline{q}$-RO′. Let $h = v_{\underline{p}}(\underline{q})$ and let $a \in U'_{\underline{p}}(\underline{q})$. There is nothing to prove unless $h > 0$, and if $v_{\underline{p}}(1 - a) = h$ then $\underline{q}$-RO′ agrees with $\underline{q}$-RO. But if $v_{\underline{p}}(1 - a) > h$ then $a \equiv 1 \bmod \underline{pq}$ so $\chi_{\underline{p}}(a) = 1$ already, in this case.

$\underline{q}$-RO′ $\Rightarrow$ $\underline{q}$-RO. Let h and a be as above. First suppose $h = 0$. If $v_{\underline{p}}(1 - a) = 0$ there is nothing to prove. If $v_{\underline{p}}(1 - a) > 0$ then $a \equiv 1 \bmod \underline{p}$, and hence $a \equiv 1 \bmod \underline{pq}$, so $\chi_{\underline{p}}(a) = 1$.

Next suppose $h > 0$. Then, just as above, $\underline{q}$-RO and $\underline{q}$-RO′ agree if $v_{\underline{p}}(1 - a) = h$, and otherwise $a \equiv 1 \bmod \underline{pq}$, so that $\chi_{\underline{p}}(a) = 1$ already.

$\underline{q}$-RO$'$ $\Longleftrightarrow$ $\underline{q}$-RO". Neither axiom asserts anything non trivial if $\underline{p} \nmid \underline{q}$, so suppose $\underline{p}|\underline{q}$. In this case $U_{\underline{p}}(\underline{q}) \simeq A/\underline{p}$. the additive group (see (2) and (3) above), and $\underline{q}$-RO$'$ asserts that $\mathrm{Im}(\chi_{\underline{p}})$ has exponent $h = v_{\underline{p}}(\underline{q}) > 0$. If char$(A/\underline{p})$ = 0 then $A/\underline{p}$ is divisible, so it has no non trivial quotients of finite exponent. If char$(A/\underline{p}) = p > 0$ then $A/\underline{p}$ can have a non trivial quotient of exponent h if and only if p divides h. This establishes the equivalence of $\underline{q}$-RO$'$ and $\underline{q}$-RO". q.e.d.

(6.3) THEOREM. *Let C be an abelian group. There is a bijective correspondence between Mennicke symbols* $[\ ]: W_{\underline{q}} \longrightarrow C$ *and* $\underline{q}$-*reciprocities* $\{\chi_{\underline{p}}\}$ *with values in C, defined by:*

$$\chi_{\underline{p}}(a) = \begin{bmatrix} \underline{p}\ \underline{q} \\ a \end{bmatrix} \qquad (a \in U'_{\underline{p}}(\underline{q}))$$

and

$$\begin{bmatrix} b \\ a \end{bmatrix} = \Pi_{\underline{p}|b}\ \chi_{\underline{p}}(a)^{v_{\underline{p}}(b)} \qquad ((a, b) \in W_{\underline{q}},\ a \neq 0 \neq b).$$

Note that we have made use here of (5.1) which makes available the symbols $\begin{bmatrix} \underline{p}\ \underline{q} \\ a \end{bmatrix}$ above. This is legitimate because, since A is Dedekind, all non zero ideals are invertible, so we have the hypothesis of (5.1) for all $\underline{p}\ \underline{q}$.

Proof. Suppose first that $[\ ]: W_{\underline{q}} \longrightarrow C$ is a Mennicke symbol, and extend it to $[\ ]: \overline{W}_{\underline{q}} \longrightarrow C$ as in (5.1). Suppose $a \equiv 1 \bmod \underline{q}$ and $a \neq 0$. If $\underline{b}_1$ and $\underline{b}_2$ are comaximal with a (and $\neq 0$) then, since $\begin{bmatrix} \underline{q} \\ a \end{bmatrix} = 1$ (see M1 (a) and M1 (b) of (5.1)), we have

$$\begin{bmatrix} \underline{b}_1\ \underline{b}_2\ \underline{q} \\ a \end{bmatrix} = \begin{bmatrix} \underline{b}_1\ \underline{b}_2\ \underline{q} \\ a \end{bmatrix} \begin{bmatrix} \underline{q} \\ a \end{bmatrix} = \begin{bmatrix} \underline{b}_1\ \underline{q}\ \underline{b}_2\ \underline{q} \\ a \end{bmatrix}$$

$$= \begin{bmatrix} \underline{b}_1 & \underline{q} \\ & a \end{bmatrix} \begin{bmatrix} \underline{b}_2 & \underline{q} \\ & a \end{bmatrix}.$$

Therefore, for any $\underline{b} \neq 0$ which is comaximal with a,

$$\begin{bmatrix} \underline{b} & \underline{q} \\ & a \end{bmatrix} = \Pi_{\underline{p}|\underline{b}} \begin{bmatrix} \underline{p} & \underline{q} \\ & a \end{bmatrix}^{v_{\underline{p}}(\underline{b})}$$

$$= \Pi_{\underline{p}|\underline{b}} \chi_{\underline{p}}(a)^{v_{\underline{p}}(\underline{b})},$$

where we define $\chi_{\underline{p}}(a)$ here to be $\begin{bmatrix} \underline{p} & \underline{q} \\ & a \end{bmatrix}$. Note that $a \in U'_{\underline{p}}(\underline{q})$ and that $\chi_{\underline{p}}(a)$ depends on a only modulo $\underline{p}\,\underline{q}$ by M1 ($\underline{b}$). Hence we can view $\chi_{\underline{p}}$ as a map

$$\chi_{\underline{p}}: U_{\underline{p}}(\underline{q}) \longrightarrow C,$$

and M2 ($\underline{b}$) implies it is a homomorphism. In case $\underline{b} \subset \underline{q}$ above then we have

$$\begin{bmatrix} \underline{b} \\ a \end{bmatrix} = \begin{bmatrix} \underline{b} & \underline{q} \\ & a \end{bmatrix} = \Pi_{\underline{p}|\underline{b}} \chi_{\underline{p}}(a)^{v_{\underline{p}}(\underline{b})}$$

as well.

We must show that $\{\chi_{\underline{p}}\}$ is a $\underline{q}$-reciprocity. We first establish $\underline{q}$-R0. As pointed out in (6.2) this is automatic if $\underline{p} \nmid \underline{q}$. Assume therefore that $h = v_{\underline{p}}(\underline{q}) > 0$. An element of $U_{\underline{p}}(\underline{q})$ can be represented by an element $a \in U'_{\underline{p}}(\underline{q})$ such that $a \equiv 1 \bmod \underline{p}'\underline{q}$ for all primes $\underline{p}' \neq \underline{p}$ that divide $\underline{q}$. Therefore $\chi_{\underline{p}'}(a) = 1$ for these $\underline{p}'$. Moreover we have already remarked that $\chi_{\underline{p}_1}(a)^{v_{\underline{p}_1}(1-a)} = 1$ if $\underline{p}_1 \nmid \underline{q}$. Therefore we have

$$1 = \begin{bmatrix} 1-a \\ a \end{bmatrix} = \Pi_{\underline{p}'|1-a}\, \chi_{\underline{p}'}(a)^{v_{\underline{p}'}(1-a)}$$

$$= \chi_{\underline{p}}(a)^{v_{\underline{p}}(1-a)} .$$

Next we must establish $\underline{q}$-R1. Given $a \equiv 1 \bmod \underline{q}$ and $b \neq 0$ such that $aA + bA = A$ we claim that

$$\Pi_{\underline{p}|b}\, \chi_{\underline{p}}(a)^{v_{\underline{p}}(b)} = \Pi_{\underline{p}|a}\, \chi_{\underline{p}}(b)^{v_{\underline{p}}(a)} .$$

This is trivial if $a = 1$, so assume $t = 1 - a \neq 0$. Then it follows from formula (*) in the proof of (1.7) that

$$\begin{bmatrix} -at \\ a+bt \end{bmatrix} = \begin{bmatrix} bt^2 \\ a \end{bmatrix} = \begin{bmatrix} bt \\ a \end{bmatrix}.$$

Expanding each side of this equation we obtain

$$\begin{bmatrix} bt \\ a \end{bmatrix} = \Pi_{\underline{p}|bt}\, \chi_{\underline{p}}(a)^{v_{\underline{p}}(bt)} = \Pi_{\underline{p}|b}\, \chi_{\underline{p}}(a)^{v_{\underline{p}}(b)} \begin{bmatrix} t \\ a \end{bmatrix},$$

with $\begin{bmatrix} t \\ a \end{bmatrix} = 1$, and,

$$\begin{bmatrix} -at \\ a+bt \end{bmatrix} = \Pi_{\underline{p}|at}\, \chi_{\underline{p}}(a+bt)^{v_{\underline{p}}(at)}$$

$$= \Pi_{\underline{p}|a}\, \chi_{\underline{p}}(a+bt)^{v_{\underline{p}}(a)} \begin{bmatrix} t \\ a+bt \end{bmatrix},$$

with $\begin{bmatrix} t \\ a+bt \end{bmatrix} = 1$. If $\underline{p}|a$ then $\chi_{\underline{p}}$ depends only on the residue class modulo $\underline{p}$, and therefore only modulo a. Since $a + bt = a + b(1-a) \equiv b \bmod a$ we conclude that $\chi_{\underline{p}}(a+bt) = \chi_{\underline{p}}(b)$ if $\underline{p}|a$. Therefore the three equations displayed above imply $\underline{q}$-R1. q.e.d.

For the converse, suppose $\{\chi_{\underline{p}}\}$ is a $\underline{q}$-reciprocity with values in C. If $a \equiv 1 \bmod \underline{q}$, and if $\underline{b} \neq 0$ is comaximal with a, put

$$\begin{bmatrix} \underline{b} \\ a \end{bmatrix} = \Pi_{\underline{p}|\underline{b}} \, \chi_{\underline{p}}(a)^{v_{\underline{p}}(\underline{b})} = \Pi_{\underline{p} \nmid a} \, \chi_{\underline{p}}(a)^{v_{\underline{p}}(\underline{b})}.$$

Evidently $\begin{bmatrix} \underline{b} \\ 1 \end{bmatrix} = 1$ and $\begin{bmatrix} \underline{b} \\ a \end{bmatrix}$ is bimultiplicative in $(a, \underline{b})$.

Next we claim that, if $\underline{b} \subset \underline{q}$, then

$$\begin{bmatrix} \underline{b} \\ a + b \end{bmatrix} = \begin{bmatrix} \underline{b} \\ a \end{bmatrix} \quad \text{for all } b \in \underline{b}.$$

In fact we will show that $\chi_{\underline{p}}(a + b)^{v_{\underline{p}}(\underline{b})} = \chi_{\underline{p}}(a)^{v_{\underline{p}}(\underline{b})}$ for all $\underline{p}$ that divide $\underline{b}$. If $\underline{p} \nmid \underline{q}$ this follows from $a \equiv a + b \bmod \underline{p}$, because $\chi_{\underline{p}}$ depends only on the residue class mod $\underline{p}$ in this case. Suppose, therefore, that $v_{\underline{p}}(\underline{q}) = h > 0$. If $v_{\underline{p}}(\underline{b}) > h$ then $a + b \equiv a \bmod \underline{p}\,\underline{q}$ so $\chi_{\underline{p}}(a + b) = \chi_{\underline{p}}(a)$. If $v_{\underline{p}}(\underline{b}) = h$ then $\underline{q}$-RO′ implies $\chi_{\underline{p}}(a + b)^h = 1 = \chi_{\underline{p}}(a)^h$.

Now if $(a, b) \in W_{\underline{q}}$ define

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{cases} \begin{bmatrix} bA \\ a \end{bmatrix} & \text{if } b \neq 0 \\ 1 & \text{if } b = 0. \end{cases}$$

It follows easily from the remarks above that this symbol is multiplicative in a (MS2b) and depends on a only modulo b (MS1a), even allowing for the case b = 0. Moreover it is clear that, if (a, b_1), $(a, b_2) \in W_{\underline{q}}$, then $\begin{bmatrix} b_1 b_2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 \\ a \end{bmatrix}$ $\begin{bmatrix} b_2 \\ a \end{bmatrix}$ provided either $b_1 \neq 0 \neq b_2$ or $b_1 = 0 = b_2$. Suppose, therefore, that $b_1 \neq 0 = b_2$. Then the left side of the equation is 1 and the right side is $\begin{bmatrix} b_1 \\ a \end{bmatrix}$, where now $a \in U(A)$ since $(a, 0) \in W_{\underline{q}}$. It follows therefore from $\underline{q}$-R1 that

$$\begin{bmatrix} b_1 \\ a \end{bmatrix} = \Pi_{\underline{p}|b_1} \chi_{\underline{p}}(a)^{v_{\underline{p}}(b_1)} = \Pi_{\underline{p}|a} \chi_{\underline{p}}(b_1)^{v_{\underline{p}}(a)} = 1.$$

This establishes MS2a, so we have all the axioms for a Mennicke symbol except MS1a.

We must show that, if $(a, b) \in W_{\underline{q}}$ and if $t \in \underline{q}$, then $\begin{bmatrix} b + ta \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$. If either b or $b + ta$ is zero then a is a unit, and we saw above that both symbols equal 1 in this case. Moreover, if $a = 0$ the equation is an identity. Otherwise we can apply $\underline{q}$-R1 to obtain

$$\begin{aligned} \begin{bmatrix} b + ta \\ a \end{bmatrix} &= \Pi_{\underline{p}|b+ta} \chi_{\underline{p}}(a)^{v_{\underline{p}}(b+ta)} \\ &= \Pi_{\underline{p}|a} \chi_{\underline{p}}(b + ta)^{v_{\underline{p}}(a)} . \end{aligned}$$

If $\underline{p}|a$ then $\underline{p} \nmid \underline{q}$ so $\chi_{\underline{p}}$ depends only on the residue class modulo $\underline{p}$ and therefore only modulo a. For such $\underline{p}$, therefore, we have $\chi_{\underline{p}}(b + ta) = \chi_{\underline{p}}(b)$. Therefore the formula above together with the corresponding formula for $\begin{bmatrix} b \\ a \end{bmatrix}$ shows that $\begin{bmatrix} b + ta \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$, as claimed.

What we have shown now is that the formulas in Theorem (6.3) do, indeed, define functions from Mennicke symbols to $\underline{q}$-reciprocities, as well as in the opposite direction. It is evident from the arguments above that these two functions are each other's inverse, so this completes the proof of (6.3).

Certain reciprocity laws witnessed in number theory and in algebraic geometry are conveniently expressed as "product formulas". In order to find a similar description for the $\underline{q}$-reciprocities encountered here we shall now introduce some local symbols (cf. Serre [3], Chapter III, no. 1).

Let A, $\underline{q}$, and X = max(A) be as above, and let L be the field of fractions of A. Put

$$U_{\underline{q}}(L) = \{a \in U(L) \mid v_{\underline{p}}(a - 1) \geq v_{\underline{p}}(\underline{q}) \text{ whenever } v_{\underline{p}}(\underline{q}) > 0\}.$$

This is a subgroup of U(L). For $\underline{p} \in X$ we define the local $\underline{q}$-symbol at $\underline{p}$ to be the following antisymmetric bilinear (i.e. bimultiplicative) pairing,

$$(\ , \ )_{\underline{p}}: \ U_{\underline{q}}(L) \times U(L) \longrightarrow U_{\underline{p}}(\underline{q});$$

$$(a, b)_{\underline{p}} = \text{the residue class in } U_p(\underline{q}) \text{ of } c, \text{ where}$$

$$c = (-1)^{\alpha\beta} a^{\beta}/b^{\alpha}; \quad \alpha = v_{\underline{p}}(a), \quad \beta = v_{\underline{p}}(b).$$

This definition requires some comment, to insure that c does have a residue class in $U_{\underline{p}}(\underline{q})$. We have

$$v_{\underline{p}}(c) = \beta v_{\underline{p}}(a) - \alpha v_{\underline{p}}(b) = 0, \text{ so } c \in U(A_{\underline{p}})$$

and $U_{\underline{p}}(\underline{q}) = U(A/\underline{p}) = U(A_{\underline{p}}/\underline{p} A_{\underline{p}})$ if $\underline{p} \nmid \underline{q}$. Moreover

$$(5) \qquad c = \begin{cases} a^{\beta} & \text{if } \alpha = 0 \\ b^{-\alpha} & \text{if } \beta = 0. \end{cases}$$

Finally suppose $v_{\underline{p}}(\underline{q}) = h > 0$. Then $v_{\underline{p}}(1 - a) \geq h > 0$ so $\alpha = 0$ and $c = a^{\beta} \in 1 + \underline{p}^{h} A_{\underline{p}}$. Therefore it has a residue class in $U_{\underline{p}}(\underline{q}) \simeq 1 + (\underline{p}^h/\underline{p}^{h+1})$.

The factor $(-1)^{\alpha\beta}$ will play no role in this section. It is inserted to make our notation compatible with that of

Serre [3], Chapter III, no. 4).

If $\underline{a}$ is a fractional ideal of A in L then (see (III, §7)) we have $\mathrm{div}(\underline{a}) = \Sigma\, v_{\underline{p}}(\underline{a})\underline{p} \in D(A)$. In particular, for $a \in U(L)$ we have $\mathrm{div}(a) = \mathrm{div}(aA)$. The support of a divisor $\underline{d} = \Sigma\, n_{\underline{p}}\underline{p}$ is the set of $\underline{p} \in X$ such that $n_{\underline{p}} \neq 0$; it is denoted $\mathrm{supp}(\underline{d})$.

Suppose $(a, b) \in U_{\underline{q}}(L) \times U(L)$. Then it follows from (5) above that $(a, b)_{\underline{p}} = 1$ if $\underline{p} \notin \mathrm{supp}(\mathrm{div}\,(a)) \cup \mathrm{supp}(\mathrm{div}\,(b))$, and the latter is a finite set. Hence we can define

$$((a, b)) = ((a, b)_{\underline{p}})_{\underline{p} \in X} \in \Sigma,$$

where

$$\Sigma = \prod_{\underline{p} \in X} U_{\underline{p}}(\underline{q}).$$

(6.4) THEOREM. *Let* C *be an abelian group, and let* $\chi: \Sigma \longrightarrow C$ *be a homomorphism corresponding to a family of homomorphisms* $\{\chi_{\underline{p}}: U_{\underline{p}}(\underline{q}) \longrightarrow C \mid \underline{p} \in X\}$. *The following conditions are equivalent.*

(a) $\{\chi_{\underline{p}}\}$ *is a* $\underline{q}$*-reciprocity*

(b) (0) *If* $a \in U_{\underline{q}}(L)$ *and* $a \neq 1$ *then* $\chi_{\underline{p}}((a, 1 - a)_{\underline{p}}) = 1$ *for all* $\underline{p} \in X$.

(1) *For all* $(a, b) \in V = \{(a, b) \in U_{\underline{q}}(L) \times U(L) \mid \mathrm{supp}(\mathrm{div}\,(a)) \cap \mathrm{supp}(\mathrm{div}\,(b)) = \phi\}$

(6) $$\prod_{\underline{p} \in X} \chi_{\underline{p}}((a, b)_{\underline{p}}) = 1.$$

(b′) (0′) $\chi_{\underline{p}}$ *is trivial unless* $v_{\underline{p}}(\underline{q})$ *is a multiple of* $\mathrm{char}(A/\underline{p})$.

(1′) *Formula* (6) *holds for all* $(a, b) \in W_{\underline{q}}$.

(6.5) COROLLARY. *There is a canonical epimorphism* $\chi(\underline{q}): \Sigma \longrightarrow SK_1(A, \underline{q})$ *whose kernel is generated by all* $U_{\underline{p}}(\underline{q})$ *for which* $v_{\underline{p}}(\underline{q})$ *is not a multiple of* $\mathrm{char}(A/\underline{p})$ *together with all* $((a, b))$ *with* $(a, b) \in W_{\underline{q}}$.

Proof. There is a universal Mennicke symbol $[\ ]_{\underline{q}}: W_{\underline{q}} \longrightarrow SK_1(A, \underline{q})$ (Theorem (2.3)). To this corresponds a universal $\underline{q}$-reciprocity,

$$\{\chi_{\underline{p}}(\underline{q}): U_{\underline{p}}(\underline{q}) \longrightarrow SK_1(A, \underline{q}) \mid \underline{p} \in X\},$$

by Theorem (6.3).

These $\chi_{\underline{p}}(\underline{q})$ define a homomorphism $\chi(\underline{q}): \Sigma \longrightarrow SK_1(A, \underline{q})$. The universality of $\{\chi_{\underline{p}}(\underline{q})\}$ implies that, if $\chi: \Sigma \longrightarrow C$ is any other homomorphism corresponding to a $\underline{q}$-reciprocity, then $\chi = h \cdot \chi(\underline{q})$ for a unique homomorphism $h: SK_1(A, \underline{q}) \longrightarrow C$. But Theorem (6.4) says that the projection of Σ onto its quotient by the subgroup with the generators indicated above is the solution of the last universal problem. The corollary follows immediately from this observation.

Proof of (6.4). If $a \in U_{\underline{q}}(L)$ and $a \neq 1$ then

$$(a, 1-a)_{\underline{p}} = \begin{cases} a^{v_{\underline{p}}(1-a)} & \text{if } v_{\underline{p}}(a) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Therefore (b) (0) is just $\underline{q}$-R0, and clearly (b´) (0´) is just $\underline{q}$-R0" (see (6.2)).

If $(a, b) \in V$ we can write (6) more explicitly as

$$1 = \Big(\prod_{\underline{p} \in \mathrm{supp}(\mathrm{div}\,(b))} \chi_{\underline{p}}(a^{v_{\underline{p}}(b)})\Big)$$

$$\cdot \Big(\prod_{\underline{p} \in \mathrm{supp}(\mathrm{div}\,(a))} \chi_{\underline{p}}(b^{-v_{\underline{p}}(a)})\Big),$$

or, transposing the second factor,

$$(7)\qquad \prod_{\underline{p}\,\varepsilon\,\mathrm{supp}(\mathrm{div}\,(b))} \chi_{\underline{p}}(a)^{v_{\underline{p}}(b)}$$

$$= \prod_{\underline{p}\,\varepsilon\,\mathrm{supp}(\mathrm{div}\,(a))} \chi_{\underline{p}}(b)^{v_{\underline{p}}(a)}$$

Thus axiom $\underline{q}$-R1 says precisely that (7) (i.e. (6)) is valid for $(a, b)\ \varepsilon\ W_{\underline{q}}$, so (b) (1) $\Rightarrow$ $\underline{q}$-R1 $\Leftrightarrow$ (b′) (1′).

It remains to be shown that, if (6) is valid for all $(a, b)\ \varepsilon\ W_{\underline{q}}$ then it is valid for all $(a, b)\ \varepsilon\ V$. Formula (6) can be written, more briefly, as $\chi((a, b)) = 1$. Since $\chi((a, b))$ is a bilinear (i.e. bimultiplicative) expression in (a, b) the theorem will be proved once we establish:

(6.6) LEMMA. <u>If</u> $(a, b)\ \varepsilon\ V$ <u>we can write</u> $a = a_1 a_2^{-1}$ <u>and</u> $b = b_1 b_2^{-1}$ <u>so that</u> $(a_i, b_j)\ \varepsilon\ W_{\underline{q}}$ $(1 \le i, j \le 2)$.

For then we have $\chi((a, b)) = \chi((a_1, b_1))\ \chi((a_2, b_2))\ \chi((a_1, b_2))^{-1}\ \chi((a_2, b_1))^{-1} = 1$.

<u>Proof of (6.6)</u>. We first seek an $a_2\ \varepsilon\ A$ such that

(i) $v_{\underline{p}}(a_2) = -v_{\underline{p}}(a)$ if $v_{\underline{p}}(a) < 0$,

(ii) $v_{\underline{p}}(a_2) = 0$ if $v_{\underline{p}}(b) \neq 0$

(iii) $v_{\underline{p}}(1 - a_2) \ge v_{\underline{p}}(\underline{q})$ if $v_{\underline{p}}(\underline{q}) > 0$.

Since $v_{\underline{p}}(1 - a) \ge v_{\underline{p}}(\underline{q})$ if $v_{\underline{p}}(\underline{q}) > 0$ we have $v_{\underline{p}}(a) = 0$ for these $\underline{p}$. Therefore the sets of primes in (i) and (iii) are disjoint, and those in (i) and (ii) are disjoint by hypothesis. Those in (ii) and (iii) need not be, but the condition in (iii) implies the condition in (ii) for any prime occurring in both. Therefore we can solve for a_2 (Chinese Remainder Theorem). Put $a_1 = aa_2$, so that $v_{\underline{p}}(a_1) = v_{\underline{p}}(a) + v_{\underline{p}}(a_2) \ge 0$ for all $\underline{p}$ (by (i)). Since $A = \bigcap_{\underline{p}\,\varepsilon\,X} A_{\underline{p}}$ we have $a_1\ \varepsilon\ A$. Moreover $v_{\underline{p}}(1 - a_1) \ge v_{\underline{p}}(\underline{q})$ if $v_{\underline{p}}(\underline{q}) \ge 0$ since this is true of a and of a_2 (by (iii)). Thus $a_1 \equiv 1 \equiv a_2 \bmod \underline{q}$. Moreover, (ii) implies $\text{supp}(\text{div}(a_2)) \cap \text{supp}(\text{div}(b)) = \phi$, so the same is true of a_1.

Next we seek $b_2 \in A$ such that

$$v_{\underline{p}}(b_2) = -v_{\underline{p}}(b) \qquad \text{if } v_{\underline{p}}(b) < 0$$

$$v_{\underline{p}}(b_2) = 0 \qquad \text{if } v_{\underline{p}}(a_1) \neq 0 \text{ or } v_{\underline{p}}(a_2) \neq 0.$$

These conditions are independent, as we remarked above, so b_2 exists, by the Chinese Remainder Theorem. Arguing just as above we see that $b_1 = bb_2 \in A$ and that $\text{supp}(\text{div}(b_j)) \cap \text{supp}(\text{div}(a_i)) = \phi$ $(1 \leq i, j \leq 2)$. This proves the lemma.

§7. RECIPROCITY LAWS IN NUMBER FIELDS.

As in §6, A <u>is a Dedekind ring</u>, X = max(A), L <u>is the field of fractions of</u> A, <u>and</u> $\underline{q}$ <u>is a non zero ideal in</u> A. The "classical" $\underline{q}$-reciprocities, which we discuss in this and the next section, arise from the following type of "reciprocity laws".

Let $V = \{v_{\underline{p}} \mid \underline{p} \in X\}$ and let S_∞ be a set of independent valuations of L inequivalent to those in V. Write $\overline{V} = V \cup S_\infty$. If $v \in \overline{V}$ write L_v for the completion of L in the topology defined by v, and, in case v is non-archimedean, write A_v for the valuation ring of v in L_v. In the latter case we also write, for $t \geq 0$,

$$U_v(t) = \{a \in U(L_v) \mid v(1 - a) \geq t\}.$$

Thus $U_v(0) = U(A_v)$ and $U_v(t)$ is a subgroup of $U(A_v)$ for $t \geq 0$.

(7.1) DEFINITION. A <u>reciprocity law on</u> $\overline{V}$ with values in an abelian group C is a collection of antisymmetric bilinear pairings

$$\left(\frac{\,,\,}{v}\right): \quad U(L_v) \times U(L_v) \longrightarrow C \qquad (v \in \overline{V})$$

such that $\left(\frac{a,1-a}{v}\right) = 1$ if a, $1 - a \in U(L_v)$, and satisfying the following "product formula": If a, $b \in U(L)$ then $\left(\frac{a,\ b}{v}\right)$ = 1 for all but finitely many $v \in \overline{V}$, and

$$(1) \qquad \prod_{v \in \overline{V}} \left(\frac{a,\ b}{v}\right) = 1.$$

In order to obtain from such a reciprocity law a $\underline{q}$-reciprocity, we introduce the conditions:

$(0)_{\underline{p}}$ *If* $\underline{p} \in X$ *and* $v = v_{\underline{p}}$ *then, with* $h = v(\underline{q})$, *we have*

$$\left(\frac{U_v(h+1),\ U(L_v)}{v}\right) = \{1\} = \left(\frac{U_v(h),\ U_v(0)}{v}\right).$$

We also put

C_∞ = the subgroup of C generated by all

$$\left(\frac{a,b}{v}\right) \qquad ((a,\ b) \in W_{\underline{q}};\ v \in S_\infty).$$

(7.2) PROPOSITION. *Condition* $(0)_{\underline{p}}$ *implies that there is a unique homomorphism* $f_{\underline{p}}: U_{\underline{p}}(\underline{q}) \longrightarrow C$ *such that* $\left(\frac{a,\ b}{v_{\underline{p}}}\right)$ $= f_{\underline{p}}((a,\ b)_{\underline{p}})$ *for* $(a,\ b) \in W_{\underline{q}}$. *Let* $\chi_{\underline{p}}$ *be the composite,*

$$U_{\underline{p}}(\underline{q}) \xrightarrow{\ f_{\underline{p}}\ } C \xrightarrow{\text{nat. proj.}} C/C_\infty.$$

If $(0)_{\underline{p}}$ *is satisfied for all* $\underline{p} \in X$ *then* $\{\chi_{\underline{p}} \mid \underline{p} \in X\}$ *is a* $\underline{q}$-*reciprocity. It therefore induces a homomorphism* $SK_1(A,\ \underline{q}) \longrightarrow C/C_\infty$ *whose image is generated by* $\{\mathrm{Im}(\chi_{\underline{p}}) \mid \underline{p} \in X\}$.

Proof. Let $v = v_{\underline{p}}$ and $h = v(\underline{q})$. Choose a generator π for the maximal ideal in A_v. It follows from $(0)_{\underline{p}}$ that

$\alpha \longmapsto \left(\frac{\alpha,\ \pi}{v}\right)$ $(\alpha \in U_{\underline{p}}(h))$ induces a homomorphism $f_{\underline{p}} : U_{\underline{p}}(\underline{q}) = U_{\underline{p}}(h)/U_{\underline{p}}(h+1) \longrightarrow C$, which is independent of the choice of π. Moreover, if $b \in U(L_v)$ and $\beta = v(b)$ then $\left(\frac{\alpha,\ b}{v}\right) = \left(\frac{\alpha,\ \pi}{v}\right)^{\beta} = \left(\frac{\alpha^{\beta},\ \pi}{v}\right)$.

Now suppose $(\alpha, b) \in W_{\underline{q}}$. If $\alpha \in U_v(h)$ (which is automatic if $h > 0$) then $(\alpha, b)_{\underline{p}}$ = the residue class in $U_{\underline{p}}(\underline{q})$ of α^{β}, so $\left(\frac{\alpha,\ b}{v}\right) = f_{\underline{p}}((\alpha, b)_{\underline{p}})$. If $\alpha \notin U_{\underline{p}}(h)$ we must have $h = 0$ and $b \in U(0)$. Since both $\left(\frac{\ ,\ }{v}\right)$ and $(\ ,\ )_{\underline{p}}$ are antisymmetric we have $\left(\frac{\alpha,\ b}{v}\right) = \left(\frac{b,\ \alpha}{v}\right)^{-1} = f_{\underline{p}}((b, \alpha)_{\underline{p}})^{-1} = f_{\underline{p}}((\alpha, b)_{\underline{p}})$. This establishes the first assertion of the proposition. Moreover, it follows from (6.6), that $\left(\frac{\alpha,\ b}{v}\right) = f_{\underline{p}}((\alpha, b)_{\underline{p}})$ whenever $(\alpha, b) \in U_{\underline{q}}(L) \times U(L)$ and $\operatorname{supp}(\operatorname{div}(\alpha)) \cap \operatorname{supp}(\operatorname{div}(b)) = \phi$, because the two sides of the equation are bilinear in (α, b).

Suppose we have $(0)_{\underline{p}}$ for all $\underline{p} \in X$, and we define $\{\chi_{\underline{p}}\}$ as above. To show that this a $\underline{q}$-reciprocity we will verify conditions (b) (0) and (b) (1) of (6.4). For (b) (0) we take $\alpha \in U_{\underline{q}}(L)$ and $\alpha \neq 1$. Then from the definition (7.1) and the formula above we have $1 = \left(\frac{\alpha,\ 1-\alpha}{v_{\underline{p}}}\right) = f_{\underline{p}}((\alpha, 1-\alpha)_{\underline{p}})$, and hence $\chi_{\underline{p}}((\alpha, 1-\alpha)_{\underline{p}}) = 1$ for all $\underline{p} \in X$.

Condition (b) (1) requires that $\prod_{\underline{p} \in X} \chi_{\underline{p}}((\alpha, b)_{\underline{p}}) = 1$ for $(\alpha, b) \in W_{\underline{q}}$. By formula (1) in (7.1) plus the formula above we have

$$1 = \prod_{v \in \overline{V}} \frac{\alpha,\ b}{v} = \Big(\prod_{\underline{p} \in X} f_{\underline{p}}((\alpha, b)_{\underline{p}})\Big) \cdot \prod_{v \in S_\infty} \left(\frac{\alpha,\ b}{v}\right).$$

This is an equation in C. Passing to C/C_∞, the last factor evaporates and $f_{\underline{p}}$ becomes $\chi_{\underline{p}}$. q.e.d.

Remark. For this proof it sufficed that formula (1) of (7.1) hold only for $(a, b) \in W_{\underline{q}}$.

Now assume that L is a number field, i.e. a finite extension of $\underline{\underline{Q}}$, and let S_∞ be the set of all (inequivalent) valuations of L not among those of V. Suppose also that L contains

$$\mu_m,$$

the group of m^{th} roots of unity. Then there is a reciprocity law on $\overline{V}$ with values in μ_m called the m^{th} power reciprocity law. When m = 2 it is the usual quadratic reciprocity law in number fields. Its local symbols will be denoted $\left(\frac{\;,\;}{v}\right)_m$, and we shall now describe them in certain (in fact "most") cases. (The omnibus reference for all material of this section is Bass-Milnor-Serre [1], appendix).

v complex: $L_v = \underline{\underline{C}}$ and $\left(\frac{\;,\;}{v}\right)_m$ is trivial.

v real : $L_v = \underline{\underline{R}}$, and we must have $m \leq 2$.

$$\left(\frac{a,\ b}{v}\right)_2 = \begin{cases} -1 & \text{if } a,\ b < 0 \\ 1 & \text{otherwise.} \end{cases}$$

Now let v be non archimedean, and write k(v) for the residue class field of A_v. It is a finite field of characteristic p with $q = p^f$ elements.

v non archimedean and char(k(v)) $\nmid$ m: Then $\mu_m \subset A_v$ maps injectively into k(v), so U(k(v)) is a cyclic group of order $q - 1 = m \cdot e$ (this defines e).

$$\left(\frac{a,\ b}{v}\right)_m \equiv ((-1)^{\alpha\beta}\ a^\beta/b^\alpha) \mod (\text{rad } A_v),$$

where $\alpha = v(a)$, $\beta = v(b)$. This congruence, and the fact that

$\left(\frac{a,\ b}{v}\right)_m \in \mu_m$ defines the symbol. For example if $\alpha = 0$ then $\left(\frac{a,\ b}{v}\right)_m \equiv a^{\beta e}$, so, if b generates rad A_v, $\left(\frac{a,\ b}{v}\right)_m = 1$ if and only if a becomes an m^{th} power in k(v). Thus we recover the "m^{th} power residue symbol".

The case when $\mathrm{char}(k(v)) \mid m$ is much more complicated. Nevertheless the symbols exist also in that case, and the product formula (1) holds.

With our ideal $\underline{q}$ now, we want to see for what m the conditions $(0)_{\underline{p}}$ hold, and what the group C_∞ is. The last question has an easy answer: C_∞ = <u>all of</u> μ_m <u>unless every</u> $v \in S_\infty$ <u>is complex</u>.

Moreover Proposition (A.17), (cf. also (3.1)) of Bass-Milnor-Serre [1] asserts that $(0)_{\underline{p}}$ holds precisely when

$$\frac{v_{\underline{p}}(\underline{q})}{v_{\underline{p}}(p)} - \frac{1}{p-1} \geq v_p(m),$$

where $p = \mathrm{char}(A/\underline{p})$. In this way one discovers, with the aid of (7.2) certain $\underline{q}$-reciprocities; one of the main theorems (see (7.3) below) states that there are no others. We shall quote this theorem here for future reference and use in these notes.

(7.3) THEOREM (Bass-Milnor-Serre). *Let* A *be a Dedekind ring whose field of fractions* L *is a finite extension of* $\mathbb{Q}$.

(a) *Unless* L *is totally imaginary* (i.e. $\mathbb{R} \otimes_{\mathbb{Q}} L \approx \mathbb{C}^r$ *for some* r) *and* A *is the ring of algebraic integers in* L, *we have* $SK_1(A, \underline{q}) = 0$ *for all ideals* $\underline{q}$ *in* A. *Hence there are no non trivial* $\underline{q}$*-reciprocities*.

(b) *Assume* L *is totally imaginary and* A *is its ring of algebraic integers. Let* m *denote the number of roots of* unity in L, and let $\underline{q} \neq 0$ be an ideal in A. For each (rational) prime p dividing m let j_p be the nearest integer in the interval $[0, v_p(m)]$ to

(2) $$\min_{\underline{p}|p \text{ in } A} \left[\frac{v_{\underline{p}}(\underline{q})}{v_{\underline{p}}(p)} - \frac{1}{p-1} \right],$$

where [x] denotes the integral part of x for $x \in \underline{\underline{R}}$. Then

$$SK_1(A, \underline{q}) \cong \mu_r \qquad (r^{th} \text{ roots of unity}),$$

where

$$r = r(\underline{q}) = \Pi_{p|m}\, p^{j_p}.$$

The universal $\underline{q}$-reciprocity is that induced, as in (7.2), by the r^{th} power reciprocity law in L. If $0 \neq \underline{q}' \subset \underline{q}$ and $r' = r(\underline{q}')$ then the natural homomorphism $SK_1(A, \underline{q}') \longrightarrow SK_1(A, \underline{q})$ corresponds to the $(r'/r)^{th}$ power map $\mu_{r'} \longrightarrow \mu_r$ $(\subset \mu_{r'})$.

It follows easily from formula (2) above that $j_p = 0$ if, for some $\underline{p}$ dividing p, $v_{\underline{p}}(\underline{q}) \leq v_{\underline{p}}(p)$. At the other extreme we have, for example, $j_p = v_p(m)$ if $p^{2v_p(m)}$ divides $\underline{q}$. Thus, in case (b),

(3) $SK_1(A, \underline{q})$ has no p-torsion if, for some $\underline{p}$ dividing p, we have $v_{\underline{p}}(\underline{q}) \leq v_{\underline{p}}(p)$.

(4) If $p^{2v_p(m)}$ divides $\underline{q}$ (e.g. if m^2 divides $\underline{q}$) then the p-primary part of $SK_1(A, \underline{q})$ is isomorphic to that of μ_m.

(7.4) COROLLARY. Let A be as in (7.3). Then $SK_1(A) = 0$ and, for all $n \geq 3$, $SL_n(A) = E_n(A)$ and it is a finitely generated group.

The vanishing of $SK_1(A)$, even in case (b) of (7.3), follows from (7.3) and (3) above. The remaining assertions follow from (V, 4.5) and (V, 1.3).

Note that Theorem (7.3) can also be used in conjunction with Theorem (V, 4.1) to give a determination of the normal subgroups of $SL_n(A)$ for $n \geq 3$. In turn this information solves the "congruence subgroup problem" for $SL_n(A)$, i.e. it decides when there exist subgroups of finite index in $SL_n(A)$ which contain no congruence subgroup. The latter occurs precisely when A is the ring of integers in a totally imaginary number field.

§8. RECIPROCITY LAWS ON ALGEBRAIC CURVES

We shall presume here the basic facts about function fields in one variable.

Consider a ground field k and a finitely generated field extension L of transcendence degree one over k. We assume, for all field extensions k' of k, that $L_{k'} = k' \otimes_k L$ remains a field.

X denotes the following set: $\underline{p} \in X$ if and only if $\underline{p}$ is the maximal ideal of a discrete valuation ring, $A_{\underline{p}}$, such that $k \subset A_{\underline{p}} \subsetneq L$, and L is the field of fractions of $A_{\underline{p}}$. We also write $k(\underline{p}) = A_{\underline{p}}/\underline{p}$; it is a finite extension of k of degree

$$\deg(\underline{p}) = [k(\underline{p}) : k].$$

The valuation corresponding to $A_{\underline{p}}$ is denoted $v_{\underline{p}}$. For lack of a better name, in this ad hoc notation, we will call X the set of "closed points" of L/k. Similarly, if k' is an extension of k, we have the set $X_{k'}$ of closed points of $L_{k'}/k'$, and there is a natural projection $X_{k'} \longrightarrow X (= X_k)$ defined by $\underline{p}' \longmapsto \underline{p}' \cap L$. In case k' is separable over $k(\underline{p})$ we have

$$k' \otimes_k k(\underline{p}) = \prod_{\substack{\underline{p}' \in X_{k'} \\ \underline{p}' \cap L = \underline{p}}} k'(\underline{p}').$$

In the general case we obtain the right side by factoring out the nil radical on the left.

The divisor group, D(X), is the free abelian group generated by X, and there is an exact sequence

$$(1) \qquad 0 \longrightarrow U(k) \longrightarrow U(L) \xrightarrow{\text{div}} D(X)$$

where $\text{div}(a) = \Sigma_{\underline{p} \in X} v_{\underline{p}}(a)\, \underline{p}$. Thus $v_{\underline{p}}(a) = 0$ for almost all $\underline{p}$, and $v_{\underline{p}}(a) = 0$ for all $\underline{p} \in X \iff a \in U(k)$. We also have

$$(2) \qquad \Sigma_{\underline{p} \in X} v_{\underline{p}}(a) \deg(\underline{p}) = 0 \qquad (a \in U(L)).$$

For $\underline{p} \in X$ we define

$$(\ ,\ )_{\underline{p}} : U(L) \times U(L) \longrightarrow U(k)$$

by

$$(a, b)_{\underline{p}} = N_{k(\underline{p})/k}(c), \text{ where}$$

(3) $\quad c =$ the residue class in $k(\underline{p})$ of $(-1)^{\alpha\beta} a^{\beta}/b^{\alpha}$, and

$$\alpha = v_{\underline{p}}(a),\ \beta = v_{\underline{p}}(b).$$

Because of the norm here this does not coincide with the symbol $(a, b)_{\underline{p}}$ introduced in §7.

(8.1) PROPOSITION. _The $(\ ,\ )_{\underline{p}}$ are antisymmetric bilinear pairings with the following properties_:

(a) _If_ a, $1 - a \in U(L)$ _then_ $(a, 1 - a)_{\underline{p}} = 1$ _for all_ $\underline{p} \in X$.

(b) If $a, b \in U(L)$ _then_ $(a, b)_{\underline{p}} = 1$ _for almost all_ $\underline{p} \in X$.

(c) If k' is an extension of k then

$$(a, b)_{\underline{p}} = \prod_{\underline{p}' \in X_{k'},\ \underline{p}' \cap L = \underline{p}} (a, b)_{\underline{p}'}.$$

Proof. $(\ ,\ )_{\underline{p}}$ is obviously bilinear and antisymmetric.

(a) If $v_{\underline{p}}(a) > 0$ then, in (3) above, $\alpha = 0$ and $1 - a \equiv 1 \bmod \underline{p}$, i.e. $c = 1$. Similary, $c = 1$ if $v_{\underline{p}}(1 - a) > 0$, since $a = 1 - (1 - a)$, by antisymmetry. Also $c = 1$ if $\alpha = 0 = \beta$, so assume $\alpha < 0$. Then, the "ultrametric inequality" for valuations shows that $\beta = \alpha < 0$ also. Therefore c^{-1} is the residue class modulo $\underline{p}$ of $(-1)^{\alpha^2} (1 - a/a)^{\alpha} = (-1)^{\alpha^2} (a^{-1} - 1)^{\alpha}$, so $c = (-1)^{\alpha^2} (-1)^{\alpha} = 1$.

(b) It follows from (1) above that $\alpha = 0 = \beta$ for most $\underline{p}$, and for these we have $(a, b)_{\underline{p}} = 1$ clearly.

(c) We have $(a, b)_{\underline{p}} = N_{k(\underline{p})/k}(c) = N_{k' \otimes_k k(\underline{p})/k'}(c)$. Now $k' \otimes_k k(\underline{p}) = \Pi B_{\underline{p}'}$ $(\underline{p}' \in X_{k'}, \underline{p}' \cap L = \underline{p})$ where $B_{\underline{p}'} = A_{\underline{p}'}/\underline{p}A_{\underline{p}'}$, so $N_{k' \otimes_k k(\underline{p})/k'}(c) = \Pi_{\underline{p}'} N_{B_{\underline{p}'}/k'}(c)$. The norm here is the determinant of multiplication by c, and $B_{\underline{p}'}$ has a Jordan-Holder series of length $v_{\underline{p}'}(\underline{p})$ with quotients $k(\underline{p}')$. Thus we can deduce that $N_{B_{\underline{p}'}/k'}(c) = N_{k(\underline{p}')/k'}(c)^{v_{\underline{p}'}(\underline{p})}$.

Now c is the residue class of $(-1)^{\alpha\beta} a^{\beta}/b^{\alpha}$ so $c^{v_{\underline{p}'}(\underline{p})}$ is the residue class of $(-1)^{\alpha'\beta'} a^{\beta'}/b^{\alpha'}$ where $\alpha' = \alpha v_{\underline{p}'}(\underline{p}) = v_{\underline{p}}(a)\, v_{\underline{p}'}(\underline{p}) = v_{\underline{p}'}(a)$, and similarly $\beta' = v_{\underline{p}'}(b)$. Therefore $N_{B_{\underline{p}'}/k'}(c) = (a, b)_{\underline{p}'}$. q.e.d.

(8.2) THEOREM (Weil). If $a, b \in U(L)$ then

(4) $$\prod_{\underline{p} \in X} (a, b)_{\underline{p}} = 1.$$

The key point in the proof is the following lemma, which reduces the theorem to the case of a rational function field. We first note that (8.1) (c) makes it sufficient to prove the theorem when k is algebraically closed, so we shall assume this is the case for the rest of the proof.

(8.3) LEMMA. Suppose K is a subfield of L of transcendence degree one over k, and let Y be the set of "closed points" of K/k. If $(a, b) \in U(K) \times U(L)$ and if $\underline{q} \in Y$ then

$$(a, N_{L/K}(b))_{\underline{q}} = \prod_{\substack{\underline{p} \in X \\ \underline{p} \cap K = \underline{q}}} (a, b)_{\underline{p}}.$$

Proof. Let $K_{\underline{q}}$ be the completion of K with respect to $v_{\underline{q}}$. Then $K_{\underline{q}} \otimes_K L = \prod_{\substack{p \in X \\ \underline{p} \cap K = \underline{q}}} L_{\underline{p}}$, where $L_{\underline{p}}$ is the $v_{\underline{p}}$ completion of L, and $N_{L/K}(b) = \prod N_{L_{\underline{p}}/K_{\underline{q}}}(b)$. Since the symbols $(\, , \,)_{\underline{q}}$ and $(\, , \,)_{\underline{p}}$ clearly extend to the completions we see that it suffices to prove each of the local formulas,

$$(*) \qquad (a, N_{L_{\underline{p}}/K_{\underline{q}}}(b))_{\underline{q}} = (a, b)_{\underline{p}}.$$

It is known that these completions are power series field in one variable over the residue class fields of $A_{\underline{q}}$ and $A_{\underline{p}}$, respectively. Since we are now assuming k is algebraically closed they are power series fields over k.

Any unit of $A_{\underline{q}}$ is a quotient of two local parameters, so $U(K_{\underline{q}})$ is generated by local parameters. The same is true of $L_{\underline{p}}$. Since (*) is bilinear in (a, b) it suffices to establish (*) for $(a, b) = (t, s)$ where $K_{\underline{q}} = k((t))$ and $L_{\underline{p}} = k((s))$. Put $e = [L_{\underline{p}} : K_{\underline{q}}]$, and let

$$(**) \qquad s^e + a_{e-1} s^{e-1} + \dots + a_1 s + a_o = 0$$

be the minimal equation of s over $K_{\underline{q}}$. Then

$$v_{\underline{p}}(a_i\, s^i) = v_{\underline{q}}(a_i)e + i \equiv i \mod e.$$

Therefore the $v_{\underline{p}}(a_i\, s^i)$ are distinct except, possibly, that $v_{\underline{p}}(s^e)$ (= e) is the same as $v_{\underline{p}}(a_o) = e\, v_{\underline{q}}(a_o)$. But then (**) implies the latter must be equal, so $v_{\underline{q}}(a_o) = 1$, and further $v_{\underline{p}}(a_i\, s^i) > e$ $(0 < i < e)$. Now $a_o = (-1)^e N_{L_{\underline{p}}/K_{\underline{q}}}(s)$ so $(t, N_{L_{\underline{p}}/K_{\underline{q}}}(s))_{\underline{q}}$ is the residue class mod t (or s) of $(-1)^{1\cdot 1}\, t^1 / ((-1)^e\, a_o)^1 = (-1)^{1-e}\, (t/a_o)$. On the other hand $(t, s)_{\underline{p}}$ is the residue class mod s of $(-1)^{e\cdot 1}\, t^1/s^e$. Therefore we must show that

$$(-1)^{1-e}\, (t/a_o) \equiv (-1)^e\, t/s^e \mod s,$$

i.e. that

$$a_o/s^e \equiv -1 \mod s.$$

If we divide (**) by s^e we obtain

$$1 + x + a_o/s^e = 0,$$

where $x = s^{-e}\, (\Sigma_{0<i<e}\, a_i\, s^i)$, and we saw above that $v_{\underline{p}}(x) > 0$. q.e.d.

Proof of (8.2). As remarked above, we can assume k is algebraically closed. Let a, b $\in$ U(L). If $a \in k$ then, in the notation of (3), $\alpha = 0$ for all $\underline{p}$ and $(a, b)_{\underline{p}} = a^{v_{\underline{p}}(b)}$. In this case, therefore, (4) reduces to the formula (2): $\Sigma v_{\underline{p}}(b) = 0$. (Deg($\underline{p}$) = 1 for all $\underline{p}$ because k is algebraically closed).

If $a \notin k$ we can apply (8.3) to K = k(a) and then we are reduced to the case L = k(t), t an indeterminate, and a = t. Moreover $b = b_o\, \Pi_i\, (t - x_i)^{n_i}$ with $b_o \in U(k)$ all x_i

ε k, and $n_i \in \underline{\underline{Z}}$. By linearity, therefore, and the case of constants treated above, we can assume $b = t - x$ $(x \in k)$.

Case $x = 0$. X now corresponds to the points of $k \cup \infty$ (the projective line over k). The only non trivial symbols are

$$(t, t)_0 = ((-1)^{1 \cdot 1} \; t^1/t^1 \bmod t)$$
$$= -1$$

and

$$(t, t)_\infty = ((-1)^{(-1)\,(-1)} \; t^{-1}/t^{-1} \bmod t^{-1})$$
$$= -1.$$

Therefore (4) is valid because $(-1)(-1) = 1$.

Case $x \neq 0$. The only non trivial symbols now are

$$(t, t - x)_0 = ((-1)^{1 \cdot 0} \; t^0/(t - x)^1 \bmod t)$$
$$= -x^{-1}$$

$$(t, t - x)_x = ((-1)^{0 \cdot 1} \; t^1/(t - x)^0 \bmod t - x)$$
$$= x$$

$$(t, t - x)_\infty = ((-1)^{(-1)(-1)} \; t^{-1}/(t - x)^{-1} \bmod t^{-1})$$
$$= -1.$$

Since $(-x^{-1}) \cdot (x) \cdot (-1) = 1$ we have established (4) also in this case, thus completing the proof of (8.2).

By virtue of (8.1) and (8.2) the symbols $(\; , \;)_{\underline{p}}$ define a reciprocity law on X in the sense of (7.1). Moreover the symbol $(\; , \;)_{\underline{p}}$ evidently satisfies the condition $(0)_{\underline{p}}$ of §7 for all $h \geq 0$. We can therefore apply this reciprocity law, as in (7.2), to Dedekind rings of the following type:

Let S_∞ be a finite, non-empty, subset of X, and set

$$A = \bigcap_{\underline{p} \notin S_\infty} A_{\underline{p}} = \{\alpha \in L \mid v_{\underline{p}}(\alpha) \geq 0 \text{ for all } \underline{p} \in S_\infty\}.$$

We shall write $A = k[X - S_\infty]$ when we want the notation to be more precise. It is known that A is a Dedekind ring whose localizations at maximal ideals are precisely the $A_{\underline{p}}$ ($\underline{p} \notin S_\infty$). Thus we can identify max(A) with $X - S_\infty$.

It follows now from (7.2) that we obtain an A-reciprocity (i.e. $\underline{q} = A$ in (7.2)) with values in

$$U(k)/N_\infty,$$

where N_∞ is the group generated by $\{\mathrm{Im}(N_{k(\underline{p})/k}) \mid \underline{p} \in S_\infty\}$.

(8.4) COROLLARY. *Let* $A = k[X - S_\infty]$ *as above, and let* N_∞ *be the group just defined. Then there is a homomorphism*

$$SK_1(A) \longrightarrow U(k)/N_\infty$$

whose image is generated by the images of $N_{k(\underline{p})/k}(U(k(\underline{p})))$ *for all* $\underline{p} \notin S_\infty$.

Note that $N_{k(\underline{p})/k}(U(k)) \supset U(k)^{\deg(\underline{p})}$ so that $U(k)/N_\infty$ is a torsion group of exponent g.c.d. $\{\deg(\underline{p}) \mid \underline{p} \in S_\infty\}$.

Corollary (8.4) represents the only classical source of reciprocity laws on curves of the type which occur here in connection with $SK_1(A)$. Of course it gives nothing non trivial if the norms $N_{k(\underline{p})/k}$ are always surjective. This is the case when k is *finite*, and, indeed, in that case we have:

(8.5) THEOREM (Bass-Milnor-Serre [1]). *Suppose* k *is a finite field, and let* A *be as in* (8.4). *Then, for all ideals* $\underline{q}$ *in* A, $SK_1(A, \underline{q}) = 0$.

Just as in (7.4) this implies:

(8.6) COROLLARY. *With* A *as in* (8.5) *we have* $SL_n(A) = E_n(A)$ *for all* $n \geq 3$, *and these are finitely generated groups.*

One is now further tempted to conjecture that $SK_1(A) = 0$ also if k is algebraically closed, for again $N_\infty = U(k)$ in this case. This question was posed by Mumford. We shall see in Chapter XIII that this is not the case. In fact

$SK_1(A)$ can be quite large even when k is algebraically closed, and we can use the theory of this chapter to go backward, then, and deduce the existence of non classical reciprocity laws on curves.

We will close this section now by showing how (8.4) can be used to help compute $SK_1(A)$ in some simple examples.

Let $k = \mathbb{R}$ and let $L_1 = \mathbb{R}(x, y)$ where x and y are subject to the single relation

$$x^2 + y^2 = 1.$$

Thus $A_1 = \mathbb{R}[x, y]$ is the real coordinate ring of the unit circle $S^1 \subset \mathbb{R}^2$. In fact S^1 is precisely the "real locus" of X, i.e. the set of $\underline{p} \in X$ such that $k(\underline{p}) = \mathbb{R}$. All the other points are complex. It follows that the group N_∞ in (8.4) is $N_{\mathbb{C}/\mathbb{R}}(U(\mathbb{C}))$ = the positive reals, and we have an exact sequence

$$U(\mathbb{C}) \xrightarrow{N_{\mathbb{C}/\mathbb{R}}} U(\mathbb{R}) \xrightarrow{\text{sign}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

For technical reasons we want to write signs additively, so that sign (x) = 0 if x > 0 and 1 if x < 0.

If a, b $\in$ U(L) write

$$[a, b]_{\underline{p}} = \text{sign }(a, b)_{\underline{p}} \quad \in \quad \mathbb{Z}/2\mathbb{Z}.$$

Then $[a, b]_{\underline{p}} = 0$ if $\underline{p}$ is complex, so the product formula (4) yields a reciprocity law on the real locus,

(5) $$\Sigma_{t \in S^1} [a, b]_t = 0$$

Moreover we have the homomorphism

$$SK_1(R[x, y]) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

as in (8.4), and it is clearly surjective (because $\max(A_1)$ contains real points). We shall see in Chapter XIII that this is even an isomorphism.

Formula (5) can be made more explicit in a special case. Let f and g be non vanishing real rational functions on S^1 with no common zeros or poles on S^1. Then

$$\Pi_{v_t(g) \neq 0} f(t)^{v_t(g)} \qquad (t \in S^1)$$

and

$$\Pi_{v_t(f) \neq 0} g(t)^{v_t(f)} \qquad (t \in S^1)$$

are non zero real numbers with the same sign!

We shall now give a direct proof of (5) which has the advantage of giving certain $\underline{q}$-reciprocities, when $\underline{q}$ is not the unit ideal, as well.

If f and g are real valued functions which are meromorphic (and not identically zero) in a neighborhood of $t \in \mathbb{R}$ then we can write, just as above,

$$[f, g]_t = \text{sign}\ ((-1)^{nm}\ (f^n/g^m)\ (t)) \in \mathbb{Z}/2\mathbb{Z}$$

where $n = v_t(g)$ and $m = v_t(f)$. Suppose $a \leq b$ and that f and g are meromorphic in an open interval containing a and b. Then they have only a finite number of zeros and poles in $(a \leq t \leq b)$, so we can define

$${}_a[f, g]_b = \Sigma_{a \leq t < b}\ [f, g]_t.$$

These symbols are antisymmetric and bimultiplicative in (f, g) and satisfy the analogue of the property in (8.1) (a). Moreover, if $a \leq b \leq c$ then evidently

(6) $${}_a[f, g]_c = {}_a[f, g]_b + {}_b[f, g]_c.$$

(8.7) PROPOSITION. *Let* f *and* g *be non vanishing real meromorphic functions on an open interval containing* a *and* b $(a \leq b)$. *Then*

(7) $${}_a[f, g]_b = {}_af_ag + {}_bf_bg.$$

Here ${}_xf$ denotes the sign of $f(x - \varepsilon)$ for all sufficiently small $\varepsilon > 0$, and the expression on the right is computed in the ring $\mathbb{Z}/2\mathbb{Z}$. (It is for this reason that we have written signs additively).

Proof. Moving a and b a small amount to the left will change neither side of (7), clearly, so we can assume f and g each have neither a zero or pole at a or at b. Then, we can cut the interval into small subintervals with the same property, and such that, in the interior of each one, at most one point is a singularity of either function. It follows from (6) that the left side of (7) is additive over intervals, and the right side is also because we are adding in $\mathbb{Z}/2\mathbb{Z}$. Therefore it suffices to prove the proposition when there is at most one point in $(a \leq t \leq b)$ which is a singularity of f or of g, and it is not an end point. Further, since the two sides of (7) are bimultiplicative in (f, g) we can assume that the singularities are at most a zero of order one. Therefore we have only the following three cases to consider:

(i) Neither f nor g has a singularity. Then ${}_a f = {}_b f$ and ${}_a g = {}_b g$ so ${}_a f {}_a g + {}_b f {}_b g = 2\ {}_a f {}_a g = 0$, and clearly also ${}_a[f, g]_b = 0$.

(ii) There is a c, $a < c < b$, such that f (or g) has a zero of order one at c, and the other has no singularities. By the symmetry in f and g of both side of (7) we can assume $f(c) = 0$. Then ${}_b f = 1 - {}_a f$ (in $\mathbb{Z}/2\mathbb{Z}$; i.e. ${}_a f$ and ${}_b f$ have opposite signs) and ${}_a g = {}_b g$. Therefore the right side of (7) is ${}_a f {}_a g + (1 - {}_a f) {}_a g = {}_a g$. The left side of (7) is sign $(g(c)) = {}_a g$ also.

(iii) f and g both have a zero of order one at c. Then ${}_b f = 1 - {}_a f$ and ${}_b g = 1 - {}_a g$ so the right side of (7) is ${}_a f {}_a g + (1 - {}_a f)(1 - {}_a g) = 1 - {}_a f - {}_a g$. The left side of (7) is sign $((-1)(f/g)(c)) = 1 + \text{sign}((f/g)(c))$. Clearly the latter term equals ${}_a f - {}_a g$ since each function has the singularities of a linear function in the interval. Thus (7) is established in case (iii), and this concludes the proof of (8.7).

The reciprocity formula (5) is a corollary of (8.7) since we can cut the circle into intervals, each of which is analytically equivalent to a real interval. Then we can apply (8.2) and add up over the intervals. The sum of the terms on the right side of (7) will cancel, and those on the left add up to the left side of (5).

Of more interest, however, is the fact that (8.7) yields $\underline{q}$-reciprocities on the affine line for certain $\underline{q}$. Specifically, let $L_o = \underline{\underline{R}}(T)$, where T is an indeterminate, let $A_o = \underline{\underline{R}}[T]$, and let $\underline{q} = (T^2 - T)A_o$. If $f, g \in U(L)$ and if $\underline{p} \in X$ (the "closed points" of $L_o/\underline{\underline{R}}$) we define $\left(\frac{f, g}{\underline{p}}\right)$ to be trivial unless $\underline{p}$ corresponds to a point t, $0 \leq t < 1$, and in that case we define $\left(\frac{f, g}{\underline{p}}\right) = [f, g]_t$ as above. Then we have $\Sigma_{\underline{p} \in X} \left(\frac{f, g}{\underline{p}}\right) = \Sigma_{0 \leq t < 1} [f, g]_t = {}_0[f, g]_1 = {}_0f_0g + {}_1f_1g$, by (8.7). If $(f, g) \in W_{\underline{q}}$ then $f \equiv 1 \bmod (T^2 - T)A_o$ so $f(0) = 1 = f(1) > 0$. Therefore ${}_0f = 0 = {}_1f$ and we have the reciprocity formula (1) of (7.1) for $(f, g) \in W_{\underline{q}}$. It follows therefore from (7.2) that there is an induced homomorphism

(8) $$SK_1(A_o, \underline{q}) \longrightarrow \underline{\underline{Z}}/2\underline{\underline{Z}}$$

whose image is generated by the symbols $[f, g]_t$ $(0 \leq t < 1;\ (f, g) \in W_{\underline{q}})$. If we take $f(T) = 1 + 8(T^2 - T)$ and $g(T) = (T^2 - T)(T - (1/2))$ then $(f, g) \in W_{\underline{q}}$ and $[f, g]_{1/2} = \text{sign}\,(f(1/2))$. Since $f(1/2) = 1 + 8(1/2)(1/2 - 1) = -1$ we see that (8) <u>is an epimorphism</u>. In contrast, note that $SK_1(A_o) = 0$ since $A_o = \underline{\underline{R}}[T]$ is a euclidean ring.

Finally, we shall give a topological method for constructing the homomorphisms

$$SK_1(R[x, y]) \longrightarrow \underline{\underline{Z}}/2\underline{\underline{Z}}$$

and

$$SK_1(\underline{\underline{R}}[T], (T^2 - T)) \longrightarrow \underline{\underline{Z}}/2\underline{\underline{Z}}$$

above. If $\alpha \in SL_n(A_1)$ then, for each $t \in S^1$, $\alpha(t) \in SL_n(\underline{\underline{R}})$, and α induces a continuous function

$$S^1 \longrightarrow SL_n(\underline{\underline{R}}).$$

Let $[\alpha]$ denote the homotopy class of this function, in $\pi_1(SL_n(\mathbb{R}))$. Then $\alpha \mapsto [\alpha]$ defines a homomorphism

$$SL_n(A_1) \longrightarrow \pi_1(SL_n(\mathbb{R})).$$

The latter is isomorphic to $\mathbb{Z}$ for $n = 2$ and $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. When $n \geq 3$ the resulting homomorphism $SK_1(A_1) = SL_n(A_1)/E_n(A_1) \longrightarrow \pi_1(SL_n(\mathbb{R})) \simeq \mathbb{Z}/2\mathbb{Z}$ is the same as the one constructed above. In fact $\alpha = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in SL_2(A_1)$ maps onto a a generator on the right.

If $\alpha \in SL_n(A_o, \mathfrak{q})$ then, if $0 \leq t \leq 1$, $\alpha(t) \in SL_n(\mathbb{R})$ and $\alpha(0) = I_n = \alpha(1)$. Again, therefore, if we identify S^1 with the unit interval modulo identification of its two end points, then we obtain, for $n \geq 3$, an epimorphism $SL_1(A_o, \mathfrak{q}) \longrightarrow \pi_1(SL_n(\mathbb{R})) \simeq \mathbb{Z}/2\mathbb{Z}$ which coincides with (8) above. This example was first pointed out by Stallings, using this topological construction.

HISTORICAL REMARKS

The material of §§1-5 is taken from Bass-Milnor-Serre [1], with some technical improvements due to T.Y. Lam. Similarly the review of the situation in number fields, in §7, is based on the same source. The reciprocity law of Weil in §8 is taken from Serre [3], and the last example in §8 is due to Stallings. The axiomatization of reciprocity laws in §6, and the proof of their equivalence with Mennicke symbols, is published here for the first time.

Part 3

ALGEBRAIC K-THEORY

Chapter VII
K-THEORY EXACT SEQUENCES

In this and the following chapter we develop an axiomatic theory of Grothendieck groups (K_0) and Whitehead groups (K_1). What is needed, to start with, is simply a category $\underline{\underline{A}}$ in which objects can be multiplied by an operation enjoying all of properties of, say $\oplus$ for modules. Then $\underline{\underline{A}}$ is called a "category with product". If $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ is a product preserving functor, the basic objective is to associate with it an exact sequence of the form

$$(1) \qquad K_1(\underline{\underline{A}}) \longrightarrow K_1(\underline{\underline{A}}') \longrightarrow K_0(F) \longrightarrow K_0(\underline{\underline{A}}) \longrightarrow K_0(\underline{\underline{A}}')$$

for a suitable "relative group" $K_0(F)$. Such a sequence, in a more general setting, has been constructed by Heller [1].

The approach here is based upon an idea of Milnor. This is to associate to a "fibre product diagram" (see §3),

$$(2) \qquad \begin{array}{ccc} \underline{\underline{A}} & \xrightarrow{G_2} & \underline{\underline{A}}_2 \\ {\scriptstyle G_1}\big\downarrow & & \big\downarrow{\scriptstyle F_2} \\ \underline{\underline{A}}_1 & \xrightarrow[F_1]{} & \underline{\underline{A}}' \end{array}$$

of product preserving functors, a Mayer-Vietoris sequence

$$(3) \qquad K_1\underline{\underline{A}} \longrightarrow K_1\underline{\underline{A}}_1 \oplus K_1\underline{\underline{A}}_2 \longrightarrow K_1\underline{\underline{A}}' \longrightarrow K_0\underline{\underline{A}} \longrightarrow$$
$$K_0\underline{\underline{A}}_1 \oplus K_0\underline{\underline{A}}_2 \longrightarrow K_0\underline{\underline{A}}'.$$

This is done in §4. The exact sequence (1) is then deduced from (3) in the special case $F_1 = F = F_2$. The fibre product, $\underline{\underline{A}}$, in this case is denoted co(F), since it plays a role here analogous to that of the mapping cone in topology.

Finally, in §6, we establish excision isomorphism theorem. Under suitable hypotheses on the square (2) it asserts that $K_0(G_2) \simeq K_0(F_1)$.

§1. GROTHENDIECK AND WHITEHEAD GROUPS OF CATEGORIES WITH A PRODUCT

A product on a category $\underline{\underline{A}}$ is a functor

$$\perp : \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow \underline{\underline{A}}$$

which is "coherently associative and commutative" in the sense of MacLane [2]. This means that is supplied with natural isomorphisms

$$\perp \circ (1_{\underline{\underline{A}}} \times \perp) \simeq \perp \circ (\perp \times 1_{\underline{\underline{A}}}) : \underline{\underline{A}} \times \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow \underline{\underline{A}}$$

and

$$\perp \circ t \simeq \quad : \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow \underline{\underline{A}},$$

where t is the transposition of $\underline{\underline{A}} \times \underline{\underline{A}}$. The "coherence" of these isomorphisms requires that isomorphisms of products of several factors, obtained from the above by a succession of threefold reassociations and twofold permutations, are all the same. This permits us to write, unambiguously up to canonical isomorphism, expressions like $A_1 \perp \dots \perp A_n = \perp_{i=1}^{n} A_i$. We shall also write $A^n = A \perp \dots \perp A$ (n terms) for $A \in \underline{\underline{A}}$, $n > 0$.

A product preserving functor $(\underline{\underline{A}}, \perp) \longrightarrow (\underline{\underline{A}}', \perp')$ is a functor $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ supplied with a natural isomorphism

(1) $$F \circ \perp \simeq \perp' \circ (F \times F) : \underline{\underline{A}} \times \underline{\underline{A}} \text{ —— } \underline{\underline{A}}'.$$

The latter is required to be compatible, in an obvious sense, with the associativity and commutativity isomorphisms in $\underline{\underline{A}}$

and $\underline{\underline{A}}'$. Moreover natural transformations of product preserving functors will be understood to respect the isomorphisms (1) for the two functors.

In practice we shall denote all products by the same symbol, $\perp$, except in examples where a standard notation is available. Moreover we will allow expressions like: "F: $\underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ is a product preserving functor between categories with product". Under these circumstances we shall usually use the (implicit) natural isomorphism to identify $F(A \perp B)$ ($A, B \in \underline{\underline{A}}$) with $FA \perp FB$ in $\underline{\underline{A}}'$. We shall say that F is cofinal if, given $A' \in \underline{\underline{A}}'$, these exist $A \in \underline{\underline{A}}$ and $B' \in \underline{\underline{A}}'$ such that $A' \perp B' \simeq FA$.

(1.1) EXAMPLES. Let R be a commutative ring and let A be an R-algebra. Then we have:

1. $\underline{\underline{P}}(A)$, the category of finitely generated projective right A-modules, and A-homomorphisms, with $\perp = \oplus$. One can use other categories of modules just as well.

2. $\underline{\underline{FP}}(R)$, the category of faithfully projective R-modules, and R-homomorphisms, with $\perp = \otimes_R$. (See (II, §1)).

3. $\underline{\underline{Pic}}_R(A)$, the category of invertible left $A \otimes_R A^o$-modules, with $\perp = \otimes_A$ (cf (II, §5)). Note that $\underline{\underline{Pic}}_R(R)$ is a subcategory of $\underline{\underline{FP}}(R)$.

4. $\underline{\underline{Quad}}(R)$, the category of pairs (P, q) with $P \in \underline{\underline{P}}(R)$ and q a non-singular quadratic form on P. The morphisms are isometries, and $\perp$ is orthogonal direct sum. One obtains similar categories by using other types of forms (alternating, hermitian, ...).

5. A is an Azumaya R-algebra if there is another R-algebra, B, such that $A \otimes_R B$ is a full matrix algebra over R. These, and their algebra homomorphisms, constitute a category $\underline{\underline{Az}}(R)$ with product $\perp = \otimes_R$.

6. In $\underline{\underline{P}}(A)$ the free modules are the objects of a cofinal subcategory. This is also true, but less obvious, in $\underline{\underline{FP}}(R)$ (see (IX, 4.6)). If we restrict the morphisms in $\underline{\underline{FP}}(R)$ to be isomorphisms, then $P \longmapsto \mathrm{End}_R(P)$ defines a product preserving functor $\underline{\underline{FP}}(R) \longrightarrow \underline{\underline{Az}}(R)$, and the last remark implies that it is cofinal.

7. Let $\underline{\underline{A}}(R)$ be any of examples 2-5, and let $R \longrightarrow S$ be a homomorphism of commutative rings. Then $\otimes_R S: \underline{\underline{A}}(R) \longrightarrow \underline{\underline{A}}(S)$ is a cofinal product preserving functor. In example 1 it induces $\underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}(A \otimes_R S)$.

(1.2) DEFINITION OF K_0. Let $\underline{\underline{A}}$ be a category with product. Its <u>Grothendieck group</u> is an abelian group $K_0\underline{\underline{A}}$ supplied with a map,

$$[\ ]_{\underline{\underline{A}}}: \text{ob } \underline{\underline{A}} \longrightarrow K_0\underline{\underline{A}},$$

which is universal for maps into an abelian group satisfying:

$$\begin{array}{lll} \text{Ka} & \text{If } A \simeq B \text{ then } [A]_{\underline{\underline{A}}} = [B]_{\underline{\underline{A}}} & \\ & & (A, B \in \underline{\underline{A}}) \\ \text{Kb} & [A \perp B]_{\underline{\underline{A}}} = [A]_{\underline{\underline{A}}} + [B]_{\underline{\underline{A}}} & \end{array}$$

This means that any map f: ob $\underline{\underline{A}} \longrightarrow G$ (G on abelian group) satisfying the analogues of Ka and Kb is of the form $fA = f_0[A]_{\underline{\underline{A}}}$ for a unique homomorphism $f_0: K_0\underline{\underline{A}} \longrightarrow G$.

To construct $K_0\underline{\underline{A}}$ we form the free abelian group with the isomorphism classes of ob $\underline{\underline{A}}$ as a basis, and then factor out the subgroup generated by the relations corresponding to Kb.

It follows immediately from the definition that $K_0\underline{\underline{A}}$ is a functor of $\underline{\underline{A}}$ with respect to product preserving functors $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$. Thus $K_0A \longrightarrow K_0\underline{\underline{A}}'$. is defined by $[A]_A \longmapsto [\bar{F}A]_{\underline{\underline{A}}'}$. We will not denote this map by $K_0(F)$, since the symbol $K_0(F)$ will be used to denote a "relative group" to be introduced in §5.

When the category $\underline{\underline{A}}$ is clear from the context we shall often drop the subscript from $[A]_{\underline{\underline{A}}}$.

(1.3) PROPOSITION. <u>Let</u> $\underline{\underline{A}}$ <u>be a category with product and let</u> $\underline{\underline{B}}$ <u>be a cofinal subcategory</u>.

(a) <u>Every element of</u> $K_0\underline{\underline{A}}$ <u>is of the form</u> $[A]_{\underline{\underline{A}}} - [B]_{\underline{\underline{A}}}$ <u>with</u> $A \in \underline{\underline{A}}$ <u>and</u> $B \in \underline{\underline{B}}$.

(b) If $A_1, A_2 \in \underline{\underline{A}}$ then $[A_1]_{\underline{\underline{A}}} = [A_2]_{\underline{\underline{A}}} \iff A_1 \perp B \simeq A_2 \perp B$ for some $B \in \underline{\underline{B}}$.

Proof. Let F be the free abelian group generated by the isomorphism classes, (A), of $A \in \underline{\underline{A}}$, and let R be the subgroup generated by all elements $(A, A') = (A \perp A') - (A) - (A')$. Then $(A) \longmapsto [A]$ induces an isomorphism $F/R \simeq K_0\underline{\underline{A}}$.

Since the [A]'s generate $K_0\underline{\underline{A}}$ any element is of the form $\Sigma_i[A_i] - \Sigma_j[A_j'] = [A] - [A']$, where $A = \perp_i A_i$ and $A' = \perp_j A_j'$. Since $\underline{\underline{B}}$ is cofinal we can solve $A' \perp A'' \simeq B$ for $A'' \in \underline{\underline{A}}$ and $B \in \underline{\underline{B}}$. Therefore $[A] - [A'] = [A \perp A''] - [A' \perp A''] = [A \perp A''] - [B]$, and this proves (a).

As for (b), the implication $\Leftarrow$ is trivial. Conversely, assume $[A_1] = [A_2]$. Then, in F we have an equation of the form $(A_1) - (A_2) = \Sigma_i(C_i, C_i') - \Sigma_j(D_j, D_j')$ so $(A_1) + \Sigma_i (C_i) + (C_i') + \Sigma_j(D_j \quad D_j') = (A_2) + \Sigma_i(C_i \quad C_i') + \Sigma_j(D_j) + (D_j')$. Since F is free on the isomorphism classes, this equation implies that

$$A_1 \perp E \simeq A_2 \perp E,$$

where $E = \perp_i(C_i \perp C_i') \perp \perp_j(D_j \perp D_j')$. Solving $E \perp E' \simeq B$ for $E' \in \underline{\underline{A}}$ and $B \in \underline{\underline{B}}$ we obtain $A_1 \perp B \simeq A_2 \perp B$, as claimed.

EXAMPLES. (cf. (1.1). For an R-algebra A the group $K_0A = K_0\underline{\underline{P}}(A)$ will be studied in detail in Chapter IX. The Picard group, $\mathrm{Pic}_R(A) = K_0\underline{\underline{\mathrm{Pic}}}_R(A)$, has already been introduced in (II, §5). The group $K_0\underline{\underline{\mathrm{Quad}}}(R)$ has, via $\otimes_R$, a commutative ring structure, and it has a natural quotient which is classically called the Witt ring of quadratic forms. The functor End_R: $\underline{\underline{FP}}(R) \longrightarrow \underline{\underline{Az}}(R)$ induces a homomorphism $K_0\underline{\underline{FP}}(R) \longrightarrow K_0\underline{\underline{Az}}(R)$ whose cokernel is called the Brauer group of R.

(1.4) DEFINITION OF $K_1(\underline{\underline{A}}, F)$. Let $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ be a product preserving functor. There is an induced functor

$$\Sigma F : \Sigma\underline{\underline{A}} \longrightarrow \Sigma\underline{\underline{A}}',$$

where $\Sigma\underline{\underline{A}} = \underline{\underline{A}}^{\underline{\underline{Z}}}$ is the category of automorphisms of objects of $\underline{\underline{A}}$ (cf. (I, §1)). It inherits a product from $\underline{\underline{A}}$,

$$(A, \alpha) \perp (B, \beta) = (A \perp B, \alpha \perp \beta),$$

and similarly for $\Sigma A'$. Moreover ΣF preserves this product. We shall write

$$\text{Ker } \Sigma F \subset \Sigma\underline{\underline{A}}$$

for the full subcategory of objects (A, α) such that $F\alpha = 1_{FA}$. Now the <u>Whitehead group</u> of $\underline{\underline{A}}$ relative to F is a group $K_1(\underline{\underline{A}}, F)$, supplied with a map

$$[\ ]_{(\underline{\underline{A}}, F)}: \text{ob Ker } \Sigma F \longrightarrow K_1(\underline{\underline{A}}, F)$$

which is universal for maps into an abelian group which satisfy:

Ka. If $(A, \alpha) \simeq (B, \beta)$ then $[A, \alpha]_{(\underline{\underline{A}}, F)} = [B, \beta]_{(\underline{\underline{A}}, F)}$;

Kb. $[A \perp B, \alpha \perp \beta]_{(\underline{\underline{A}}, F)} = [A, \alpha]_{(\underline{\underline{A}}, F)} + [B, \beta]_{(\underline{\underline{A}}, F)}$; and

Kc. $[A, \alpha\alpha']_{(\underline{\underline{A}}, F)} = [A, \alpha]_{(\underline{\underline{A}}, F)} + [A, \alpha']_{(\underline{\underline{A}}, F)}$,

for $A, B \in \underline{\underline{A}}$, $\alpha, \alpha' \in \text{Aut}_{\underline{\underline{A}}}(A, F)$, and $\beta \in \text{Aut}_{\underline{\underline{A}}}(B, F)$. Here we write

$$\text{Aut}_{\underline{\underline{A}}}(A, F) = \text{Ker}(\text{Aut}_{\underline{\underline{A}}}(A) \longrightarrow \text{Aut}_{\underline{\underline{A}}'}(FA)).$$

In case F is a constant functor we have $\Sigma F = \Sigma\underline{\underline{A}}$, and in that case we shall write

$$K_1(\underline{\underline{A}})$$

in place of $K_1(\underline{\underline{A}}$, constant functor). The functors Ker ΣF $\Sigma\underline{\underline{A}} \xrightarrow{\Sigma F} \Sigma\underline{\underline{A}}'$ induce homomorphisms

(1.5) $K_1(\underline{\underline{A}}, F) \xrightarrow{j} K_1(\underline{\underline{A}}) \longrightarrow K_1(\underline{\underline{A}}')$

whose composite is evidently zero. Proposition (2.5) below will give a criterion for the sequence to be exact.

In §5 we shall construct a sequence of the form

$$K_1(F) \longrightarrow K_1(\underline{\underline{A}}) \longrightarrow K_1(\underline{\underline{A}}') \longrightarrow K_0'(F)$$
$$K_0(\underline{\underline{A}}) \longrightarrow K_0(\underline{\underline{A}}')$$

for a cofinal product preserving functor $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$. We shall see then that the homomorphism j factors through a homomorphism $h: K_1(\underline{\underline{A}}, F) \longrightarrow K_1(F)$, which is sometimes an isomorphism.

(1.6) PROPOSITION. *Let* $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ *be a product preserving functor. If* $(A, \alpha) \in \text{Ker}\, \Sigma F$ write $[\alpha]$ for $[A, \alpha]_{(\underline{\underline{A}}, F)} \in K_1(\underline{\underline{A}}, F)$.

(a) *Every element of* $K_1(\underline{\underline{A}}, F)$ *is of the form* $[\alpha]$ *for some* $(A, \alpha) \in \text{Ker}\, \Sigma F$.

(b) *We have* $[\alpha] = [\beta]$ in $K_1(\underline{\underline{A}}, F)$ *if and only if there exist* $\gamma, \delta_0, \delta_1, \varepsilon_0, \varepsilon_1$, *such that* $\delta_0\delta_1$ *and* $\varepsilon_0\varepsilon_1$ *are defined and such that*

$$\alpha \perp \gamma \perp \delta_0 \perp \delta_1 \perp \varepsilon_0\varepsilon_1 \cong \beta \perp \gamma \perp \delta_0\delta_1 \perp \varepsilon_0 \perp \varepsilon_1$$

as objects of Ker ΣF.

Proof. Since $K_1(\underline{\underline{A}}, F)$ is a quotient, say $K_0(\text{Ker}\, \Sigma F)/M$, of $K_0(\text{Ker}\, \Sigma F)$, it follows from (1.3) that every element has the form $[\alpha] - [\beta]$. Axiom Kc implies that $0 = [1] = [\beta\, \beta^{-1}] = [\beta] + [\beta^{-1}]$, so that $[\alpha] - [\beta] = [\alpha] + [\beta^{-1}] = [\alpha \perp \beta^{-1}]$. This proves (a).

To prove (b) note first that M above is generated by the elements $\langle\alpha, \beta\rangle = [\alpha\beta]' - [\alpha]' - [\beta]'$, where $[\ ]'$ denotes the class of an element in $K_0(\text{Ker}\, \Sigma F)$. If $\langle\alpha', \beta'\rangle$ is another such element then, since $(\alpha \perp \alpha')(\beta \perp \beta') = (\alpha\beta \perp \alpha'\beta')$ ($\perp$ is a functor of two variables) it follows that $\langle\alpha, \beta\rangle + \langle\alpha', \beta'\rangle = \langle\alpha \perp \alpha', \beta \perp \beta'\rangle$. This implies that any element of M is a difference, $\langle\delta_0, \delta_1\rangle - \langle\varepsilon_0, \varepsilon_1\rangle$.

Now if $[\alpha] = [\beta]$ in $K_1(\underline{\underline{A}}, F)$ then $[\alpha]' - [\beta]'$ is an element of M, therefore of the form $\langle\delta_0, \delta_1\rangle - \langle\varepsilon_0, \varepsilon_1\rangle$ as above. Thus

$$[\alpha]' + [\delta_0]' + [\delta_1]' + [\varepsilon_0\varepsilon_1]' = [\beta]' + [\delta_0\delta_1]' + [\varepsilon_0]' + [\varepsilon_1]'$$

in $K_0(\mathrm{Ker}\Sigma\ F)$. If we apply (1.3) (b) to this equation we obtain a γ satisfying the conclusion of the proposition. q.e.d.

The commutativity of $\perp$ gives us, for any permutation s of $\{1,\ldots,n\}$, and any $A_1,\ldots,A_n \in \underline{\underline{A}}$, an isomorphism

$$A_1\perp\ldots\perp A_n \xrightarrow{\ s\ } A_{s(1)}\perp\ldots\perp A_{s(n)}.$$

If $\alpha_i\colon A_i \longrightarrow B_i$ are morphisms in $\underline{\underline{A}}$ then the diagram

$$\begin{array}{ccc} A_1\perp\ldots\perp A_n & \xrightarrow{\alpha_1\perp\ldots\perp\alpha_n} & B_1\perp\ldots\perp B_n \\ \downarrow s & & \downarrow s \\ A_{s(1)}\perp\ldots\perp A_{s(n)} & \xrightarrow[\alpha_{s(1)}\perp\ldots\perp\alpha_{s(n)}]{} & B_{s(1)}\perp\ldots\perp B_{s(n)} \end{array}$$

commutes, i.e.

$$s(\alpha_1\perp\ldots\perp\alpha_n)s^{-1} = \alpha_{s(1)}\perp\ldots\perp\alpha_{s(n)}$$

Suppose now that we have isomorphisms $\alpha_i\colon A_i \longrightarrow A_{i+1}$, $i \le i < n$, and $\alpha_n\colon A_n \longrightarrow A_1$. Let

(2) $\qquad s(i) = i - 1 \pmod{n}$,

and set $\alpha = \alpha_1\perp\ldots\perp\alpha_n$. Then $(A_1\perp\ldots\perp A_n, s\alpha) \in \Sigma\underline{\underline{A}}$. Let

(3) $\qquad \beta = (1_{A_1} \perp \alpha_1^{-1}\perp\ldots\perp(\alpha_{n-1}\cdots\alpha_1)^{-1})\colon A_1 \perp A_2\perp\ldots\perp A_n \longrightarrow A_1 \perp A_1\perp\ldots\perp A_1.$

Then $\beta\colon (A_1\perp\ldots\perp A_n, s\alpha) \longrightarrow (A_1\perp\ldots\perp A_1, \beta(s\alpha)\beta^{-1})$ is an isomorphism in $\Sigma\underline{\underline{A}}$. We have $\alpha\,\beta^{-1} = \alpha_1 \perp \alpha_2\alpha_1 \perp \ldots \perp (\alpha_n \ldots \alpha_1)$, and, by the formula above, $\beta s = s(\alpha_1{}^{-1} \perp (\alpha_2\alpha_1)^{-1} \perp \ldots \perp (\alpha_{n-1} \ldots \alpha_1)^{-1} \perp 1_{A_1})$. Hence

$$(4) \qquad \beta s \alpha \beta^{-1} = s(1_{A_1} \perp \ldots \perp 1_{A_1} \perp (\alpha_n \ldots \alpha_1)).$$

This proves:

(1.7) LEMMA ("Abstract Whitehead Lemma"). <u>Let $\underline{\underline{A}}$ be a category with product and let</u> $\alpha_i : A_i \longrightarrow A_{i+1}$, $1 \leq i < n$ <u>and</u> $\alpha_n : A_n \longrightarrow A_1$ <u>be isomorphisms in $\underline{\underline{A}}$. Let</u> s <u>be the cyclic permutation</u> $s(i) = i - 1 \pmod n$. <u>Then we have a $\Sigma\underline{\underline{A}}$ - isomorphism</u>

$$(A_1 \perp \ldots \perp A_n,\ s(\alpha_1 \perp \ldots \perp \alpha_n)) \simeq (A_1 \perp \ldots \perp A_1,\ s(1_{A_1} \perp \ldots \perp 1_{A_1} \perp (\alpha_n \ldots \alpha_1)).$$

<u>In particular, if</u> $\alpha : A \longrightarrow B$ <u>and</u> $\beta : B \longrightarrow C$ <u>are isomorphisms then</u>

$$(A \perp B,\ t(\alpha \perp \alpha^{-1})) \simeq (A \perp A,\ t)$$

and

$$(A \perp B \perp C,\ s(\alpha \perp \beta \perp (\beta\alpha)^{-1})) \simeq (A \perp A \perp A,\ s)$$

<u>in $\Sigma\underline{\underline{A}}$ where</u> t <u>and</u> s <u>are a transposition and three cycle, respectively</u>.

Suppose that all the A_i above are the same object A, and assume also that $\alpha_n \ldots \alpha_1 = 1_A$. Then equation (4) implies that $\alpha = s^{-1}\,\beta^{-1} s \beta$. Thus:

(1.8) LEMMA. <u>Suppose</u> $\alpha_1, \ldots, \alpha_n \in \mathrm{Aut}_{\underline{\underline{A}}}(A)$ <u>are such that</u> $\alpha_n \ldots \alpha_1 = 1_A$. <u>Then</u>

$$\alpha_1 \perp \ldots \perp \alpha_n = s^{-1}\,\beta^{-1} s \beta,$$

a commutator, where s *and* β *are as in (2) and (3) above.*

(1.9) PROPOSITION. *Suppose* $[A, \alpha] = 0$ *in* $K_1\underline{\underline{A}}$. *Then there is an* $(F, \phi) \in \Sigma\underline{\underline{A}}$ *such that* $\alpha \perp \phi \perp \phi^{-1}$ *is a commutator in* $\mathrm{Aut}_{\underline{\underline{A}}}(A \perp F \perp F)$. *Moreover* $\alpha \perp 1_{F \perp F}$ *is a product of two commutators.*

Proof. Since $[\alpha] = 0 = [1_A]$ we have

$$\alpha \perp \gamma \perp \delta_0 \perp \delta_1 \perp \varepsilon_0\varepsilon_1 \simeq 1_A \perp \gamma \perp \delta_0\delta_1 \perp \varepsilon_0 \perp \varepsilon_1$$

as in (1.5) (b). Denoting the domain of each automorphism by the corresponding Latin letter, this implies, in particular, that $A \perp C \perp D \perp D \perp E \simeq A \perp C \perp D \perp E \perp E$. Let $X = A \perp C \perp D \perp E$. Then $D' = D \quad X \simeq E' = E \quad X$. Moreover the isomorphism (5) above is preserved if we replace δ_i by $\delta_i \perp 1_X$ and ε_i by $\varepsilon_i \perp 1_X$ $(i = 0, 1)$, for this amounts to adding three 1_X's to each side. After changing notation, therefore, we can assume that $D = E$. If we further add 1_D to both sides we obtain an isomorphism of

$$\alpha_1 = \alpha \perp \gamma \perp \delta_0 \perp \delta_1 \perp 1_D \perp \varepsilon_0\varepsilon_1$$

with

$$\alpha_2 = 1_A \perp \gamma \perp 1_D \perp \delta_0\delta_1 \perp \varepsilon_0 \perp \varepsilon_1.$$

Here $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\underline{\underline{A}}}(A \perp C \perp D^4)$. The existence of the above isomorphism just means that α_1 and α_2 are conjugate, so $\alpha_1\alpha_2^{-1}$ is a commutator. We have

$$\alpha_1\alpha_2^{-1} = \alpha \perp 1_C \perp \delta_0 \perp \delta_0^{-1} \perp \varepsilon_0^{-1} \perp \varepsilon_0.$$

Set $F = C \perp D \perp D$ and $\phi = 1_C \perp \delta_0 \perp \varepsilon_0$. Then

$$\alpha_1\alpha_2^{-1} \perp 1_C \simeq \alpha \perp \phi \perp \phi^{-1},$$

and this is also a commutator, clearly. Finally, Lemma (1.8) implies that $1_A \perp \phi^{-1} \perp \phi$ is a commutator, so

$$\alpha \perp 1_{F \perp F} = (\alpha \perp \phi \perp \phi^{-1})\ (1_A \perp \phi^{-1} \perp \phi)$$

is a product of two commutators. q.e.d.

(1.10) COROLLARY. Suppose α is in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}}(A)$. Then there is a $\phi \in \mathrm{Aut}_{\underline{\underline{A}}}(A^n)$ for some $n \geq 0$, such that $\alpha \perp \phi \perp \phi^{-1}$ is a commutator, and $\alpha \perp 1_{A^{2n}}$ is a product of two commutators.

Proof. Let $\underline{\underline{B}}$ be the full subcategory of $\underline{\underline{A}}$ whose objects are the $A^n = A \perp \ldots \perp A$ (n terms). Then since $\mathrm{Aut}_{\underline{\underline{B}}}(A) \longrightarrow K_1\underline{\underline{B}}$ is a homomorphism into an abelian group we have $[\alpha]_{\underline{\underline{B}}} = 0$ in $K_1\underline{\underline{B}}$. The corollary now follows from Proposition (1.9).

§2. COFINAL FUNCTORS, AND K_1 AS A DIRECT LIMIT

Let $\underline{\underline{A}}$ be a category with product. Then the set $M(\underline{\underline{A}})$ of isomorphism classes, (A), of $A \in \underline{\underline{A}}$ is a commutative monoid, with $(A) + (B) = (A \perp B)$. We shall write

$$G(A) = \mathrm{Aut}_{\underline{\underline{A}}}(A)$$

and

$$G((A)) = G(A)/[G(A),\ G(A)],$$

the commutator factor group of $G(A)$, for $A \in \underline{\underline{A}}$. The notation is justified because two $\underline{\underline{A}}$-isomorphisms $A \longrightarrow B$ induce isomorphisms $G(A) \longrightarrow G(B)$ which differ by an inner automorphism. Hence they induce the same isomorphism $G((A)) \longrightarrow G((B))$. This shows that $G((A))$ depends, indeed, only on (A).

More generally, let $F\colon \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ be a product preserving functor. Write

$$G(A,\ F) = \mathrm{Aut}_{\underline{\underline{A}}}(A,\ F) = \mathrm{Ker}(\mathrm{Aut}_{\underline{\underline{A}}}(A) \xrightarrow{F} \mathrm{Aut}_{\underline{\underline{A}}'}(FA)),$$

and write

$$G((A),\ F) = G(A,\ F)/[G(A),\ G(A,\ F)].$$

If F is a constant functor we just recover the definitions above. Moreover, G((A), F) depends, just as above, only on (A).

An object $B \in \underline{\underline{A}}$ induces a group homomorphism

$$G(A, F) \xrightarrow{\perp B} G(A \perp B, F), \quad \alpha \longmapsto \alpha \perp 1_B,$$

and this induces a homomorphism

$$G((A), F) \xrightarrow{\perp B} G((A \perp B), F).$$

Moreover, it is clear that the homomorphism $\perp(B \perp C)$ from $G(A, F)$ to $G(A \perp B \perp C, F)$ is the composite of $\perp B$ and $\perp C$. It follows that $(A) \longmapsto G((A), F)$ is a functor

$$G: \mathrm{Tran}(M(\underline{\underline{A}})) \longrightarrow \text{(abelian groups)},$$

where $\mathrm{Tran}(M(\underline{\underline{A}}))$ is the translation category of the monoid $M(\underline{\underline{A}})$, in the sense of (I, §8).

(2.1) PROPOSITION. <u>The natural homomorphisms</u>

$$G(A, F) \longrightarrow K_1(\underline{\underline{A}}, F)$$

(see (1.6) and the definitions above) <u>induce an isomorphism</u>

$$\phi: \underset{\rightarrow}{G} = \text{colimit } G((A), F) \longrightarrow K_1(\underline{\underline{A}}, F).$$

<u>Proof</u>. If $\alpha \in G(A, F)$ and $\beta \in G(A)$ then, in the category Ker ΣF from which $K_1(\underline{\underline{A}}, F)$ is constructed, α and $\beta^{-1}\alpha\beta$ are isomorphic. Therefore $[\alpha^{-1}\beta^{-1}\alpha\beta] = [\beta^{-1}\alpha\beta] - [\alpha] = 0$ in $K_1(\underline{\underline{A}}, F)$, so $G(A, F) \longrightarrow K_1(\underline{\underline{A}}, F)$ factors through the quotient $G((A), F) = G(A, F)/[G(A), G(A, F)]$.

If $B \in \underline{\underline{A}}$ then $[\alpha \perp 1_B] = [\alpha]$ so the maps $G((A), F) \longrightarrow K_1(\underline{\underline{A}}, F)$ above are compatible with the direct system homomorphisms $G((A), F) \longrightarrow G((A \perp B), F)$. We thus obtain ϕ as above, and ϕ is clearly surjective. To show that ϕ is an isomorphism it suffices to show that $\alpha \longmapsto \langle\alpha\rangle$, where $\langle\alpha\rangle$ is the class of α in $\underset{\rightarrow}{G}$, satisfies axioms Ka, Kb, and Kc for K_1. For then the universality of K_1 gives us the required inverse. Axiom Ka is already built into the fact that G((A), F) depends only on the isomorphism class (A) of A. Thus, if $(A, \alpha) \simeq (B, \beta)$ in Ker ΣF then α and β are already identified in G((A), F), via any isomorphism $A \longrightarrow B$. Axiom Kc is clear since $G(A, F) \longrightarrow \underset{\rightarrow}{G}$ is a homomorphism. Finally, given (A, α) and (B, β), we must establish that $\langle\alpha \perp \beta\rangle = \langle\alpha\rangle + \langle\beta\rangle$. By definition of the direct system, $\langle\alpha\rangle = \langle\alpha \perp 1_B\rangle$ and $\langle\beta\rangle = \langle\beta \perp 1_A\rangle = \langle 1_A \perp \beta\rangle$, the last because $\beta \perp 1_A \simeq 1_A \perp \beta$. Hence $\langle\alpha \perp \beta\rangle = \langle(\alpha \perp 1_B)(1_A \perp \beta)\rangle = \langle\alpha \perp 1_B\rangle + \langle 1_A \perp \beta\rangle = \langle\alpha\rangle + \langle\beta\rangle$. q.e.d.

(2.2) COROLLARY. *Let* $\underline{\underline{A}}_0$ *be a full cofinal subcategory of* $\underline{\underline{A}}$, *and let* $F_0 = F|\underline{\underline{A}}_0$. *Then the inclusion functor induces an isomorphism*

$$K_1(\underline{\underline{A}}_0, F_0) \longrightarrow K_1(\underline{\underline{A}}, F).$$

Proof. If $A, B \in \underline{\underline{A}}_0$ then $\mathrm{Aut}_{\underline{\underline{A}}_0}(A) = \mathrm{Aut}_{\underline{\underline{A}}}(A)$ and $A \simeq B$ in $\underline{\underline{A}}_0 \iff A \simeq B$ in $\underline{\underline{A}}$. This is because $\underline{\underline{A}}_0$ is full in $\underline{\underline{A}}$. Therefore $M(\underline{\underline{A}}_0)$ is a cofinal submonoid of $M(\underline{\underline{A}})$. Moreover $K_1(\underline{\underline{A}}_0, F_0) = \underset{\rightarrow}{G}_0$ where G_0 is the restriction of G above to $\mathrm{Tran}\, M(\underline{\underline{A}}_0) \subset \mathrm{Tran}\, M(\underline{\underline{A}})$. By (I, 8.5) the induced map $\underset{\rightarrow}{G}_0 \longrightarrow \underset{\rightarrow}{G}$ is an isomorphism.

(2.3) COROLLARY. *Let* $A_1, A_2, \ldots, A_n, \ldots$ *be a sequence of objects of* $\underline{\underline{A}}$. *Write* $A_{n,m} = A_{n+1} \perp \ldots \perp A_m$ *for* $0 \leq n < m$, *and* $S_n = A_{o,n}$. *Assume*:

(1) *Given* $A \in \underline{\underline{A}}$ *and* $n \geq 0$, *there is a* $B \in \underline{\underline{A}}$ *and an* $m > n$ *such that* $A \perp B \simeq A_{n,m}$.

Let $G(\underline{\underline{A}}, F)$ *be the direct limit of the groups* $G(S_n, F) = \mathrm{Aut}_{\underline{\underline{A}}}(S_n, F)$, *with respect to the homomorphisms* $\perp 1_{A_{n,m}}$: $G(S_n, F) \longrightarrow G(S_m, F)$ *for* $n < m$. *Let* $G(\underline{\underline{A}})$ *be the corresponding direct limit of the groups* $G(S_n) = \mathrm{Aut}_{\underline{\underline{A}}}(S_n)$. *Then the natural homomorphisms* $G(S_n, F) \longrightarrow K_1(A, F)$ *induce an isomorphism*

$$G(\underline{\underline{A}}, F)/[G(\underline{\underline{A}}), G(\underline{\underline{A}}, F)] \longrightarrow K_1(\underline{\underline{A}}, F).$$

Proof. The left end of the arrow is just the direct limit of the groups $G((S_n), F) = G(S_n, F)/[G(S_n), G(S_n, F)]$, with respect to the homomorphisms "$\perp(A_{n,m})$", in the notation introduced above. Now the corollary follows immediately from (2.1) together with (I, 8.6). (The hypothesis (2) of (I, 8.6) corresponds exactly to the hypothesis (1) above). q.e.d.

(2.4) DEFINITION. A product preserving functor $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ will be called E-surjective if, given $A \in \underline{\underline{A}}$ and α' in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}'}(FA)$, there exists a $B \in \underline{\underline{A}}$ and an α in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}}(A \perp B)$ such that $F\alpha = \alpha' \perp 1_{FB}$.

(2.5) PROPOSITION. *Let $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ be a cofinal product preserving functor. Then $K_1\underline{\underline{A}}'$ is the direct limit over Tran $(M(\underline{\underline{A}}))$ of the groups $G((FA))$, the commutator factor group of $G(FA) = \mathrm{Aut}_{\underline{\underline{A}}'}(FA)$, $(A \in \underline{\underline{A}})$ with respect to the morphisms $G((FA)) \longrightarrow G((F(A \perp B)))$ induced by $\alpha \longmapsto \alpha \perp 1_{FB}$ for $\alpha \in G(FA)$. If F is E-surjective the sequence*

$$K_1(\underline{\underline{A}}, F) \longrightarrow K_1(\underline{\underline{A}}) \longrightarrow K_1(\underline{\underline{A}}')$$

is exact. If, moreover, there is a product preserving functor $F': \underline{\underline{A}}' \longrightarrow \underline{\underline{A}}$ such that $F \circ F' \simeq \mathrm{Id}_{\underline{\underline{A}}'}$ then it is a split short exact sequence.

Proof. According to Proposition (2.1), $K_1(\underline{\underline{A}}') = \underrightarrow{G}$ where G: Tran $(M(\underline{\underline{A}}')) \longrightarrow$ (abelian groups) is defined by $A' \longmapsto G((A'))$. The functor F, being cofinal, induces a cofinal homomorphism $M(F): M(\underline{\underline{A}}) \longrightarrow M(\underline{\underline{A}}')$, and hence a cofinal functor Tran $(M(F))$: Tran $(M(\underline{\underline{A}})) \longrightarrow$ Tran $(M(\underline{\underline{A}}'))$. Under these circumstances (I, 4.5) says that $\underrightarrow{G \circ \mathrm{Tran}\,(M(F))} \longrightarrow \underrightarrow{G}$ is an isomorphism, The first part of the proposition at hand just makes this assertion explicit.

Suppose now that F is E-surjective and that $[A, \alpha]_{\underline{\underline{A}}} \in \mathrm{Ker}\, K_1F$. We must show that $[A, \alpha]_{\underline{\underline{A}}} = [B, \beta]_{\underline{\underline{A}}}$ for some β

such that $F\beta = 1_{FB}$, i.e. such that $(B, \beta) \in \operatorname{Ker} \Sigma F$. (It was already noted, and is clear, that the composite of the two maps in question is zero). According to (1.9) we can choose $B' \in \underline{\underline{A}}'$ so that $F\alpha \perp 1_{B'}$ is in the commutator subgroup of $\operatorname{Aut}_{\underline{\underline{A}}'}(FA \perp B')$. The cofinality of F allows us further to take B' of the form FB. By augmenting B still further, if necessary, the E surjectivity of F provides us with an ε in the commutator subgroup of $\operatorname{Aut}_{\underline{\underline{A}}}(A \perp B)$ such that $F\varepsilon = F\alpha \perp 1_{FB}$. Then, in $K_1(\underline{\underline{A}})$, we have $[\bar{\alpha}] = [\alpha \perp 1_B] + [\varepsilon^{-1}] = [\beta]$, where $\beta = (\alpha \perp 1_B)\varepsilon^{-1}$ is such that $F\beta = 1_{F(A \perp B)}$. This shows that the sequence is exact.

Proposition (2.1) and the first part of this proposition show that the sequence $K_1(\underline{\underline{A}}, F) \longrightarrow K_1(\underline{\underline{A}}) \longrightarrow K_1(\underline{\underline{A}}')$ is a direct limit of sequences

$$G((A), F) \longrightarrow G((A)) \longrightarrow G((FA)) \qquad (A \in \underline{\underline{A}})$$

which are quotients of the sequences

$$G(A, F) \longrightarrow G(A) \longrightarrow G(FA).$$

The existence of a functor F' such that $F \circ F' \simeq \mathrm{Id}_{A'}$ implies that the latter are split group extensions. Hence the final assertion of the proposition follows from:

(2.6) LEMMA. <u>Let</u> $1 \longrightarrow N \longrightarrow G \xrightarrow{p} G' \longrightarrow 1$ <u>be a split group extension. Then</u>

$$0 \longrightarrow N/[G, N] \longrightarrow G/[G, G] \longrightarrow G'/[G', G'] \longrightarrow 0$$

<u>is a split short exact sequence of abelian groups</u>.

<u>Proof</u>. If $h: G' \longrightarrow G$ splits p ($ph = 1_{G'}$) then we need only check that $x \longmapsto x(hp(x))^{-1}$ induces a homomorphism $G/[G, G] \longrightarrow N/[G, N]$. For this will split the left half of the sequence, while h splits the right half. Set $e = hp: G \longrightarrow G$ and let $x, y \in G$. Then $xy\, e(xy)^{-1} = x(e(x)^{-1}y\, e(y)^{-1})\,(e(y)\, y^{-1}\, e(x))y\, e(y)^{-1}\, e(x)^{-1} = (x\, e(x)^{-1})\,(y\, e(y)^{-1})\,((e(y)y^{-1})\, e(x)\,(e(y)y^{-1})^{-1}\, e(x)^{-1}) \equiv (x\, e(x)^{-1})\,(y\, e(y)^{-1})$

mod [G, N]. This proves the lemma.

§3. FIBRE PRODUCT CATEGORIES

(3.1) DEFINITION. Given a diagram of functors

(1)
$$\begin{array}{ccc} & & \underline{\underline{A}}_2 \\ & & \big\downarrow{\scriptstyle F_2} \\ \underline{\underline{A}}_1 & \xrightarrow[F_1]{} & \underline{\underline{A}}' \end{array}$$

we define the <u>fibre product category</u>,

$$\underline{\underline{A}} = \underline{\underline{A}}_1 \times_{\underline{\underline{A}}'} \underline{\underline{A}}_2 = \mathrm{co}(F_1, F_2)$$

as follows: Its objects are triples (A_1, α, A_2) with $A_i \in A_i$ and $\alpha: F_1A_1 \longrightarrow F_2A_2$ an isomorphism in $\underline{\underline{A}}'$. A morphism $(A_1, \alpha, A_2) \longrightarrow (B_1, \beta, B_2)$ in $\underline{\underline{A}}$ is a pair of morphisms $f_i: A_i \longrightarrow B_i$ in $\underline{\underline{A}}_i$ $(i = 1, 2)$, such that

$$\begin{array}{ccc} F_1A_1 & \xrightarrow{\alpha} & FA_2 \\ {\scriptstyle F_1f_1}\big\downarrow & & \big\downarrow{\scriptstyle F_2f_2} \\ F_1B_1 & \xrightarrow[\beta]{} & F_2B_2 \end{array}$$

commutes. There are canonical functors

$$G_i: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}_i; \quad \begin{array}{ccc} (A_1, \alpha, A_2) & \longmapsto & A_i \\ (f_1, f_2) & \longmapsto & f_i \end{array} \quad (i = 1, 2).$$

Moreover the square

(2)
$$\begin{array}{ccc} \underline{\underline{A}} & \xrightarrow{G_2} & \underline{\underline{A}}_2 \\ {\scriptstyle G_1}\big\downarrow & & \big\downarrow{\scriptstyle F_2} \\ \underline{\underline{A}}_1 & \xrightarrow[F_1]{} & \underline{\underline{A}}' \end{array}$$

is commutative up to the natural isomorphism

$$\alpha: F_1G_1 \longrightarrow F_2G_2$$

which maps $F_1G_1(A_1, \alpha, A_2) = F_1A_1$ to $F_2G_2(A_1, \alpha, A_2) = F_2A_2$ by α.

This construction solves the following <u>universal problem</u>: Given a square

$$(3) \qquad \begin{array}{ccc} \underline{\underline{B}} & \xrightarrow{H_2} & \underline{\underline{A}}_2 \\ {\scriptstyle H_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \underline{\underline{A}}_1 & \xrightarrow[F_1]{} & \underline{\underline{A}}' \end{array} \quad , \text{ and } \beta: F_1H_1 \xrightarrow{\cong} F_2H_2,$$

there is a <u>unique</u> (not just up to isomorphism) functor $T: \underline{\underline{B}} \longrightarrow \underline{\underline{A}}$ such that $G_iT = H_i$ (equality, not isomorphism) ($i = 1, 2$) and such that

$$\beta = \alpha \cdot T : F_1H_1 = F_1G_1T \longrightarrow F_2H_2 = F_2G_2T.$$

Namely, we must have

$$T(B) = (H_1B, \beta_B, H_2B)$$

$$T(f) = (H_1f, H_2f),$$

and this T clearly works.

We shall refer to the above data,

$$\begin{array}{ccc} \underline{\underline{A}} & \xrightarrow{G_2} & \underline{\underline{A}}_2 \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \underline{\underline{A}}_1 & \xrightarrow[F_1]{} & \underline{\underline{A}}' \end{array} , \quad \alpha: F_1G_1 \longrightarrow F_2G_2,$$

as a <u>cartesian square</u>. If $A \in \underline{\underline{A}}$ then, as a triple, we have $A = (G_1A, \alpha_A, G_2A)$.

Suppose that (1) above is a diagram of product preserving functors between categories with product. Then we can introduce a product on $\underline{\underline{A}} = \underline{\underline{A}}_1 \times_{\underline{\underline{A}}'} \underline{\underline{A}}_2$ by:

$$(A_1, \alpha, A_2) \perp (B_1, \beta, B_2) = (A_1 \perp B_1, \alpha\perp\beta, A_2 \perp B_2)$$
$$(f_1, f_2) \perp (g_1, g_2) = (f_1 \perp g_1, f_2 \perp g_2).$$

Implicit in this definition are the identifications $F_i(A_i \perp B_i) = F_iA_i \perp F_iB_i$ $(i = 1, 2)$. Evidently the functors G_i in (2) preserve this product. Finally, if (3) is a diagram of product preserving functors then the functor $T: \underline{\underline{B}} \longrightarrow \underline{\underline{A}}$ constructed above is likewise product preserving. We shall now investigate conditions which will guarantee that the functors G_i and T are cofinal. The results below prepare for certain arguments in §4 to follow.

(3.2) DEFINITION. Let (1) be a diagram of product preserving functors. We say that F_1 _is cofinal relative to_ F_2 if, given $A_2 \in \underline{\underline{A}}_2$, we can find $A_2' \in \underline{\underline{A}}_2$ and $A_1 \in \underline{\underline{A}}_1$ such that $F_2(A_2 \perp A_2') \simeq F_1A_1$. We say that (F_1, F_2) is a _cofinal pair_ if each F_i is a cofinal functor and if each is cofinal rel the other.

Suppose $A = (A_1, \alpha_A, A_2) \in \underline{\underline{A}}$, $\beta \in \mathrm{Aut}_{\underline{\underline{A}}'}(F_1A_1)$, and $\gamma \in \mathrm{Aut}_{\underline{\underline{A}}'}(F_2A_2)$. Then we shall write

$$\gamma A\beta = (A_1, \gamma\alpha_A\beta, A_2).$$

(3.3) DEFINITION. A diagram (3) of product preserving functors will be called _E-surjective_ if the following condition is satisfied: Given $B \in \underline{\underline{B}}$ and ε in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}'}(F_1H_1B)$, there is a $B' \in \underline{\underline{B}}$ and ε_i in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}_i}(H_i(B \perp B'))$ $(i = 1, 2)$ such that

$$(\varepsilon_1, \varepsilon_2) : (TB)\varepsilon \perp TB' \longrightarrow TB \perp TB'$$

is an isomorphism in $\underline{\underline{A}}$.

(3.4) PROPOSITION. _Let_ (3) _be a diagram of product preserving functors_.

(a) _If_ H_1 _and_ H_2 _are cofinal then the objects_ $(TB)\alpha$ $(B \in \underline{\underline{B}}), \alpha \in \mathrm{Aut}_{\underline{\underline{A}}'}(F_1G_1B))$ _are cofinal in_ $\underline{\underline{A}}$.

(b) *If, further,* (3) *is* E-*surjective* (see (3.3)) *then* $T : \underline{\underline{B}} \longrightarrow \underline{\underline{A}}$ *is cofinal.*

(c) *If* F_2 *is cofinal relative to* F_1 (see (3.2)) *and if* F_1 *is* E-*surjective* (see (2.4)) *then the cartesian square* (2) *is* E-*surjective in the sense of* (3.3) *above.*

(d) *Suppose* (F_1, F_2) *is a cofinal pair* (see (3.2)). *Then given* $A_i \in \underline{\underline{A}}_i$ (i = 1, 2) *there exists a* $B \in \underline{\underline{A}}$ *and* $A_i' \in \underline{\underline{A}}_i$ (i =1, 2) *such that* $G_iB \simeq A_i \perp A_i'$ (i = 1, 2). *In particular, the functors* G_i, *and therefore also* F_iG_i, *are cofinal.*

Note that, by symmetry, we can interchange F_1 and F_2 in part (c).

Proof. (a) Given $A = (A_1, \alpha, A_2) \in \underline{\underline{A}}$ it suffices to find $B \in \underline{\underline{B}}$ and $A' = (A_1', \alpha', A_2')$ $\underline{\underline{A}}$ such that $A_i \perp A_i' \simeq H_iB$ (i = 1, 2). For then we will have $A \perp A' \simeq (H_1B, \gamma, H_2B)$ for some γ, and $(H_1B, \gamma, H_2B) = (TB)\beta$, where $\beta = \beta_B^{-1}\gamma$.

Since H_i is cofinal we can find $C_i \in \underline{\underline{A}}_i$ and $B_i \in \underline{\underline{B}}$ such that $A_i \perp C_i \simeq H_iB_i$ (i = 1, 2). Now set $A_1' = C_1 \perp H_1B_2$ and $A_2' = C_2 \perp H_2B_1$. Then $F_1A_1' = F_1C_1 \perp F_1H_1B_2 \simeq F_1C_1 \perp F_2H_2B_2 \simeq F_1C_1 \perp F_2A_2 \perp F_2C_2 \simeq F_1C_1 \perp F_1A_1 \perp F_2C_2$ (using α) $\simeq F_1H_1B_1 \perp F_2C_2 \simeq F_2H_2B_1 \perp F_2C_2 \simeq F_2A_2'$. Thus there is an isomorphism $\alpha' : F_1A_1' \longrightarrow F_2A_2'$. Moreover $A_i \perp A_i' \simeq H_iB$ (i = 1, 2), where $B = B_1 \perp B_2$. This completes the construction.

(b) Thanks to part (a) it suffices, given $(TB)\alpha$ as in the statement of (a), to find $A \in A$ and $B' \in \underline{\underline{B}}$ such that $(TB)\alpha \perp A \simeq TB'$. First form $(TB)\alpha \perp (TB)\alpha^{-1} = T(B \perp B)\varepsilon$, where $\varepsilon = \alpha \perp \alpha^{-1}$. Since, by (1.8), ε is a commutator, it follows from the definition of E-surjectivity (3.3) that $T(B \perp B)\varepsilon \perp TB' \simeq T(B \perp B) \perp TB'$ for some $B' \in \underline{\underline{B}}$. q.e.d.

(c) Given $A = (A_1, \alpha, A_2) \in \underline{\underline{A}}$ and ε in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}'}(F_1A_1)$ we must find $B = (B_1, \beta, B_2) \in \underline{\underline{A}}$ and ε_i in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}'}(F_i(A_i \perp B_i))$ $(i = 1, 2)$ such that $(\varepsilon_1, \varepsilon_2) : (A_1, \alpha\varepsilon, A_2) \perp B \longrightarrow A \perp B$ is an isomorphism in $\underline{\underline{A}}$.

Since F_1 is E-surjective there is a $B_1 \in \underline{\underline{A}}_1$ and a δ in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}_1}(A_1 \perp B_1)$ such that $F_1\delta = \varepsilon \perp 1_{F_1B_1}$. Since F_2 is cofinal relative to F_1 we can, after augmenting B_1 and δ if necessary, assume that there is a $B_2 \in \underline{\underline{A}}_2$ and an isomorphism $\beta: F_1B_1 \longrightarrow F_2B_2$. This constructs $B = (B_1, \beta, B_2)$. Moreover, $(\delta, 1_{A_2 \perp B_2}) : (A_1, \alpha\varepsilon, A_2) \perp B \longrightarrow (A_1 \perp B_1, (\alpha\varepsilon \perp B)(F_1\delta)^{-1}, A_2 \perp B_2)$. Since $\alpha\varepsilon \perp \beta = (\alpha \perp \beta)(\varepsilon \perp 1_{F_1B_1}) = (\alpha \perp \beta)(F_1\delta)$ the right side of the above isomorphism is $A \perp B$, as required. q.e.d.

(d) We are given $A_i \in \underline{\underline{A}}_i$ $(i = 1, 2)$. Since each F_i is cofinal relative to the other we can find $A_i', A_i'' \in \underline{\underline{A}}_i$ $(i = 1, 2)$ such that $F_1(A_1 \perp A_1') \simeq F_2A_2''$ and $F_2(A_2 \perp A_2') \simeq F_1A_1''$. Set $B_i = A_i \perp A_i' \perp A_i''$. Then clearly $F_1B_1 \simeq F_2B_2$, so there is a $B = (B_1, \beta, B_2) \in \underline{\underline{A}}$. This proves the first part of (d), and the cofinality of the G_i is an immediate consequence. Since the F_i are, by hypothesis, cofinal, the F_iG_i are also.

§4. THE MAYER-VIETORIS SEQUENCE OF A FIBRE PRODUCT

In this section we propose to associate with a <u>cartesian square</u> (see §3)

(1)
$$\begin{array}{ccc} \underline{\underline{A}} & \xrightarrow{G_2} & \underline{\underline{A}}_2 \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \underline{\underline{A}}_1 & \xrightarrow[F_1]{} & \underline{\underline{A}}' \end{array} \quad , \ \alpha: F_1G_1 \longrightarrow F_2G_2,$$

an exact sequence. This is done in Theorem (4.3) below.

If $A = (A_1, \alpha, A_2) \in \underline{\underline{A}}$, $\beta \in \mathrm{Aut}_{\underline{\underline{A}}}(F_1A_1)$, and $\gamma \in \mathrm{Aut}_{\underline{\underline{A}}'}(F_2A_2)$ we shall write (as in (3.3))

$$\gamma A\beta = (A_1, \gamma\alpha\beta, A_2).$$

Moreover, if $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\underline{\underline{A}}'}(F_1A_1)$ we shall write

$$\langle A, \alpha_1, \alpha_2\rangle = [A\alpha_1\alpha_2] + [A] - [A\alpha_1] - [A\alpha_2] \in K_o\underline{\underline{A}}.$$

(4.1) DEFINITION OF $K_o{}'\underline{\underline{A}}$. We define

$$K_o{}'\underline{\underline{A}} = K_o\underline{\underline{A}}/M$$

where M is the group generated by all $\langle A, \alpha_1, \alpha_2\rangle$, $(A = (A_2, \alpha, A_2) \in \underline{\underline{A}};\ \alpha_1, \alpha_2 \in \mathrm{Aut}_{\underline{\underline{A}}'}(F_1A_1))$. We denote the class of A in $K_o{}'A$ by $[A]'$.

Note that, if $\langle B, \beta_1, \beta_2\rangle$ is a second such element, then

$$\langle A, \alpha_1, \alpha_2\rangle + \langle B, \beta_1, \beta_2\rangle = \langle A \perp B, \alpha_1 \perp \beta_1, \alpha_2 \perp \beta_2\rangle.$$

From this it follows that any element of M is of the form $\langle A, \alpha_1, \alpha_2\rangle - \langle B, \beta_1, \beta_2\rangle$.

(4.2) LEMMA. Every element of $K_o{}'\underline{\underline{A}}$ is of the form $[A]' - [B]'$. If $[A]' = [B]'$ then there exist elements $\langle C, \gamma_1, \gamma_2\rangle$ and $\langle D, \delta_1, \delta_2\rangle$ as above, and an $E \in A$ such that

$$A \perp C\gamma_1 \perp C\gamma_2 \perp D\delta_1\delta_2 \perp D \perp E$$

and

$$B \perp C\gamma_1\gamma_2 \perp C \perp D\delta_1 \perp D\delta_2 \perp E$$

are isomorphic in $\underline{\underline{A}}$. In case the cartesian square (1) is E-surjective (see (3.3)) then the natural projection $K_o\underline{\underline{A}} \longrightarrow K_o{}'\underline{\underline{A}}$ is an isomorphism.

Proof. The first assertion is clear since $K_o'\underline{\underline{A}}$ is a quotient of $K_o\underline{\underline{A}}$. If $[A]' = [B]'$ then $[A] - [B] \in M$ so, as remarked above, we can write $[A] - [B] = \langle C, \gamma_1, \gamma_2\rangle - \langle D, \delta_1, \delta_2\rangle$. Transpose this equation so that all terms on each side have coefficient + 1, and then apply (1.3) (b) to obtain an E yielding the above isomorphism.

For the last assertion we must show that all elements $\langle A, \alpha_1, \alpha_2\rangle$ are zero. According to (1.8), $\varepsilon = \alpha_2^{-1} \perp \alpha_2$ is a commutator in $\mathrm{Aut}_{\underline{\underline{A}}'}(F_1(A_1 \perp A_1))$. Moreover $(A\alpha_1\alpha_2 \perp A)\varepsilon = A\alpha_1 \perp A\alpha_2$. The definition of E-surjectivity now implies that, for some $B \in \underline{\underline{A}}$, $A\alpha_1\alpha_2 \perp A \perp B \simeq A\alpha_1 \perp A\alpha_2 \perp B$. Hence $[A\alpha_1\alpha_2] + [A] - [A\alpha_1] - [A\alpha_2] = 0$, as required.

Let (1) be a cartesian square. We propose to construct a Mayer-Vietoris sequence.

$$K_1\underline{\underline{A}} \xrightarrow{g_1} K_1\underline{\underline{A}}_1 \oplus K_1\underline{\underline{A}}_2 \xrightarrow{f_1} K_1\underline{\underline{A}}' \xrightarrow{\partial} K_o'\underline{\underline{A}} \xrightarrow{g_o} K_o\underline{\underline{A}}_1 \oplus K_o\underline{\underline{A}}_2 \xrightarrow{f_o} K_o\underline{\underline{A}}' \tag{2}$$

under suitable hypotheses. If we write $(T)_i$ for the homomorphism on K_i induced by a functor T, then we define

$$f_i(x_1, x_2) = (F_1)_i\,(x_1) + (F_2)_i\,(x_2) \qquad (i = 0, 1),$$

$$g_1(x) = ((G_1)_1\,(x), -(G_2)_1\,(x)), \text{ and}$$

$$g_o(x) = ((G_1)_o'\,(x), -(G_2)_o'\,(x)).$$

In the latter $(G_j)_o'$ is the homomorphism on $K_o'(\underline{\underline{A}}) = K_o(\underline{\underline{A}})/M$ induced by $(G_j)_o$. This exists because $(G_j)_o$ evidently kills the generators $\langle A, \alpha_1, \alpha_2\rangle$ of M (see (4.1)). It is clear from these definitions and the commutativity of (1) (up to isomorphism) that

$$f_i g_i = 0 \qquad (i = 0, 1). \tag{3}$$

(4.3) THEOREM. ("Mayer-Vietoris Sequence"). Let (1) be a cartesian square in which (F_1, F_2) is a cofinal pair of functors (see (3.2)). Then there is a unique homomorphism $\partial: K_1\underline{\underline{A}}' \longrightarrow K_0'\underline{\underline{A}}$ such that

$$\partial[F_1G_1A, \alpha] = [A\alpha]' - [A]' \qquad (A \in \underline{\underline{A}}, \alpha \in \mathrm{Aut}_{\underline{\underline{A}}'}(F_1G_1A)).$$

The resulting sequence (2) above is exact except perhaps at $K_1\underline{\underline{A}}_1 \oplus K_1\underline{\underline{A}}_2$. If (1) is E-surjective (see (3.3)) then (2) is exact and the natural projection $K_0(\underline{\underline{A}}) \longrightarrow K_0'(\underline{\underline{A}})$ is an isomorphism. This is the case, in particular, if one of the functors F_i is E-surjective (see (2.4)). Finally, the sequence is natural with respect to functors between cartesian squares.

Proof. The last assertion, which we leave the reader to make precise, will be clear from the definition of ∂ and of the f_i and g_i above. The fact that (1) is E-surjective if one of the F_i is, is just (3.4) (c). The fact that $K_0(\underline{\underline{A}}) \longrightarrow K_0'(\underline{\underline{A}})$ is an isomorphism when (1) is E-surjective is contained in (4.2) above.

There remains for us now only the construction of ∂ and the proof of the alleged exactness properties of (2). We have already shown in (3) above that $f_ig_i = 0$ $(i = 0, 1)$. Note that the assumption that (F_1, F_2) is a cofinal pair implies, thanks to (3.4) (d), that the functors G_i and F_iG_i $(i = 1, 2)$ are cofinal.

(a). Existence and uniqueness of ∂: Suppose $A = (A_1, \alpha_A, A_2) \in \underline{\underline{A}}$ and $\alpha \in G(F_1A_1) = \mathrm{Aut}_{\underline{\underline{A}}'}(F_1A_1)$. Set $d(A, \alpha) = [A\alpha]' - [A]' \in K_0'\underline{\underline{A}}$. If $\alpha_1, \alpha_2 \in \overline{\overline{G}}(F_1A_1)$ then $d(A, \alpha_1\alpha_2) = [A\alpha_1\alpha_2]' - [A]' = d(A, \alpha_1) + d(A, \alpha_2)$, as we see directly from definition (4.1). Thus $d(A, \ ): G(F_1A_1) \longrightarrow K_0'\underline{\underline{A}}$ is a homomorphism into an abelian group, so it factors through the commutator quotient group, $G((F_1A_1))$, of $G(F_1A_1)$. If h =

$(h_1, h_2)\colon A \longrightarrow B$ is an isomorphism in $\underline{\underline{A}}$ it induces an isomorphism $(F_1A_1, \alpha) \simeq (F_1B_1, (F_1h_1)\alpha\,(F_1h_1)^{-1})$ in $\Sigma\underline{\underline{A}}'$, and we have $\alpha_B = (F_2h_2)\alpha_A\,(F_1h_1)^{-1}$. It follows that h induces an isomorphism $A\alpha \longrightarrow B(F_1h_1)\,\alpha(F_1h_1)^{-1}$ in $\underline{\underline{A}}$. Consequently $d(B, (F_1h_1)\,\alpha(F_1h_1)^{-1}) = [B(F_1h_1)\,\alpha(F_1h_1)^{-1}]' - [B]' = [A\alpha]' - [A]' = d(A, \alpha)$. This shows that d is insensitive to isomorphisms $A \longrightarrow B$ in $\underline{\underline{A}}$, so d depends only on the isomorphism class (A) of A in $\underline{\underline{A}}$. Finally, if $A, B \in \underline{\underline{A}}$ and $\alpha \in G(F_1A_1)$ then $d(A \perp B, \alpha \perp 1_{F_1B_1}) = [(A \perp B)\,(\alpha \perp 1_{F_1B_1})]' - [A \perp B]' = [A\alpha \perp B]' - [A]' - [B]' = [A\alpha]' - [A]' = d(A, \alpha)$. Thus d defines a morphism into $K_o'\underline{\underline{A}}$ from the direct system of groups $G((F_1G_1A))$, indexed by the isomorphism classes (A) of $A \in \underline{\underline{A}}$, and with maps $G((F_1G_1A)) \longrightarrow G((F_1G_1(A \perp B)))$ induced by $\alpha \longmapsto \alpha \perp 1_{F_1G_1B}$. Since we know that F_1G_1 is a cofinal functor, it follows from (2.5) that $K_1\underline{\underline{A}}'$ is the direct limit of the above system, so the existence and uniqueness of ∂, as the homomorphism induced by d, is established.

(b) $g_o\partial = 0$ and $\partial f_1 = 0$: If $A = (A_1, \alpha_A, A_2) \in \underline{\underline{A}}$ and $\alpha \in G(F_1A_1)$ as above then $g_o\partial[F_1A_1, \alpha] = g_o([A\alpha]' - [A]') = ([A_1] - [A_1], [A_2] - [A_2]) = 0$. If $\alpha = F_1\beta$, $\beta \in \mathrm{Aut}_{\underline{\underline{A}}_1}(A_1)$, then $(\beta, 1_{A_2})\colon A\beta \longrightarrow A$ is an isomorphism in $\underline{\underline{A}}$ so $0 = \partial[F_1A_1, F_1\beta] = \partial f_1([A_1, \beta], 0)$. Since $G_1\colon \underline{\underline{A}} \longrightarrow \underline{\underline{A}}_1$ is cofinal it follows from (2.2) that every element of $K_1\underline{\underline{A}}_1$ is of the form $[A_1, \beta] = [G_1A, \beta]$ for some $A \in \underline{\underline{A}}$. Arguing similarly with respect to the second coordinate in $K_1\underline{\underline{A}}_1 \oplus K_1\underline{\underline{A}}_2$ we conclude that $\partial f_1 = 0$ as required.

(c) $\operatorname{Ker} f_o \subset \operatorname{Im} g_o$: Suppose $(x_1, x_2) \in \operatorname{Ker} f_o$. Since the G_i are cofinal we can write $x_1 = [B_1]_{\underline{\underline{A}}_1} - [G_1A]_{\underline{\underline{A}}_1}$ and $-x_2 = [B_2]_{\underline{\underline{A}}_2} - [G_2A']_{\underline{\underline{A}}_2}$ for some $A, A' \in \underline{\underline{A}}$. If we replace A and A' each by $A \perp A'$, and augment B_1 and B_2 correspondingly, we can further achieve the condition $A = A'$. Having done this we apply f_o and find that $[F_1B_1]_{\underline{\underline{A}}'} = [F_2B_2]_{\underline{\underline{A}}'}$. Since F_1 is cofinal it follows that there is an isomorphism $\gamma: F_1B_1 \perp F_1B_1' \longrightarrow F_2B_2 \perp F_1B_1'$ for some $B_1' \in \underline{\underline{A}}_1$. Since F_2 is cofinal relative to F_1 there is also an isomorphism $\beta: F_1B_1' \perp F_1B_1'' \longrightarrow F_2B_2'$ for some $B_1'' \in \underline{\underline{A}}_1$ and $B_2' \in \underline{\underline{A}}_2$. Now we see that (x_1, x_2) is the result of applying g_o to

$$[B_1 \perp B_1' \perp B_1'', \alpha, B_2 \perp B_2']' - [B_1' \perp B_1'', \beta, B_2']' - [A],$$

where $\alpha = (1_{F_2B_2} \perp \beta)(\gamma \perp 1_{F_1B_1''})$.

(d) Ker $g_o \subset$ Im ∂: Suppose $[B]' - [A]' \in$ Ker g_o. This means that $[B_i]_{\underline{\underline{A}}_i} = [A_i]_{\underline{\underline{A}}_i}$ $(i = 1, 2)$ and hence that $B_i \perp A_i' \simeq A_i \perp A_i'$ for suitable $A_i' \in \underline{\underline{A}}_i$ $(i = 1, 2)$. Since (F_1, F_2) is a cofinal pair it follows from (3.4) (d) that there is a $C = (C_1, \alpha_C, C_2) \in \underline{\underline{A}}$ and $A_i'' \in \underline{\underline{A}}_i$ such that $A_i' \perp A_i'' \simeq C_i$ $(i = 1, 2)$. Set $D = A \perp C$. Then $D_i = A_i \perp A_i' \perp A_i'' \simeq B_i \perp A_i' \perp A_i''$ $(i = 1, 2)$. Using such isomorphisms we find that $B \perp C \simeq D\delta$ for some $\delta \in \mathrm{Aut}_{\underline{\underline{A}}'}(F_1D_1)$. Finally then we have $[B]' - [A]' = [B \perp C]' - [A \perp C]' = [D\delta]' - [D]' = \partial[F_1D_1, \delta]$.

(e) Ker $\partial \subset$ Im f_1: Let $x \in$ Ker ∂. Since F_1G_1 is cofinal we can write $x = [F_1G_1A, \alpha]$ for some $A \in \underline{\underline{A}}$ and $\alpha \in \mathrm{Aut}_{\underline{\underline{A}}'}(F\ G\ A)$. Since $[A\alpha]' - [A]' = \partial x = 0$ it follows from Lemma (4.2) above that there is an isomorphism (h_1, h_2): $U \longrightarrow V$ in $\underline{\underline{A}}$, where $U = A\alpha \perp C\gamma_1 \perp C\gamma_2 \perp D\delta_1\delta_2 \perp D \perp E$ and $V = A \perp C\gamma_1 \perp C\gamma_2 \perp D\delta_1 \perp D\delta_2 \perp E$, as in (4.2). Writing $U = (U_1, \alpha_U, U_2)$ and $V = (V_1, \alpha_V, V_2)$ we have $U_1 = A_1 \perp W_1 = V_1$ and $U_2 = A_2 \perp W_2 = V_2$, where $W_i = C_i \perp C_i \perp D_i \perp D_i \perp E_i$ $(i = 1, 2)$. Moreover $\alpha_U = \alpha_V(\alpha \perp 1_{F_1W_1})$, and the isomorphism

(h_1, h_2) gives us $\alpha_V = (F_2h_2)\ \alpha_U(F_1h_1)^{-1}$. It follows that $\alpha \perp 1_{F_1W_1} = (F_2h_2)^{-1}\ \alpha_V(F_1h_1)\alpha_V^{-1}$ in $\mathrm{Aut}_{\underline{\underline{A}}'}(F_1(A_1 \perp W_1))$. Consequently, in $K_1\underline{\underline{A}}'$, we have $[\alpha] = [\alpha \perp 1_{F_1W_1}] = [(F_2h_2)^{-1}] + [\alpha_V(F_1h_1)\alpha_V^{-1}] = [F_1h_1] - [F_2h_2] = f_1([h_1], -[h_2])$.

(f) <u>If</u> (1) <u>is</u> E-<u>surjective then</u> Ker $f_1 \subset$ Im g_1: Suppose $x = ([A_1, \alpha_1], -[A_2, \alpha_2]) \in$ Ker f_1. Proposition (3.4) (d) gives us a $B = (B_1, \alpha_B, B_2) \in \underline{\underline{A}}$ and $A_i' \in \underline{\underline{A}}_i$ $(i = 1, 2)$ such that $B_i \simeq A_i \perp A_i'$ $(i = 1, 2)$. Then $(A_i \perp A_i', \alpha_i \perp 1_{F_iA_i}) \simeq (B_i, \beta_i)$ for some β_i $(i = 1, 2)$, and we have $x = ([B_1, \beta_1], -[B_2, \beta_2])$ clearly. Applying f_1 we find that $0 = f_1(x) = [F_1\beta_1] - [F_2\beta_2] = [\alpha_B^{-1}(F_2\beta_2)^{-1}\ \alpha_B(F_1\beta_1)]$. It follows now from (2.5) and the cofinality of F_1G_1 that there is a $B' = (B_1', \alpha_{B'}, B_2') \in \underline{\underline{A}}$ such that $\varepsilon = \alpha_B^{-1}(F_2\beta_2)^{-1}\ \alpha_B(F_1\beta_1) \perp 1_{F_1B_1'}$ is in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}'}(F_1(B_1 \perp B_1'))$. Now we have $(F_2\beta_2)^{-1}\ B(F_1\beta_1) \perp B' = (B \perp B')\varepsilon$, and it follows from the hypothesis of E-surjectivity that there is a $B'' = (B_1'', \alpha_{B''}, B_2'') \in \underline{\underline{A}}$ and ε_i in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}_i}(B_i \perp B_i' \perp B_i'')$ $(i = 1, 2)$ such that $(\varepsilon_1, \varepsilon_2)\colon (B \perp B')\varepsilon \perp B'' \longrightarrow B \perp B' \perp B''$ is an isomorphism. This means that

$$F_2\varepsilon_2^{-1}(\alpha_B \perp \alpha_{B'} \perp \alpha_{B''})F_1\varepsilon_1 = (\alpha_B \perp \alpha_{B'})\varepsilon \perp \alpha_{B''}$$

$$= (F_2\beta_2)^{-1}\ \alpha_B(F_1\beta_1) \perp \alpha_{B'} \perp \alpha_{B''}$$

$$= (F_2\gamma_2)^{-1}\ (\alpha_B \perp \alpha_{B'} \perp \alpha_{B''})\ (F_1\gamma_1),$$

where $\gamma_i = \beta_i \perp 1_{F_i(B_i' \perp B_i'')}$ $(i = 1, 2)$. Set $C = B \perp B' \perp B''$ and set $\delta_i = \gamma_i\ \varepsilon_i^{-1}$ $(i = 1, 2)$. Then the above equations imply that $(F_2\delta_2)^{-1}\,\alpha_C(F_1\delta_1) = \alpha_C$, in other words that (δ_1, δ_2) is an automorphism of C.

We conclude the proof now by showing that $x = g_o([C, (\delta_1, \delta_2)]) = ([C_1, \delta_1], -[C_2, \delta_2])$. For example, in $K_1\underline{\underline{A}}_1$, $[\delta_1] = [\gamma_1\varepsilon_1] = [\gamma_1] + [\varepsilon_1] = [\gamma_1]$ (because ε_1 is in the commutator subgroup) $= [\beta_1 \perp 1_{F_1(B_1' \perp B_1'')}] = [\beta_1]$. Similarly $[\delta_2] = [\beta_2]$ in $K_1\underline{\underline{A}}_2$. Since $x = ([\beta_1], -[\beta_2])$ the proof of part (f), and hence of Theorem (4.3), is complete.

§5. THE EXACT SEQUENCE OF A COFINAL FUNCTOR

In this section we shall show that a cofinal product preserving functor $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ induces an exact sequence of the form

$$K_1\underline{\underline{A}} \longrightarrow K_1\underline{\underline{A}}' \longrightarrow K_o'(F) \longrightarrow K_o\underline{\underline{A}} \longrightarrow K_o\underline{\underline{A}}'.$$

In order to define $K_o'(F)$ we first introduce the fibre product diagram

(1) $$\begin{array}{ccc} co(F) & \xrightarrow{G_2} & \underline{\underline{A}} \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle F} \\ \underline{\underline{A}} & \xrightarrow[F]{} & \underline{\underline{A}}' \end{array}, \quad \alpha: FG_1 \longrightarrow FG_2.$$

Since F is cofinal it is obvious that (F, F) *is a cofinal pair* (see (3.2)). Moreover, *if* F *is* E-*surjective then* it follows from (3.4) (c) that the diagram (1) *is* E-*surjective* (see (3.3)).

The identity functor from $\underline{\underline{A}}$ to its two copies in (1) induces a *diagonal functor*

(2) $$\Delta: \underline{\underline{A}} \longrightarrow co(F), \quad G_i\Delta = 1_{\underline{\underline{A}}} \quad (i = 1, 2).$$

We now define the groups $K_i(F)$ as cokernels in the short exact sequences

$$0 \longrightarrow K_i\underline{\underline{A}} \xrightarrow{\Delta} K_i(co(F)) \longrightarrow K_i(F) \longrightarrow 0$$
$$(i = 0, 1).$$

Since Δ is split by both G_1 and G_2 it follows that

$$K_i(F) \simeq \mathrm{Ker}\,(K_i(co(F)) \xrightarrow{G_j} K_i\underline{\underline{A}}) \quad (i = 0, 1;\ j = 1, 2)$$

$$K_i(co(F)) \simeq K_i\underline{\underline{A}} \oplus K_i(F). \qquad (i = 0, 1).$$

Since $co(F)$ is a fibre product we have the quotient, $K_o'(co(F)) = K_o(co(F))/M$, of $K_o(co(F))$ which occurs in the Mayer-Vietoris sequence of (1) (see (4.1)). We now define

$$K_o'(F) = K_o(co(F))/(M + \mathrm{Im}(\Delta))$$

to be the corresponding quotient of $K_o(F)$. Thus we have an exact sequence

$$K_o\underline{\underline{A}} \xrightarrow{d} K_o'(co(F)) \longrightarrow K_o'(F) \longrightarrow 0,$$

where $d = (\text{nat. proj.}) \circ \Delta$. Recall from (4.1) that M is generated by elements

$$<A, \alpha_1, \alpha_2> = [A\alpha_1\alpha_2] + [A] - [A\alpha_1] - [A\alpha_2]$$

where $A = (A_1, \alpha_A, A_2) \in co(F)$, $\alpha_i \in \mathrm{Aut}_{\underline{\underline{A}}'}(FA_1)$ $(i = 1, 2)$, and where we write $A\beta = (A_1, \alpha_A\beta, A_2)$ for $\beta \in \mathrm{Aut}_{\underline{\underline{A}}'}(FA_1)$. Since $G_iA = A_i$ $(i = 1, 2)$ it follows that $<A, \alpha_1, \alpha_2>$ above is in the kernel of the map $K_o(co(F)) \longrightarrow K_o\underline{\underline{A}}$ induced by the G_i's. Thus each G_i induces a map $K_o'(co(F)) \longrightarrow K_o\underline{\underline{A}}$, and these will both provide splittings for the homomorphism d above. This proves the first assertion of the next proposition.

If $(A, \alpha, B) \in co(F)$ we shall denote its class in $K_o'(F)$ by $[A, \alpha, B]'$, and use $[A, \alpha, B]''$ for its class in $K_o'(co(F))$.

(5.1) PROPOSITION. <u>The diagonal functor</u> Δ: $\underline{\underline{A}} \longrightarrow$

co(F) induces a split short exact sequence

$$0 \longrightarrow K_0\underline{\underline{A}} \longrightarrow K_0'(co(F)) \longrightarrow K_0'(F) \longrightarrow 0.$$

Moreover, $K_0'(F) = K_0(co(F))/N$, where N is the group generated by all elements of the form

$$[A, \beta\alpha, C] - [A, \alpha, B] - [B, \beta, C]$$

in $K_0(co(F))$. Every element of $K_0'(F)$ is of the form $[A, \alpha, B]'$.

Proof. The first assertion was proved above. To prove the second let us write $[[A, \alpha, B]]$ for the class of $[A, \alpha, B]$ modulo N. To show $M \subset N$ we must show that $[[A\alpha_1\alpha_2]] + [[A]] = [[A\alpha_1]] + [[A\alpha_2]]$ for each element $\langle A, \alpha_1, \alpha_2\rangle$ as above. This will follow immediately if we show that $[[A\beta]] = [[A]] + [[A_1, \beta, A_1]]$ for any $\beta \in Aut_{\underline{\underline{A}}'}(FA_1)$. But the latter follows directly from the definition of N.

To show that $N \subset M$ we must show that

$$[A, \beta\alpha, C]' = [A, \alpha, B]' + [B, \beta, C]'$$

in $K_0'(F)$. Suppose $A \in \underline{\underline{A}}$ and $\alpha, \beta \in Aut_{\underline{\underline{A}}'}(FA)$. Then $\Delta A = (A, 1_{FA}, A)$ and $[\Delta A]' = 0$ by the definition of $K_0'(F)$. It follows now from the definition of M that $[\Delta A\alpha\beta]' = [\Delta A\alpha\beta]' + [\Delta A]' = [\Delta A\alpha]' + [\Delta A\beta]'$. Thus $\alpha \longmapsto [\Delta A\alpha]'$ is a homomorphism, so $[\Delta A\alpha]' = 0$ if α is in the commutator subgroup.

Now let A, α, B, β, C be as above. Then $(A \perp B \perp C, \alpha \perp \beta \perp (\beta\alpha)^{-1}, B \perp C \perp A)$ is isomorphic in co(F) to $(A \perp B \perp C, s(\alpha \perp \beta \perp (\beta\alpha)^{-1}, A \perp B \perp C)$ for a suitable 3-cycle s. It follows from the Whitehead lemma (1.7) that there is an isomorphism of $A \perp B \perp C$ with $A \perp A \perp A$ carrying $s(\alpha \perp \beta \perp (\beta\alpha)^{-1})$ to t, the corresponding three cycle on $A \perp A \perp A$. Since a three cycle lies in the commutator subgroup of the symmetric group on three elements it follows that $s(\alpha \perp \beta \perp (\beta\alpha)^{-1})$ is in the commutator subgroup of $Aut_{\underline{\underline{A}}'}(F(A \perp B \perp C))$. The conclusion of the paragraph above now implies that 0 =

$[A \perp B \perp C, s(\alpha \perp \beta \perp (\beta\alpha)^{-1}), A \perp B \perp C]' = [A \perp B \perp C, \alpha \perp \beta \perp (\beta\alpha)^{-1}, B \perp C \perp A]' = [A, \alpha, B]' + [B, \beta, C]' + [C, (\beta\alpha)^{-1}, A]'$.

Now an entirely analogous argument shows that $[A, \alpha, B]' + [B, \alpha^{-1}, A]' = 0$ as well. This and the previous conclusion imply that

$$[A, \beta\alpha, C]' = [A, \alpha, B]' + [B, \beta, C]',$$

as claimed.

Any element of $K_0'(F)$ is of the form $[A_1, \alpha_1, B_1]' - [A_2, \alpha_2, B_2]'$, and we can express this as $[A_1 \perp B_2, \alpha_1 \alpha_2^{-1}, A_2 \perp B_1]$. This concludes the proof of Proposition (5.1).

We shall now investigate the group $K_1(F)$, and, in particular, compare it with the group $K_1(\underline{\underline{A}}, F) = K_1(\mathrm{Ker}\ \Sigma F)$ defined in (1.4). Recall that Ker ΣF is the full subcategory of $\Sigma\underline{\underline{A}}$ whose objects are the (A, α) such that $F\alpha = 1_{FA}$. An object of Σ co(F) is of the form $((A, \gamma, B), (\alpha, \beta))$ where (α, β) is an automorphism in co(F) of (A, γ, B). This means that $\alpha \in \mathrm{Aut}_{\underline{\underline{A}}}(A)$, $\beta \in \mathrm{Aut}_{\underline{\underline{A}}}(B)$, and

$$\begin{array}{ccc} FA & \xrightarrow{\gamma} & FB \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle F\beta} \\ FA & \xrightarrow[\gamma]{} & FB \end{array}$$

commutes. The diagonal functor $\Sigma\Delta: \Sigma\underline{\underline{A}} \longrightarrow \Sigma$ co(F) is defined by $\Sigma\Delta(A, \alpha) = (\Delta A, \Delta\alpha) = ((A, 1_{FA}, A), (\alpha, \alpha))$. It induces the split exact sequence

$$0 \longrightarrow K_1\underline{\underline{A}} \longrightarrow K_1(\mathrm{co}(F)) \longrightarrow K_1(F) \longrightarrow 0$$

which defines $K_1(F)$. It follows from (3.4) (b) and (c) that Δ <u>is cofinal provided</u> F <u>is</u> E-<u>surjective</u>.

There is also a natural functor

(3) $H: \mathrm{Ker}\ \Sigma F \longrightarrow \Sigma\ \mathrm{co}(F)$

defined by $H(A, \alpha) = (\Delta A, (\alpha, 1_A))$. The fact that $F\alpha = 1_A$ shows that $(\alpha, 1_A)$ is indeed an automorphism of $\Delta A = (A, 1_{FA}, A)$. Moreover this functor is clearly product preserving. If (β, γ) is any automorphism of ΔA then we can write $(\beta, \gamma) = (\alpha, 1_A)(\gamma, \gamma)$ where $\alpha = \beta\gamma^{-1}$ is such that $F\alpha = 1_{FA}$. This canonical factorization shows that $\mathrm{Aut}_{\mathrm{co}(F)}(\Delta A)$ is the semi-direct product of $\Delta(\mathrm{Aut}_{\underline{\underline{A}}}(A))$ with the normal subgroup $H(\mathrm{Aut}_{\underline{\underline{A}}}(A, F))$. If we abelianize $\mathrm{Aut}_{\mathrm{co}(F)}(\Delta A)$, we obtain $G((\Delta A)) = G((A), F) \oplus G((A))$, in the notation of §2 (see Lemma (2.6)). Here with first summand comes from H, the second from Δ. Now if we take the direct limit of these groups over objects ΔA ($A \in \underline{\underline{A}}$), as in §2, then we obtain $K_1(\underline{\underline{A}}, F) \oplus K_1(\underline{\underline{A}})$. In case the functor F is E-surjective then, as remarked above, Δ is cofinal. It follows therefore from (2.2) that the direct limit we have just taken is canonically isomorphic to $K_1(\mathrm{co}(F))$. We record this now:

(5.2) PROPOSITION. <u>The functors</u>

$$\Sigma\underline{\underline{A}} \xrightarrow{\Sigma\Delta} \Sigma\ \mathrm{co}(F) \xleftarrow{H} \mathrm{Ker}\ \Sigma F \quad \text{(see (2) and (3))}$$

<u>induce a homomorphism</u> $K_1(\underline{\underline{A}}, F) \oplus K_1(\underline{\underline{A}}) \longrightarrow K_1(\mathrm{co}(F))$ <u>which is an isomorphism if</u> F <u>is</u> E-<u>surjective</u>. <u>Hence</u> H <u>induces an isomorphism</u> $K_1(\underline{\underline{A}}, F) \longrightarrow K_1(F)$ <u>in this case</u>.

The exact sequence associated with F will now be constructed as the bottom row of the following diagram:

$$
(4)\qquad
\begin{array}{ccccccccccc}
K_1(\underline{\underline{A}}) & = & K_1(\underline{\underline{A}}) & \longrightarrow & 0 & \longrightarrow & K_o(\underline{\underline{A}}) & = & K_o(\underline{\underline{A}}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \Delta_1} & & \downarrow{\scriptstyle d_1} & & \downarrow & & \downarrow{\scriptstyle \Delta_o} & & \downarrow{\scriptstyle d_o} & & \downarrow \\
K_1(co(F)) & \xrightarrow{g_1} & K_1(\underline{\underline{A}}) \oplus K_1(A) & \xrightarrow{f_1} & K_1(\underline{\underline{A}}') & \xrightarrow{\partial} & K_o'(co(F)) & \xrightarrow{g_o} & K_o(\underline{\underline{A}}) \oplus K_o(\underline{\underline{A}}) & \xrightarrow{f_o} & K_o(\underline{\underline{A}}') \\
\downarrow & & \downarrow{\scriptstyle s_1} & & \| & & \downarrow & & \downarrow{\scriptstyle s_o} & & \| \\
K_1(F) & \longrightarrow & K_1(\underline{\underline{A}}) & \longrightarrow & K_1(\underline{\underline{A}}') & \xrightarrow{\partial'} & K_o'(F) & \longrightarrow & K_o(\underline{\underline{A}}) & \longrightarrow & K_o(\underline{\underline{A}}')
\end{array}
$$

$$K_1(\underline{\underline{A}}, F) \xrightarrow{H} K_1(co(F)), \qquad K_1(\underline{\underline{A}}, F) \xrightarrow{h} K_1(F)$$

The middle row is the Mayer-Vietoris sequence (4.3) of (1) above. The maps d_i and s_i are: $d_i(x) = (x, -x)$; $s_i(x, y) = x + y$, $(i = 0, 1)$. The vertical involving Δ_0 is the split exact sequence of (5.1) above. The vertical involving Δ_1 is the short exact sequence defining $K_1(F)$. Since the terms of the bottom row are each the cokernel of the corresponding vertical exact sequence, and since the top half of the diagram commutes, it follows that the horizontal arrows on the bottom are defined uniquely by commutativity of the diagram.

On the left we have $H: K_1(\underline{\underline{A}}, F) \longrightarrow K_1(co(F))$ from (5.2) above, and we define $h: K_1(\underline{\underline{A}}, F) \longrightarrow K_1(F)$ to make the triangle commute. The composite $K_1(\underline{\underline{A}}, F) \longrightarrow K_1(F) \longrightarrow K_1(\underline{\underline{A}})$ sends the class of $(A, \alpha) \in \operatorname{Ker} \Sigma F$ to $s_1(g_1([H(A, \alpha)])) = s_1 g_1[\Delta A, (\alpha, 1_A)]_{co(F)} = s_1([A, \alpha]_{\underline{\underline{A}}}, [A, 1_A]_{\underline{\underline{A}}}) = [A, \alpha]_{\underline{\underline{A}}}$, since $[A, 1_A]_{\underline{\underline{A}}} = 0$. Thus this composite is just the map $K_1(\underline{\underline{A}}, F) \longrightarrow K_1(\underline{\underline{A}})$ induced by the inclusion $\operatorname{Ker} \Sigma F \subset \Sigma \underline{\underline{A}}$ (see (1.5)).

Since the top row is acyclic and the middle row is a complex, it follows that the bottom row is a complex whose homology agrees with that of the middle - thanks to the long homology sequence (I, 5.1). Therefore the bottom row is exact everywhere that the Mayer-Vietoris sequence is. If we now invoke Theorem (4.3) we obtain from the discussion above the following conclusions:

(5.3) THEOREM. _Let_ $F: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ _be a cofinal product preserving functor, and let_

$$(5) \qquad K_1(F) \xrightarrow{f} K_1(\underline{\underline{A}}) \longrightarrow K_1(\underline{\underline{A}}') \longrightarrow K_0'(F) \longrightarrow K_0(\underline{\underline{A}}) \longrightarrow K_0(\underline{\underline{A}}')$$

be the sequence constructed above in (4). _Then_ (5) _is exact except perhaps at_ $K_1(\underline{\underline{A}})$, _and the homomorphism_ $h: K_1(\underline{\underline{A}}, F) \longrightarrow K_1(F)$ _of_ (4) _composes with_ f _to give the map_ $K_1(\underline{\underline{A}}, F)$ $K_1(\underline{\underline{A}})$ _induces by the inclusion_ $\operatorname{Ker} \Sigma F \subset \Sigma \underline{\underline{A}}$. _If_ F _is_ E-surjec-

tive then the natural projection $K_0(F) \longrightarrow K_0{}'(F)$ *is an isomorphism*, h *is an isomorphism, and the sequence* (5) *is exact*.

We now indicate the naturality of the sequence (5). For this suppose we are given a square

$$(6) \qquad \begin{array}{ccc} \underline{\underline{B}} & \xrightarrow{\;G\;} & \underline{\underline{B}}' \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J'} \\ \underline{\underline{A}} & \xrightarrow[F]{} & \underline{\underline{A}}' \end{array} \quad , \quad \alpha\colon FJ \xrightarrow{\;\simeq\;} J'G.$$

of product preserving functors. Suppose, moreover, that F and G are cofinal. Then the diagram (6) induces a morphism of sequences.

(7)

$$
\begin{array}{ccccccccccc}
K_1(G) & \longrightarrow & K_1(\underline{\underline{B}}) & \longrightarrow & K_1(\underline{\underline{B}}') & \xrightarrow{\partial'} & K_o{}'(G) & \longrightarrow & K_o(\underline{\underline{B}}) & \longrightarrow & K_o(\underline{\underline{B}}') \\
\downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J'} & & \downarrow{\scriptstyle j_o} & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J'} \\
K_1(F) & \longrightarrow & K_1(\underline{\underline{A}}) & \longrightarrow & K_1(\underline{\underline{A}}') & \xrightarrow{\partial'} & K_o{}'(F) & \longrightarrow & K_o(\underline{\underline{A}}) & \longrightarrow & K_o(\underline{\underline{A}}')
\end{array}
$$

The map j_o is defined by

$$j_o[B_1, \beta, B_2]' = [JB_1, \alpha_{B_2}{}^{-1}(J'B)\alpha_{B_1}, JB_2]'.$$

Here, of course, $\beta: GB_1 \longrightarrow GB_2$ and $\alpha_{B_i}: FJB_i \longrightarrow J'GB_i$ $(i = 1, 2)$. The definition of j_1 makes analogous use of the natural transformation α.

If $(GB, \beta) \in \Sigma\underline{\underline{B}}'$ then $j_o\partial'[GB, \beta] = j_o[B, \beta, B]' = [JB, \alpha_B{}^{-1}(J'\beta)\alpha_B, JB]' = \partial'[FJB, \alpha_B{}^{-1}(J'\beta)\alpha_B] = \partial'[J'GB, J'\beta] = \partial'J'[GB, \beta]$; i.e. $j_o\partial' = \partial'J'$. This calculation illustrates the mechanism in (7). We shall study a special type of j_o in the "excision theorem" in the next section.

We shall conclude this section now with a description of the "exact sequence of a triple". For this we suppose we are given a commutative triangle

$$\begin{array}{ccccc} & & \underline{\underline{B}} & & \\ & \overset{F}{\nearrow} & & \overset{G}{\searrow} & \\ \underline{\underline{A}} & & \xrightarrow[H = GF]{} & & \underline{\underline{C}} \end{array}$$

of cofinal product preserving functors. Then we have product and composition preserving functors.

$$\partial: co(F) \longrightarrow co(H) \;;\; \partial(A_1, \beta, A_2) = (A_1, G\beta, A_2)$$

and

$$\delta: co(H) \longrightarrow co(G) \;;\; \delta(A_1, \gamma, A_2) = (FA_1, \gamma, FA_2).$$

These induce homomorphisms

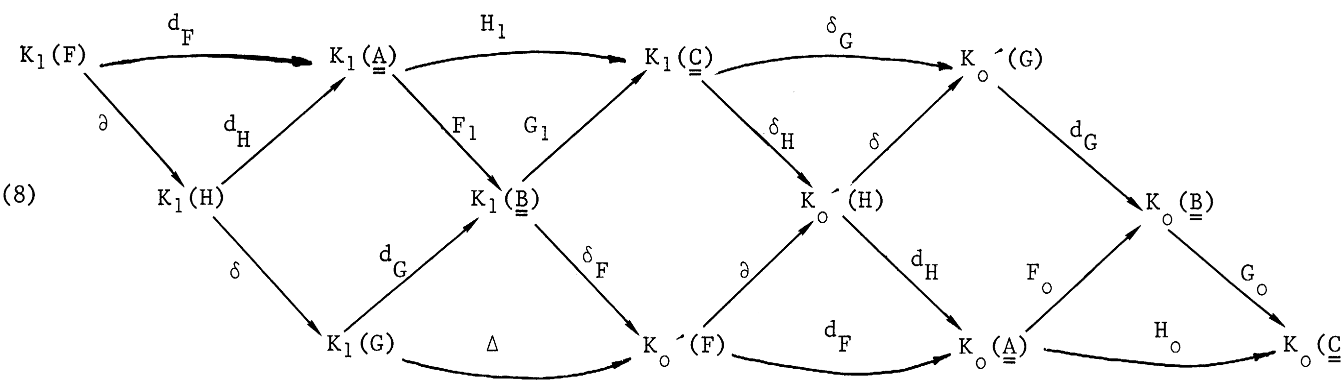
$$K_o'(F) \xrightarrow{\partial} K_o'(H) \xrightarrow{\delta} K_o'(G)$$

and

$$K_1(F) \xrightarrow{\partial} K_1(H) \xrightarrow{\delta} K_1(G)$$

with $\delta\partial = 0$ in each case. Now consider the commutative diagram

(8)


K1(F)
K1(A)
K1(C)
Ko´(G)
K1(H)
K1(B)
Ko´(H)
Ko(B)
K1(G)
Ko´(F)
Ko(A)
Ko(C)
dF
H1
δG
∂
dH
F1
G1
δH
δ
dG
δ
dG
δF
∂
dH
Fo
Go
Δ
dF
Ho

The map Δ is defined to make the diagram commute, and the remaining commutativity follows directly from the definitions.

(5.4) THEOREM. In the sequence

$$(9) \qquad K_1(F) \xrightarrow{\partial} K_1(H) \xrightarrow{\delta} K_1(G) \xrightarrow{\Delta} K_o'(F) \xrightarrow{\partial} K_o'(H) \xrightarrow{\delta} K_o'(G)$$

all composites are zero, and it is exact at $K_o'(H)$. If G is E-surjective it is exact at $K_o'(F)$. If also F is E-surjective then it is exact at $K_1(G)$.

Proof. The K-sequences (5) of the functors F, G, and H are embedded in the diagram (8), and in each of these sequences all composities are zero. It follows then from commutativity that $\Delta\delta = 0$ and $\partial\Delta = 0$ in (9), and we have already noted that $\delta\partial = 0$ in each case.

Exactness at $K_o'(H)$. This is a diagram chose, using the exactness of the K-sequences of F, G, and H. We leave it as an excercise.

Exactness at $K_o'(F)$ when G is E-surjective. If $x \in K_o'(F)$ and $dx = 0$ we must show that $x \in Im(\Delta)$. Since $d_F x = 0$ we have $x = \delta_F y$ for $y \in K_1(\underline{\underline{B}})$. Since $\delta_H G_1(y) = d\delta_F y = dx = 0$ we have $G_1(y) = H_1(z)$ for some $z \in K_1(\underline{\underline{A}})$ because G is E-surjective. Now $G_1F_1(z) = G_1(y)$ so $y - F_1(z) = d_G(u)$ for some $u \in K_1(G)$. Now $\Delta u = \delta_F d_G(u) = \delta_F(y - F_1(z)) = \delta_F(y) - 0 = x$.

Exactness at $K_1(G)$ when G and F are E-surjective. Suppose $x \in K_1(G)$ and $\Delta x = 0$. We must show that $x \in Im(\delta)$. Since $0 = \Delta(x) = \delta_F d_G(x)$ we have $d_G(x) = F_1(y)$ for some $y \in K_1(\underline{\underline{A}})$. Since $H_1(y) = 0$ and H is E-surjective (because F and G are) we have $y = d_H(z)$ for some $z \in K_1(H)$. Then $d_G\delta(z) = F_1 d_H(z) = F_1(y) = D_G(x)$ so $d_G(u) = 0$ where $u = x = x - \delta z$.

Since G is E-surjective we have $K_1(G) \simeq K_1(\underline{\underline{B}}, G)$, so we can write $u = [B, \beta]$ for some $B \in \underline{\underline{B}}$ and $\beta \in \mathrm{Aut}_{\underline{\underline{B}}}(B)$ such that $G\beta = 1_{GB}$. Since F is cofinal we can even assume $B = FA$ for some $A \in \underline{\underline{A}}$. The fact that $d_G(u) = 0$ implies, according to (1.9), that $\beta \perp 1_{B'}$ is in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{B}}}(FA \perp B')$ for some B'. We can further assume $B' = FA'$ since F is cofinal. Since F is E-surjective we can, after augmenting A' if necessary, write $\beta \perp 1_{FA'} = F\alpha$ for some α in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}}(A \perp A')$. We have $H\alpha = GF\alpha = G(\beta \perp 1_{FA'}) = G\beta \perp 1_{HA'} = 1_{H(A \perp A')}$. Now $v = [A \perp A', \alpha] \in K_1(\underline{\underline{A}}, H) \simeq K_1(H)$ is such that $\delta(v) = [F(A \perp A'), F\alpha] = u = x - \delta(z)$. Hence $x = \delta(v + z)$. q.e.d.

This proves Theorem (5.4) as formulated. We conclude with a criterion for exactness at $K_1(H)$.

(5.5) PROPOSITION. *Assume, in Theorem* (5.4), *that* F *and* G *are* E-*surjective, and that the following condition is satisfied*:

> *Given* $A \in \underline{\underline{A}}$ *and* $\alpha \in \mathrm{Aut}_{\underline{\underline{A}}}(A)$ *such that* $F\alpha \in [\mathrm{Aut}_{\underline{\underline{B}}}(FA), \mathrm{Aut}_{\underline{\underline{B}}}(FA, G)]$, *there exists a* $B = A \perp A'$ *and an* $\varepsilon \in [\mathrm{Aut}_{\underline{\underline{A}}}(B), \mathrm{Aut}_{\underline{\underline{A}}}(B, H)]$ *such that* $F\varepsilon = F\alpha$.

Then (9) *is exact at* $K_1(H)$.

Proof. Given $x \in K_1(H)$ such that $\delta(x) = 0$ we must show that $x \in \mathrm{Im}(\partial)$. Since H is E-surjective we can write $x = [A, \alpha]$ with $A \in \underline{\underline{A}}$ and $\alpha \in \mathrm{Aut}_{\underline{\underline{A}}}(A, H)$. Since $0 = \delta(x) = [FA, F\alpha]$ we can, according to (2.1), find $C' = FA \perp B'$ such that $\alpha \perp 1_{B'} \in [\mathrm{Aut}_{\underline{\underline{B}}}(C'), \mathrm{Aut}_{\underline{\underline{B}}}(C', G)]$. Since F is cofinal we can even take B' of the form FA', so that $C' = FC$ where $C = A \perp A'$. According to the hypothesis we can now augment C further, if necessary, to find $\varepsilon \in [\mathrm{Aut}_{\underline{\underline{A}}}(C), \mathrm{Aut}_{\underline{\underline{A}}}(C, H)]$ such that $F\varepsilon = F(\alpha \perp 1_{A'})$. Now $y = [C, \varepsilon^{-1}(\alpha \perp 1_{A'})]_F \in$

$K_1(\underline{\underline{A}}, F) = K_1(F)$, and $\partial(y) = [C, \varepsilon^{-1}]_H + [C, \alpha \perp 1_{A'}]_H = [A, \alpha] = x$, because $[C, \varepsilon]_H = 0$. q.e.d.

§6. EXCISION ISOMORPHISMS

(6.1) THEOREM ("Excision"). *Let*

$$(1) \qquad \begin{array}{ccc} \underline{\underline{A}} & \xrightarrow{G_2} & \underline{\underline{A}}_2 \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \underline{\underline{A}}_1 & \xrightarrow[F_1]{} & \underline{\underline{A}}' \end{array} , \quad \alpha\colon F_1G_1 \longrightarrow F_2G_2,$$

be a cartesian square of product preserving functors for which (F_1, F_2) *is a cofinal pair* (see (3.2)). *Let*

$$(2) \qquad \begin{array}{ccccccccc} K_1(\underline{\underline{A}}) & \longrightarrow & K_1(\underline{\underline{A}}_2) & \xrightarrow{\partial'} & K_0'(G_2) & \longrightarrow & K_0(\underline{\underline{A}}) & \longrightarrow & K_0(\underline{\underline{A}}_2) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow \\ K_1(\underline{\underline{A}}_1) & \longrightarrow & K_1(\underline{\underline{A}}') & \xrightarrow{\partial'} & K_0'(F_1) & \longrightarrow & K_0(\underline{\underline{A}}_1) & \longrightarrow & K_0(\underline{\underline{A}}') \end{array}$$

be the morphism of exact sequences induced by (1). *Then* ϕ *is surjective. If* (1) *is* E-*surjective* (see (3.3)), *e.g. if* F_1 *or* F_2 *is* E-*surjective* (see (2.4)), *then* ϕ *is an isomorphism.*

Remark. Since E-surjectivity of (1) is a symmetric hypothesis it also implies an isomorphism $K_0'(G_1) \longrightarrow K_0'(F_2)$. In the applications we shall make of this theorem either F_1 or F_2 will be E-surjective and hence, by (3.4) (c), (1) will be E-surjective. However, E-surjectivity of one of the F_i is no longer a symmetric hypothesis.

Proof. The map ϕ is induced by a product preserving functor:

$$F\colon co(G_2) \longrightarrow co(F_1);$$

$$F(A, \gamma, B) = (G_1A, \alpha_B^{-1}(F_2\gamma)\alpha_A, G_1B).$$

Here, of course, $A = (G_1A, \alpha_A, G_2A)$ and $B = (G_1B, \alpha_B, G_2B) \in \underline{\underline{A}} = \underline{\underline{A}}_1 \times_{\underline{\underline{A}}'} \underline{\underline{A}}_2$, and $\gamma: G_2A \longrightarrow G_2B$.

ϕ <u>is surjective</u>: Suppose $U = (A_1, \gamma, B_1) \in co(F_1)$, so $\gamma: F_1A_1 \longrightarrow F_1B_1$. Since F_2 is cofinal relative to F_1 we can find $A_1' \in \underline{\underline{A}}_1$, $A_2 \in \underline{\underline{A}}_2$, and an isomorphism $\alpha: F_1(A_1 \perp A_1') \longrightarrow F_2A_2$. Now define β to make

$$\begin{array}{ccc} F_1(A_1 \perp A_1') & \xrightarrow{\gamma \perp 1_{F_1A_1'}} & F_1(B_1 \perp A_1') \\ & {}_{\alpha}\searrow \quad \swarrow_{\beta} & \\ & F_2A_2 & \end{array}$$

commute. Then $U \perp \Delta A_1' = (A_1 \perp A_1', \gamma \perp 1_{F_1A_1'}, B_1 \perp A_1')$ $= FV$, where $V = ((A_1 \perp A_1', \alpha, A_2), 1_{A_2}, (B_1 \perp A_1', \beta, A_2))$. Therefore, in $K_o'(F_1)$, $[U]' = [U \perp \Delta A_1']' = \phi[V]'$.

ϕ <u>is injective</u>: Suppose $\phi(x) = 0$. According to (5.1) we can write $x = [U]'$ for some $U = (A, \gamma, B) \in co(G_2)$, and so $[A_1, \overline{\gamma}, B_1]' = 0$ in $K_o'(F_1)$ where we are abbreviating $A_i = G_iA$, $B_i = G_iB$ $(i = 1, 2)$ and $\overline{\gamma} = \alpha_B^{-1}(F_2\gamma)\alpha_A$. In particular $[A_1] = [B_1]$ in $K_o(\underline{\underline{A}}_1)$ so $A_1 \perp A_1' \simeq B_1 \perp A_1'$ for some $A_1' \in \underline{\underline{A}}_1$. Since the functor G_1 is cofinal (see (3.4) (d)) we can write $A_1' = G_1A'$ for some $A' \in \underline{\underline{A}}$. Since $x = [U]' = [U \perp \Delta A']'$, where $\Delta A' = (A', 1_{G_2A'}, A')$, we can replace U by $U \perp \Delta A'$ and henceforth assume that $A_1 \simeq B_1$. If we use such an isomorphism together with $\gamma: A_2 \longrightarrow B_2$ we can replace B by an isomorphic object in $\underline{\underline{A}}$ to further achieve: $A_1 = B_1$, $A_2 = B_2$, and $\gamma = 1_{A_2}$. Thus we now have $x = [U]'$ where $U = (A, 1_{A_2}, B)$, $A = (A_1, \alpha_A, A_2)$, and $B = (A_1, \alpha_B, A_2)$. If we set $\overline{\gamma} = \alpha_B^{-1}\alpha_A$ then we can write $B = A\overline{\gamma}$, so $U = (A, 1_{A_2}, A\overline{\gamma})$, $FU = (A_1, \overline{\gamma}, A_1)$.

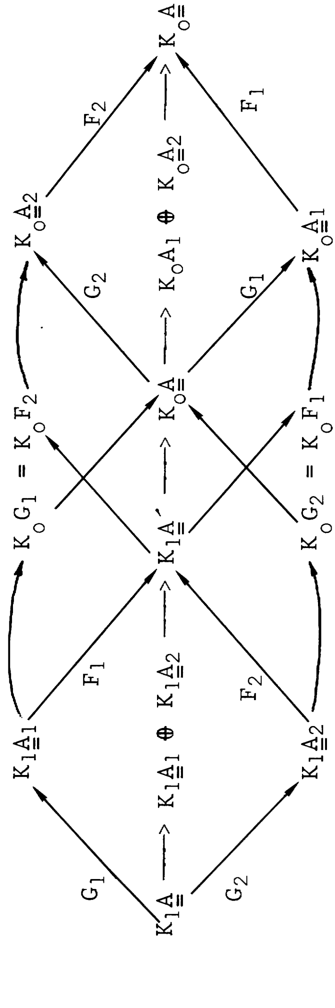
We have $0 = \phi(x) = [A_1, \gamma, A_1]' = \partial'[F_1A_1, \bar{\gamma}]$. The exact sequence of F_1 therefore implies that the element $[F_1A_1, \bar{\gamma}] \in K_1(\underline{\underline{A}}')$ belongs to Image$(K_1(\underline{\underline{A}}_1) \longrightarrow K_1(\underline{\underline{A}}'))$. Since the functor F_1G_1 is cofinal (see (3.4) (d)) it follows from (2.1) that there is an $A'' \in \underline{\underline{A}}$ and an $\alpha_1 \in \mathrm{Aut}_{\underline{\underline{A}}_1}(A_1'')$ such that $\bar{\gamma} \perp 1_{F_1A_1''} = (F_1\alpha_1)\varepsilon$, where ε is in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}'}(F_1(A_1 \perp A_1''))$. Since $x = [U]' = [U \perp \Delta A'']'$ we can replace U by $U \perp \Delta A''$. This does not affect any of the normalizations (i.e. $A_1 = B_1$, $A_2 = B_2$, $\gamma = 1_{A_2}$) made above, and it replaces $\bar{\gamma}$ by $\bar{\gamma} \perp 1_{F_1A_1''}$. Thus, after this replacement, we can assume further that $\bar{\gamma} = (F_1\alpha_1)\varepsilon$, and we still have $U = (A, 1_{A_2}, A\bar{\gamma})$.

Since the diagram (1) is, by hypothesis, E-surjective, it follows that there is a $C \in \underline{\underline{A}}$, and elements ε_i in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}_i}(A_i \perp C_i)$, such that $(\varepsilon_1, \varepsilon_2)$: $A(F_1\alpha_1)\varepsilon \perp C \longrightarrow A(F_1\alpha_1) \perp C$ is an isomorphism in $\underline{\underline{A}}$. (See definition (3.3)). Moreover $(\alpha_1, 1_{A_2})$: $A(F_1\alpha_1) \longrightarrow A$ is an isomorphism in $\underline{\underline{A}}$. From these we obtain an isomorphism in $\mathrm{co}(G_2)$:

$$\begin{array}{rl} U \perp \Delta C = & (A \perp C,\ 1_{A_2 \perp C_2},\ A(F_1\alpha_1)\varepsilon \perp C) \\ & \Big\downarrow\ ((1_{A_1 \perp C_1}, 1_{A_2 \perp C_2}), (\alpha_1\varepsilon_1, \varepsilon_2)) \\ V = & (A \perp C,\ \varepsilon_2,\ A \perp C). \end{array}$$

Since ε_2 lies in the commutator subgroup of $\mathrm{Aut}_{\underline{\underline{A}}_2}(A_2 \perp C_2)$ we have $[A_2 \perp C_2, \varepsilon_2] = 0$ in $K_1(\underline{\underline{A}}_2)$. Therefore $0 = \partial'[G_2(A \perp C), \varepsilon_2] = [V]' = [U \perp \Delta C]' = [U]' = x$. q.e.d.

Using the excision isomorphisms and the Mayer-Vietoris sequence of (1), there is a natural procedure, familiar to topologists, for constructing a commutative diagram of the following type:


K1A ——> K1A1 ⊕ K1A2 ——> K1A′ ——> K0A ——> K0A1 ⊕ K0A2 ——> K0A′
K1A1
K1A2
K0G1 = K0F2
K0G2 = K0F1
K0A2
K0A1
G1
G2
F1
F2

The middle line is Mayer-Vietoris, and the "sine curves" are the exact sequences of the four functors in (1). The equalities are the excision isomorphisms.

HISTORICAL REMARKS

A number of people have constructed exact sequences more or less related to those considered here. Examples include Heller [1], Gersten [3], and Chase [1]. There have also been several, so far unpublished, definitions of higher K-functors. In particular Milnor has defined a functor K_2 (for the category $\underline{\underline{P}}(A)$, where A is a ring) and this K_2 seems to be susceptible to many of the techniques developed in these notes (see Gersten [2]). Moreover Nobile and Villamayor [1] have recently obtained a long exact sequence for functors K_n which are related to ours for n = 0, 1.

The exposition in this chapter is derived mainly from that of Chapter I of my Tata notes [4] plus some unpublished notes of Milnor. The proof of the excision isomorphisms is adapted from that of Theorem (7.2) in Bass-Murthy [1]. The latter, in turn, generalizes a theorem of Rim and Serre on "reduction modulo the conductor" (see (IX, 5.6)).

Chapter VIII

K-THEORY IN ABELIAN CATEGORIES

If $\underline{\underline{C}}$ is an additive subcategory of an abelian category $\underline{\underline{A}}$ we can view $\underline{\underline{C}}$ as a category with product, $\oplus$, in the sense of Chapter VII. In practice, however, it is natural to define the Grothendieck and Whitehead groups of $\underline{\underline{C}}$ by introducing relations for all short exact sequences in $\underline{\underline{C}}$, not just those which split. In case all short exact sequences in $\underline{\underline{C}}$ split, i.e. if $\underline{\underline{C}}$ is "semi-simple", then the definitions coincide. In §2 we show that an "exact" functor F: $\underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ induces an exact sequence like that in Chapter VII provided we impose conditions of semi-simplicity on the given data. This result is deduced directly from its analogue in Chapter VII.

In order to relax the semi-simplicity hypotheses we then show that the groups $K_i(\underline{\underline{C}})$ can sometimes be computed on a subcategory $\underline{\underline{C}}_0 \subset \underline{\underline{C}}$, the point being that $\underline{\underline{C}}_0$ may be semi-simple even when $\underline{\underline{C}}$ is not. The first such result, called "devissage", is based on the Jordan-Holder Theorem (§3). The other, a fundamental theorem of Grothendieck (in the case of K_0), is based upon taking resolutions and the use of "Euler characteristics" (see §4).

In §5 we prove an important theorem of Heller describing the exact sequence of a localizing functor. The philosophy is, roughly, that if one views a localizing functor as defining a short exact sequence of categories, then K-theory should behave like a cohomological functor with respect to such exact sequences.

There is a closely related theorem (Theorem (5.8)), applying specifically to categories of projective objects, which seems to require additional techniques for its proof. Both of these theorems are used heavily in later chapters.

The final section (§6) contains a remarkable new theorem of Leslie Roberts: Let $\underline{\underline{A}}$ be a k-category, where k is an algebraically closed field, and assume that $\underline{\underline{A}}(A, B)$ is always finite dimensional. Then

$$K_1(A) \simeq K_o(\underline{\underline{A}}) \otimes_{\underline{\underline{Z}}} k^*.$$

This applies, notably, when $\underline{\underline{A}}$ is the category of coherent sheaves on a complete algebraic variety over k.

§1. GROTHENDIECK GROUPS AND WHITEHEAD GROUPS IN ABELIAN CATEGORIES

All categories in this chapter will be of the following type, though condition (d) below will play no role until §3.

(1.1) DEFINITION. A subcategory $\underline{\underline{C}}$ of an abelian category $\underline{\underline{A}}$ will be said to be <u>admissible</u> if it satisfies the following conditions:

(a) $\underline{\underline{C}}$ is a full subcategory of $\underline{\underline{A}}$ and it contains a zero object.

(b) $\underline{\underline{C}}$ has only a set of isomorphism classes of objects.

(c) Finite direct sums of objects in $\underline{\underline{C}}$ are again in $\underline{\underline{C}}$.

(d) If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence in $\underline{\underline{A}}$, then $A, A'' \in \underline{\underline{C}} \Rightarrow A' \in \underline{\underline{C}}$.

Clearly $\Sigma\underline{\underline{C}}$ (= $\underline{\underline{C}}^{\underline{\underline{Z}}}$, c.f. (VII, 1.4)) is then an admissible subcategory of the abelian category $\Sigma\underline{\underline{A}}$. We shall say P is <u>projective</u> in $\underline{\underline{C}}$ if $P \in \underline{\underline{C}}$ and P is projective in $\underline{\underline{A}}$. Similarly we call a sequence in $\underline{\underline{C}}$ <u>exact</u> if it is exact in $\underline{\underline{A}}$. The category of short exact sequences, $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, in $\underline{\underline{C}}$ will be denoted by

$$Ex(\underline{\underline{C}}) \quad (Ex(\underline{\underline{A}})).$$

We call $\underline{\underline{C}}$ <u>semi-simple</u> if all short exact sequences in $\underline{\underline{C}}$ split. Note that this does not imply that the objects of $\underline{\underline{C}}$ are semi-simple. Neither does it imply that the category $\Sigma\underline{\underline{C}}$ is semi-simple.

Let $\underline{\underline{C}} \subset \underline{\underline{A}}$ and $\underline{\underline{C}}' \subset \underline{\underline{A}}'$ be admissible subcategories of abelian categories. A functor $F: \underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ will be called <u>admissible</u> if it is induced by an additive functor $\overline{F}: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$. We shall say that F is <u>exact</u> if it carries short exact sequences in $\underline{\underline{C}}$ into short exact sequences in $\underline{\underline{C}}'$. In this case F induces an additive functor

$$Ex(F): Ex(\underline{\underline{C}}) \longrightarrow Ex(\underline{\underline{C}}').$$

Moreover the functor $\Sigma F: \Sigma\underline{\underline{C}} \longrightarrow \Sigma\underline{\underline{C}}'$ will be exact if F is. The category co(F) is an additive category. If $\overline{F}$ is exact then $co(\overline{F})$ is an abelian category of which co(F) is an admissible subcategory.

The direct sum, $\oplus$, gives $\underline{\underline{C}}$ the structure of a category with product, in the sense of Chapter VII. Moreover any additive functor is product preserving. In order to avoid confusion in what follows, we shall use the notation

$$(\underline{\underline{C}}, \oplus)$$

when referring to $\underline{\underline{C}}$ as a category with product. Thus we have the groups

$$K_i(\underline{\underline{C}}, \oplus) \qquad (i = 0, 1)$$

constructed in the last chapter. Similarly, if $F: \underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ is an admissible functor then we have

$$K_0'(F, \oplus) \quad \text{and} \quad K_1(F, \oplus)$$

constructed as quotients of $K_i(co(F), \oplus)$. We shall now introduce groups $K_i(\underline{\underline{C}})$ $(i = 0, 1)$ and $K_0'(F)$ which are quotients of the corresponding groups above. They are obtained by requiring the class of an object in K to be additive not only over direct sums, but over all short exact sequences. Specifically:

$$[\ ]_{\underline{\underline{C}}}: \text{ob } C \longrightarrow K_0(\underline{\underline{C}})$$

is universal for maps into an abelian group satisfying

KO. If $(0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0) \in \text{Ex}(\underline{\underline{C}})$ then

$$[A]_{\underline{\underline{C}}} = [A']_{\underline{\underline{C}}} + [A'']_{\underline{\underline{C}}}.$$

Similarly,

$$[\ ]_{\underline{\underline{C}}}: \text{ob } \Sigma\underline{\underline{C}} \longrightarrow K_1(\underline{\underline{C}})$$

is universal for maps into an abelian group satisfying KO and

K1. If $(A, \alpha), (A, \beta) \in \Sigma\underline{\underline{C}}$ then

$$[A, \alpha\beta]_{\underline{\underline{C}}} = [A, \alpha]_{\underline{\underline{C}}} + [A, \beta]_{\underline{\underline{C}}}.$$

If $F: \underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ is an exact functor then

$$[\ ]_F: \text{ob co}(F) \longrightarrow K_0'(F)$$

is universal for maps into an abelian group satisfying:

KO If $(0 \longrightarrow (A_1', \alpha', A_2') \longrightarrow (A_1, \alpha, A_2)$

$\longrightarrow (A_1'', \alpha'', A_2'') \longrightarrow 0) \in \text{Ex}(\text{co}(F))$

then

$$[A_1, \alpha, A_2]_F = [A_1', \alpha', A_2']_F + [A_1'', \alpha'', A_2'']_F;$$

and

K1 If $(A, \alpha, B), (B, \beta, C) \in \text{co}(F)$ then

$$[A, \beta\alpha, C]_F = [A, \alpha, B]_F + [B, \beta, C]_F.$$

(cf. (VII, 5.1)). From these definitions it is clear that there are canonical epimorphisms $K_i(\underline{\underline{C}}, \oplus) \longrightarrow K_i(\underline{\underline{C}})$ $(i = 0, 1)$ and $K_0'(F, \oplus) \longrightarrow K_0'(F)$. Moreover F induces homomorphisms $K_i(\underline{\underline{C}}) \longrightarrow K_i(\underline{\underline{C}}')$ $(i = 0, 1)$ via $[A]_{\underline{\underline{C}}} \longmapsto [FA]_{\underline{\underline{C}}'}$ and $[A, \alpha]_{\underline{\underline{C}}} \longmapsto [FA, F\alpha]_{\underline{\underline{C}}'}$, respectively. There is also a commutative square

$$\begin{array}{ccc} K_o'(F, \oplus) & \xrightarrow{d} & K_o(C, \oplus) \\ \downarrow & & \downarrow \\ K_o'(F) & \xrightarrow{d} & K_o(C) \end{array}$$

where $d[A, \alpha, B]_F = [A]_{\underline{\underline{C}}} - [B]_{\underline{\underline{C}}}$, and d is analogously defined.

§2. THE K-SEQUENCE OF A COFINAL EXACT FUNCTOR

Let $F: \underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ be an admissible functor between admissible subcategories of abelian categories. We say F is <u>cofinal</u> if it is cofinal with respect to $\oplus$ in the sense of Chapter VII. Recall that this means, given $A' \in \underline{\underline{C}}'$, we can find $A \in \underline{\underline{C}}$ and $B' \in \underline{\underline{C}}'$ such that $A' \oplus B' \simeq FA$. Similarly, if F is exact, then it makes sense to say that $Ex(F): Ex(\underline{\underline{C}}) \longrightarrow Ex(\underline{\underline{C}}')$ is cofinal. The latter condition clearly implies that F itself is cofinal, and the converse is true if $\underline{\underline{C}}'$ is semi-simple.

Assume now that F <u>is cofinal and exact</u>. Then, except for ∂, we have a commutative diagram,

$$
\begin{array}{ccccccccc}
K_1(C, \oplus) & \longrightarrow & K_1(C', \oplus) & \xrightarrow{\partial_{\oplus}} & K_o'(F, \oplus) & \xrightarrow{d_{\oplus}} & K_o(C, \oplus) & \longrightarrow & K_o(C', \oplus) \\
\downarrow & & \downarrow & & \downarrow f & & \downarrow & & \downarrow \\
K_1(C) & \longrightarrow & K_1(C') & \overset{\partial}{\dashrightarrow} & K_o'(F) & \xrightarrow{d} & K_o(C) & \longrightarrow & K_o(C'),
\end{array}
\qquad (1)
$$

in which the top row comes from the sequence of (VII, 5.3). The maps $K_i(\underline{\underline{C}}) \longrightarrow K_i(\underline{\underline{C}}')$ are those induced by F, and d was constructed at the end of §1. The existence of ∂ is clearly equivalent to the following condition: If $(0 \longrightarrow (A', \alpha') \longrightarrow (A, \alpha) \longrightarrow (A'', \alpha'') \longrightarrow 0) \in Ex(\Sigma\underline{\underline{C}}')$ then $f\partial\, [A, \alpha]_{(\underline{\underline{C}}', \oplus)} = f\partial\, ([A', \alpha']_{(\underline{\underline{C}}', \oplus)} + [A'', \alpha'']_{(\underline{\underline{C}}', \oplus)})$.

(2.1) PROPOSITION. _Let_ $F: \underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ _be an exact cofinal functor as above. If_ Ex(F) _is cofinal then_ $\partial: K_1(\underline{\underline{C}}') \longrightarrow K_o'(F)$, _making diagram_ (1) _commute, exists. If_ Ex(F) _is surjective on stable isomorphism classes of objects then_

$$K_o'(F) \longrightarrow K_o(\underline{\underline{C}}) \longrightarrow K_o(\underline{\underline{C}}') \longrightarrow 0$$

is exact.

The surjectivity hypothesis means that, given $A \in Ex(\underline{\underline{C}}')$ there exist $B, C \in Ex(\underline{\underline{C}})$ such that $FC \simeq A \oplus FB$.

Proof. To show that ∂ exists we must show, given an exact sequence

$$(A, \alpha) = (0 \longrightarrow (A_2, \alpha_2) \longrightarrow (A_1, \alpha_1) \longrightarrow (A_o, \alpha_o) \longrightarrow 0)$$

in $\Sigma\underline{\underline{C}}'$, that $f\partial\, [A_1, \alpha_1]_{(\underline{\underline{C}}', \oplus)} = f\partial\, ([A_o, \alpha_o]_{(\underline{\underline{C}}', \oplus)} + [A_2, \alpha_2]_{(\underline{\underline{C}}', \oplus)})$. By the hypothesis that Ex(F) is cofinal, there are exact sequences $B \in Ex(\underline{\underline{C}}')$ and $C \in Ex(\underline{\underline{C}})$ such that $A \oplus B \simeq FC$. Using such an isomorphism we obtain an isomorphism in $\Sigma\underline{\underline{C}}'$ of the form $(A \oplus B, \alpha \oplus 1_B) \simeq (FC, \gamma)$ for some γ. Since $[A_i, \alpha_i]_{(\underline{\underline{C}}', \oplus)} = [A_i \oplus B_i, \alpha_i \oplus 1_{B_i}]_{(\underline{\underline{C}}', \oplus)}$ $(0 \leq i \leq 2)$ it suffices to establish the equation above with (FC, γ) in place of (A, α). But (C, γ, C) is an exact sequence in co(F). Using axiom K0 for $K_o'(F)$, therefore, we have

$$f\partial\, [FC_1, \gamma_1]_{(\underline{\underline{C}}', \oplus)} = f[C_1, \gamma_1, C_1]_{(F, \oplus)} =$$

$$[C_1, \gamma_1, C_1]_F = [C_o, \gamma_o, C_o]_F + [C_2, \gamma_2, C_2]_F$$

$$= f\partial \left([C_o, \gamma_o]_{(\underline{\underline{C}}', \oplus)} + [C_2, \gamma_2]_{(\underline{\underline{C}}', \oplus)}\right)$$

Now that ∂ exists we can expand diagram (1) to:

$$
(2) \qquad
\begin{array}{ccccccccc}
N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(\underline{\underline{C}}, \oplus) & \longrightarrow & K_1(\underline{\underline{C}}', \oplus) & \xrightarrow{\partial} & K_o'(F, \oplus) & \xrightarrow{d} & K_o(\underline{\underline{C}}, \oplus) & \longrightarrow & K_o(\underline{\underline{C}}', \oplus) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(\underline{\underline{C}}) & \longrightarrow & K_1(\underline{\underline{C}}') & \xrightarrow{\partial} & K_o'(F) & \longrightarrow & K_o(\underline{\underline{C}}) & \longrightarrow & K_o(\underline{\underline{C}}')
\end{array}
$$

where the top row is the kernel of the epimorphism from the middle to bottom. We shall view this as a short exact sequence of complexes (the rows) whose undenoted terms we take to be zero.

Evidently N_5 is generated by all elements $\langle A\rangle_{\underline{\underline{C}}'} = [A_1]_{(\underline{\underline{C}}', \oplus)} - [A_o]_{(\underline{\underline{C}}', \oplus)} - [A_2]_{(\underline{\underline{C}}', \oplus)}$ with $A = (0 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_o \longrightarrow 0) \in Ex(\underline{\underline{C}}')$. If Ex(F) is surjective on stable isomorphism classes then we can write $A \oplus FB \simeq FC$ for some B, $C \in Ex(\underline{\underline{C}})$, so $\langle A\rangle_{\underline{\underline{C}}'} = F(\langle C\rangle_{\underline{\underline{C}}} - \langle B\rangle_{\underline{\underline{C}}})$, and so $N_4 \longrightarrow N_5$ is surjective. Moreover F itself is surjective on stable isomorphism classes of objects if Ex(F) is, and hence $K_o(\underline{\underline{C}}, \oplus) \longrightarrow K_o(\underline{\underline{C}}', \oplus)$ is likewise surjective. The middle row of (2) is acyclic at the three middle positions according to (VII, 5.3). Therefore the long homology sequence of (2) shows that $K_o'(F) \xrightarrow{d} K_o(\underline{\underline{C}}) \longrightarrow K_o(\underline{\underline{C}}') \longrightarrow 0$ is exact, as claimed. This completes the proof of (2.1).

(2.2) THEOREM. Let $F: \underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ be an exact admissible functor between admissible subcategories of abelian categories, and assume that $Ex(F): Ex(\underline{\underline{C}}) \longrightarrow Ex(\underline{\underline{C}}')$ is cofinal.

(a) If $\underline{\underline{C}}$ is semi-simple then $\underline{\underline{C}}'$ is also semi-simple, and all the vertical arrows in (1) above are isomorphisms. In particular

$$K_1(\underline{\underline{C}}) \longrightarrow K_1(\underline{\underline{C}}') \xrightarrow{\partial} K_o'(F) \longrightarrow K_o(\underline{\underline{C}}) \longrightarrow K_o(\underline{\underline{C}}')$$

is exact

(b) If $\underline{\underline{C}}'$ is semi-simple then the sequence

$$K_1(\underline{\underline{C}}) \xrightarrow{\partial} K_o'(F) \longrightarrow K_o(\underline{\underline{C}}) \longrightarrow K_o(\underline{\underline{C}}')$$

is exact.

Proof. (a). If $A \in Ex(\underline{\underline{C}}')$ there exist $B \in Ex(\underline{\underline{C}}')$ and $C \in Ex(\underline{\underline{C}})$ such that $A \oplus B \simeq FC$. Since, by hypothesis, C is split, so also is FC, and hence likewise for A. Thus C′ is also semi-simple.

The fact that $K_i(\underline{\underline{C}}, \oplus) \longrightarrow K_i(\underline{\underline{C}})$ is an isomorphism is obvious for i = 0. For i = 1 we must show that if $(A, \alpha) = (0 \longrightarrow (A_2, \alpha_2) \longrightarrow (A_1, \alpha_1) \longrightarrow (A_o, \alpha_o) \longrightarrow 0)$ $Ex(\Sigma\underline{\underline{C}})$ then $[\alpha_1] = [\alpha_2] + [\alpha_o]$ in $K(\underline{\underline{C}}, \oplus)$. Since $\underline{\underline{C}}$ is semi-simple the sequence A splits so we can identify $A_1 = A_2 \oplus A_o$. We can then write α_1 in matrix form, with respect to this decomposition, as

$$\alpha_1 = \begin{pmatrix} \alpha_2 & * \\ 0 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_o \end{pmatrix} \begin{pmatrix} 1_{A_2} & *' \\ 0 & 1_{A_o} \end{pmatrix}$$

so $\alpha_1 = (\alpha_2 \oplus \alpha_o)\varepsilon$, where ε corresponds to the right hand factor. In $K_1(\underline{\underline{C}}, \oplus)$ we have $[\alpha_1] = [\alpha_2 \oplus \alpha_o] + [\varepsilon] = [\alpha_2] + [\alpha_o] + [\varepsilon]$, so we conclude by showing that $[\varepsilon] = 0$. Set $\varepsilon' = \varepsilon \oplus 1_{A_1} \in Aut_{\underline{\underline{C}}}(A_1 \oplus A_1) \simeq GL_2(R)$, $R = End_{\underline{\underline{C}}}(A_1)$. If we exchange the two direct summands A_2 of $A_1 = A_2 \oplus A_o$ in $A_1 \oplus A_1$ we see that ε′ corresponds to an element of $GL_2(R)$ which is conjugate to one of the form $\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$. Passing to $\varepsilon \oplus 1_{A_1} \oplus 1_{A_1}$, and $GL_3(R)$, respectively, the elementary matrix above lands in $E_3(R) \subset [GL_3(R), GL_3(R)]$, the commutator subgroup (see (V, 1.5)). Thus $[\varepsilon] = [\varepsilon \oplus 1_{A_1} \oplus 1_{A_1}] = 0$, as claimed.

To see that $K_o'(F, \oplus) \longrightarrow K_o'(F)$ is an isomorphism we must show that if

$$(A, \alpha, B) = (0 \longrightarrow (A_2, \alpha_2, B_2) \longrightarrow (A_1, \alpha_1, B_1) \longrightarrow (A_o, \alpha_o, B_o) \longrightarrow 0) \in Ex(co(F))$$

then $[\alpha_1] = [\alpha_2] + [\alpha_o]$ in $K_o'(F, \oplus)$. As above, since A and

B split we can identify $A_1 = A_2 \oplus A_o$ and $B_1 = B_2 \oplus B_o$ and obtain a matrix representation

$$\alpha_1 = \begin{pmatrix} \alpha_2 & * \\ 0 & \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{pmatrix} \begin{pmatrix} 1_{FA_2} & *' \\ 0 & 1_{FA_1} \end{pmatrix}.$$

Then again we have $\alpha_1 = (\alpha_2 \oplus \alpha_o)\varepsilon$ and we seek to show that $[\varepsilon] = 0$ in $K_o'(F, \oplus)$. But $[\varepsilon] = \partial\, [F(A_2 \oplus A_o), \varepsilon]_{(\underline{\underline{C}}', \oplus)}$, and since $\underline{\underline{C}}'$ is semi-simple - we proved this above - it follows as in the last paragraph that $[F(A_1 \oplus A_o), \varepsilon]_{(\underline{\underline{C}}', \oplus)} = 0$ in $K_1(\underline{\underline{C}}', \oplus)$.

Similarly, the semi-simplicity of $\underline{\underline{C}}'$ implies that $K_i(\underline{\underline{C}}', \oplus) \longrightarrow K_i(\underline{\underline{C}}')$ is an isomorphism ($i = 0, 1$) so we have proved that all verticals in diagram (1) are isomorphisms. Since the top row is exact by (VII, 5.3), so also is the bottom. This completes the proof of part (a).

(b) We assume now only that $\underline{\underline{C}}'$ is semi-simple. Then, by virtue of part (a), the diagram (2) takes the form

$$
\begin{array}{ccccccccc}
N_1 & \longrightarrow & 0 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(\underline{\underline{C}}, \oplus) & \longrightarrow & K_1(\underline{\underline{C}}', \oplus) & \longrightarrow & K_o'(F, \oplus) & \xrightarrow{d} & K_o(\underline{\underline{C}}, \oplus) & \longrightarrow & K_o(\underline{\underline{C}}', \oplus) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(\underline{\underline{C}}) & \longrightarrow & K_1(\underline{\underline{C}}') & \longrightarrow & K_o'(F) & \longrightarrow & K_o(\underline{\underline{C}}) & \longrightarrow & K_o(\underline{\underline{C}}')
\end{array}
$$

We view the rows as complexes and write H(X) for the homology at X of the row in which X occurs. Then the long homology sequence, and the exactness of the middle row in its three middle positions shows that $H(K_o(\underline{\underline{C}})) = 0$ and that there is an exact sequence

$$0 \longrightarrow H(K_1(\underline{\underline{C}}')) \longrightarrow N_3 \longrightarrow N_4 \longrightarrow H(K_o{}'(F)) \longrightarrow 0.$$

Therefore (b) will be proved if we show that $N_3 \longrightarrow N_4$ is surjective. Suppose $\langle A\rangle_{\underline{\underline{C}}} = [A_1]_{(\underline{\underline{C}}, \oplus)} - [A_o]_{(\underline{\underline{C}}, \oplus)} - [A_2]_{(\underline{\underline{C}}, \oplus)}$ is one of the generators of N_4, where $A = (0 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_o \longrightarrow 0) \in Ex(\underline{\underline{C}})$. Let $B = (0 \longrightarrow A_2 \longrightarrow A_2 \oplus A_1 \longrightarrow A_o \longrightarrow 0)$ be the split sequence. Since F is exact and $\underline{\underline{C}}'$ is semi-simple, FA splits, so there is an isomorphism of the form

$$\alpha = (1_{FA_2}, \alpha_1, 1_{FA_o}): FA \longrightarrow FB.$$

Then $(A, \alpha, B) \in Ex(co(F))$, so it determines an element $\langle A, \alpha, B\rangle \in N_3$ such that $d\langle A, \alpha, B\rangle = \langle A\rangle_{\underline{\underline{C}}} - \langle B\rangle_{\underline{\underline{C}}}$. Since B is split $\langle B\rangle_{\underline{\underline{C}}} = 0$, so this concludes the proof.

The assumptions of semi-simplicity in the above theorem are quite restrictive. In the following sections we shall give "reduction criteria" for computing $K_i(\underline{\underline{C}})$ from a subcategory $\underline{\underline{C}}_o \subset \underline{\underline{C}}$. In practice we can often find such a $\underline{\underline{C}}_o$ which is semi-simple.

§3. REDUCTION BY "DEVISSAGE"

Let $\underline{\underline{C}}_o \subset \underline{\underline{C}}$ be admissible subcategories of an abelian category. The inclusion is exact so it induces homomorphisms

$$K_i(\underline{\underline{C}}_o) \longrightarrow K_i(\underline{\underline{C}}) \qquad (i = 0, 1).$$

In this and the following sections we shall give criteria for these to be isomorphisms. The criterion here is, roughly, that every object of $\underline{\underline{C}}$ have a "nice composition series" with factors in $\underline{\underline{C}}_o$. More precisely:

(3.1) DEFINITIONS. A $\underline{\underline{C}}_o$-filtration of an object A in $\underline{\underline{C}}$ is a finite filtration of the form $0 = A_o \subset A_1 \subset \ldots \subset A_n = A$ such that each $A_i/A_{i-1} \in \underline{\underline{C}}_o$, $(1 \leq i \leq n)$. We say that it is stable under $\alpha \in \mathrm{Aut}_{\underline{\underline{C}}}(A)$ if $\alpha A_i = A_i$ $(0 \leq i \leq n)$, and we call the filtration characteristic if it is stable under all such α. We call $\alpha \in \mathrm{Aut}_{\underline{\underline{C}}}(A)$ $\underline{\underline{C}}_o$-unipotent if there is a $\underline{\underline{C}}_o$-filtration as above such that $(1_A - \alpha)A_i \subset A_{i-1}$ $(1 \leq i \leq n)$. This means that the filtration is stable under α and that α induces the identity on each A_i/A_{i-1}. This clearly implies that α is unipotent, i.e. that $1_A - \alpha$ is nilpotent.

(3.2) PROPOSITION. Let A be an object of $\underline{\underline{C}}$ and let $\alpha \in \mathrm{Aut}_{\underline{\underline{C}}}(A)$.

(0) If $0 = A_o \subset A_1 \subset \ldots \subset A_n = A$ is a finite $\underline{\underline{C}}$-filtration then each $A_i \in \underline{\underline{C}}$ and $[A] = \Sigma[A_i/A_{i-1}]$ $(1 \leq i \leq n)$ in $K_o(\underline{\underline{C}})$.

(1) If α is $\underline{\underline{C}}$-unipotent then α is unipotent. The converse is true if $\underline{\underline{C}}$ is abelian. If α is $\underline{\underline{C}}$-unipotent then $[A, \alpha] = 0$ in $K_1(\underline{\underline{C}})$.

Proof. (0) We argue by induction on n, the case n = 1 being trivial. If n > 1 the sequence $0 \longrightarrow A_{n-1} \longrightarrow A \longrightarrow A_n/A_{n-1} \longrightarrow 0$ shows that $A_{n-1} \in \underline{\underline{C}}$ (condition (d) of (1.1)). Therefore, using the induction hypothesis, we have $[A] = [A_{n-1}] + [A_n/A_{n-1}] = \Sigma[A_i/A_{i-1}]$ $(1 \leq i \leq n)$.

(1) The first implication was noted above. Conversely, suppose $f = 1_A - \alpha$ is nilpotent, say $f^n = 0$. Let $A_i = \mathrm{Im}(f^{n-i})$, $0 \leq i \leq n$. This is a $\underline{\underline{C}}$-filtration if $\underline{\underline{C}}$ is abelian, and it then exhibits α as a $\underline{\underline{C}}$-unipotent automorphism.

If α is $\underline{\underline{C}}$-unipotent choose a $\underline{\underline{C}}$-filtration as above so that $(\alpha - 1_A)A_i \subset A_{n-1}$ $(1 \leq i \leq n)$. Then α induces 1_{B_i} on $B_i = A_i/A_{i-1}$, and it follows from part (0) (applied to $\Sigma\underline{\underline{C}}$) that $[A, \alpha) = \Sigma[B_i, 1_{B_i}]$ $(1 \leq i \leq n)$, so $[A, \alpha] = 0$.

(3.3) THEOREM. *Let $\underline{\underline{C}}_o \subset \underline{\underline{C}}$ be admissible subcategories of an abelian category and assume that $\underline{\underline{C}}_o$ is abelian.*

(0) *If every $A \in \underline{\underline{C}}$ has a $\underline{\underline{C}}_o$-filtration then $K_o(\underline{\underline{C}}_o) \xrightarrow{j_o} K_o(\underline{\underline{C}})$ is an isomorphism.*

(1) *If every $A \in \underline{\underline{C}}$ has a characteristic $\underline{\underline{C}}_o$-filtration then $K_1\underline{\underline{C}}_o \xrightarrow{j_1} K_1\underline{\underline{C}}$ is an isomorphism.*

Proof. (0) Since $\underline{\underline{C}}_o$ is abelian it follows that a refinement of a $\underline{\underline{C}}_o$-filtration is again one. According to the Zassenhaus lemma (I, 4.2) any two finite filtrations have refinements such that the successive factors of the first refinement are, up to a permutation of the order of their occurrence, isomorphic to those of the second. This shows that if $0 = A_o \subset A_1 \subset \ldots \subset A_n = A$ is any $\underline{\underline{C}}_o$-filtration of $A \in \underline{\underline{C}}$, then $J(A) = \Sigma[A_i/A_{i-1}]_{C_o}$ $(1 \leq i \leq n)$ is well defined. For, by virtue of the above remarks, we need only see that $J(A)$ is unaltered if we replace the given filtration by a refinement. This amounts to introducing a filtration of each A_i/A_{i-1}, so what we desire follows from part (0) of (3.2) above (applied in $\underline{\underline{C}}_o$).

If $(0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0) \in Ex(\underline{\underline{C}})$ we can make a $\underline{\underline{C}}_o$-filtration for A by starting at the bottom with one for A', and then continuing with the inverse image of one for A''. With such a choice we see clearly that $J(A) = J(A') + J(A'')$. It follows now that J induces a homomorphism $J_o: K_o(\underline{\underline{C}}) \longrightarrow K_o(\underline{\underline{C}}_o)$, and (3.2) (0) implies that $j_o \circ J_o$ is the identity. If $A \in \underline{\underline{C}}_o$ then the trivial $\underline{\underline{C}}_o$-filtration shows that $J_0 \circ j_0[A]_{\underline{\underline{C}}_0} = [A]_{\underline{\underline{C}}_0}$.

(1) The hypothesis of (1) implies, evidently, that every $(A, \alpha) \in \Sigma\underline{\underline{C}}$ has a $\Sigma\underline{\underline{C}}_0$-filtration. In the diagram

$$\begin{array}{ccc} K_0(\Sigma\underline{\underline{C}}_0) & \xrightarrow{\ i\ } & K_0(\Sigma\underline{\underline{C}}) \\ {\scriptstyle p_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ K_1(\underline{\underline{C}}_0) & \xrightarrow[j_1]{} & K_1(\underline{\underline{C}}) \end{array}$$

it follows from part (0) that there is an inverse I to i, defined as above, using $\Sigma\underline{\underline{C}}_0$-filtrations. If we show that $p_0 \circ I$ is multiplicative, i.e. that $p_0 \circ I\,[A, \alpha\beta]_{\Sigma\underline{\underline{C}}} = p_0 \circ I([A, \alpha]_{\underline{\underline{C}}} + [A, \beta]_{\Sigma\underline{\underline{C}}})$ for $\alpha, \beta \in \mathrm{Aut}_{\underline{\underline{C}}}(A)$, then it will follow that there is an induced $J_1: \bar{K}_1(\underline{\underline{C}}) \longrightarrow K_1(\underline{\underline{C}}_0)$ which will clearly be the required inverse to j_1.

Let $0 = A_0 \subset A_1 \subset \ldots \subset A_n = A$ be a characteristic $\underline{\underline{C}}_0$-filtration of A above. Then it is stable under α and β. Say they induce α_i and β_i, respectively, on $B_i = A_i/A_{i-1}$ $(1 \leq i \leq n)$. Then from (3.2) (0) applied in $K_0(\Sigma\underline{\underline{C}})$, and axiom K1 in $K_1(\underline{\underline{C}}_0)$, we have

$$\begin{aligned} p_0(I[A, \alpha\beta]) &= p_0(\Sigma[B_i, \alpha_i\beta_i]_{\Sigma\underline{\underline{C}}_0}) \\ &= \Sigma[B_i, \alpha_i\beta_i]_{\underline{\underline{C}}_0} = \Sigma([B_i, \alpha_i]_{\underline{\underline{C}}_0} \\ &\quad + [B_i, \beta_i]_{\underline{\underline{C}}_0}) \\ &= p_0(I([A, \alpha]_{\Sigma\underline{\underline{C}}} + [A, \beta]_{\Sigma\underline{\underline{C}}})). \end{aligned}$$

This completes the proof of Theorem (3.3).

(3.4) THEOREM. Let $\underline{\underline{A}}$ be an abelian category in which every object has finite length, and let $\underline{\underline{A}}_0$ be the full subcategory of semi-simple objects of $\underline{\underline{A}}$. Then

(a) The inclusion $\underline{\underline{A}}_o \subset \underline{\underline{A}}$ induces isomorphisms $K_i\underline{\underline{A}}_o \longrightarrow K_i\underline{\underline{A}}$ $(i = 0, 1)$.

(b) If $\{S_j \mid j \in J\}$ is a set of representatives of the isomorphism classes of simple objects in $\underline{\underline{A}}_o$ then $K_o\underline{\underline{A}}_o$ is a free abelian group with basis $\{[S_j] \mid j \in J\}$.

(c) Let $D_j = \mathrm{End}_{\underline{\underline{A}}}(S_j)$. Then D_j is a division ring, and $K_1\underline{\underline{A}}_o$ is the direct sum,

$$\coprod_{j \in J}(D_j^*/[D_j^*, D_j^*]),$$

of the commutator factor groups of the multiplicative groups D_j^*.

Remark. If we write $K_i(R) = K_i(\underline{\underline{P}}(R))$ for a ring R, a notation to be introduced in Chapter IX, then parts (b) and (c) above can be written, more suggestively, as

$$K_i(\underline{\underline{A}}_o) \simeq \coprod_{j \in J} K_i(D_j) \qquad (i = 0, 1).$$

(See (IV, §1) or (V, §2)).

Proof. If $A \in \underline{\underline{A}}$ write $s(A)$ for the largest semi-simple subobject of A. The chain condition on A plus the fact (cf. (III, 1.1)) that a finite sum of simple objects is semi-simple, shows the existence of $s(A)$. Moreover it is evidently a fully invariant subobject, and $\neq 0$ if $A \neq 0$ (look at the bottom of a Jordan-Holder series). By induction on n, now, we define $A_n \subset A$ by $A_o = 0$ and $A_{n+1}/A_n = s(A/A_n)$. The remarks above make it clear that $A_n = A$ for some n (depending on A) and that $f(A_n) \subset B_n$ for any $f: A \longrightarrow B$ in $\underline{\underline{A}}$. In particular, every object of $\underline{\underline{A}}$ has a characteristic $\underline{\underline{A}}_o$-filtration, so part (a) of the theorem follows from Theorem (3.3).

Since $\underline{\underline{A}}_o$ is semi-simple and each object A has finite length, it follows (cf. (III, §1)) that $A \simeq \coprod S_j^{n_j}$, with almost all $n_j = 0$, and $\mathrm{End}_{\underline{\underline{A}}_o}(A) = \prod M_{n_j}(D_j)$. Hence $\mathrm{Aut}_{\underline{\underline{A}}_o}(A)$

$= \prod GL_{n_j}(D_j)$. If we abelianize these groups and pass to the limit, as in (VII, §2), by enlarging A, then we find that

$$K_1(\underline{\underline{A}}_o) = \coprod_j GL(D_j)/[GL(D_j), GL(D_j)]$$
$$= \coprod_j K_1(D_j).$$

According to Dieudonné's Theorem (V, 9.5) the natural map $D_j^*/[D_j^*, D_j^*] \longrightarrow K_1(D_j)$ is an isomorphism. (If $d \in D_j$ the image of d corresponds to $[S_j, d] \in K_1(\underline{\underline{A}}_o)$). q.e.d.

(3.5) COROLLARY. Let A be a commutative ring and let $\underline{\underline{A}}$ be the category of A-modules of finite length. Then

$$K_i(\underline{\underline{A}}) \simeq \coprod_{\underline{m} \in \max(A)} K_i(A/\underline{m}) \qquad (i = 0, 1).$$

Here $K_o(A/\underline{m}) \simeq \underline{\underline{Z}}$ and $K_1(A/\underline{m}) \simeq U(A/\underline{m})$ for each $\underline{m} \in \max(A)$.

The notational convention here is that $K_i(B) = K_i(\underline{\underline{P}}(B))$ for a ring B.

§4. REDUCTION BY RESOLUTION

Let $\underline{\underline{C}}_o \subset \underline{\underline{C}}$ be admissible subcategories of an abelian category $\underline{\underline{A}}$. The aim of this section is to show that if objects in $\underline{\underline{C}}$ have "nice" resolutions by objects in $\underline{\underline{C}}_o$ then $K_i(\underline{\underline{C}}_o) \longrightarrow K_i(\underline{\underline{C}})$ (i = 0, 1) are isomorphisms.

Let $0 \longrightarrow A_n \longrightarrow \dots \longrightarrow A_1 \xrightarrow{d} A_o \longrightarrow 0$ be an exact sequence in $\underline{\underline{A}}$ such that $A_i \in \underline{\underline{C}}$ for $0 \leq i < n$. Then $A_n \in \underline{\underline{C}}$ also. For n = 1 this is trivial and for n = 2 it is condition (d) in the definition (1.1) of admissible subcategory. The general case follows by applying induction to $0 \longrightarrow A_n \longrightarrow \dots \longrightarrow A_2 \longrightarrow \ker d \longrightarrow 0$.

If $C = (C_n)_{n \in \underline{\underline{Z}}}$ is a finite graded object (i.e. $C_n = 0$ for almost all n) in $\underline{\underline{C}}$ we shall write

$$\chi(C) = \chi^{\underline{\underline{C}}}(C) = \Sigma(-1)^n\ [C_n] \in K_o(\underline{\underline{C}}).$$

(4.1) PROPOSITION ("Euler Characteristics").

(a) _If_ $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ _is an exact sequence of finite graded objects in_ $\underline{\underline{C}}$ _then_ $\chi(C) = \chi(C') + \chi(C'')$.

(b) _If_ C _is a finite complex in_ $\underline{\underline{C}}$ _such that_ H(C) _is also in_ $\underline{\underline{C}}$ _then_ $\chi(C) = \chi(H(C))$.

(c) _If_ $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ _is an exact sequence of complexes each of whose homologies is finite and in_ $\underline{\underline{C}}$ _then_ $\chi(H(C)) = \chi(H(C')) + \chi(H(C''))$.

(d) _Let_ $f: C' \longrightarrow C$ _be a morphism of finite complexes in_ $\underline{\underline{C}}$, _with mapping cone_ MC(f). _Then_ MC(f) _is a finite complex in_ $\underline{\underline{C}}$ _and_ $\chi(MC(f)) = \chi(C) - \chi(C')$. _If_ H(f) _is an isomorphism then_ $\chi(C') = \chi(C)$.

Proof. (a) is trivial.

(b) Consider the exact sequences

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n \longrightarrow 0.$$

Suppose $B_{n-1} \in \underline{\underline{C}}$. Then $Z_n \in \underline{\underline{C}}$ also, thanks to the first sequence. We have assumed $H_n \in \underline{\underline{C}}$ so the second sequence implies further that $B_n \in \underline{\underline{C}}$. Now we can continue the same reasoning. Since $B_{n-1} = 0$ for all sufficiently small n we can start with such an n and the argument above shows that $B_n, C_n, Z_n, H_n \in \underline{\underline{C}}$ for all n. Applying (a) to the exact sequences above we find that $\chi(C) = \chi(Z) - \chi(B)$ and $\chi(Z) = \chi(H) + \chi(B)$. Hence $\chi(C) = \chi(H)$, thus proving (b).

(c) Let L denote the long homology sequence of $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$, graded, say, so that $L_0 = H_0(C'')$. Then L is a finite acyclic complex in $\underline{\underline{C}}$ so part (b) implies $\chi(L) = \chi(H(L)) = 0$. Since, clearly, $\chi(L) = \chi(H(C'')) - \chi(H(C)) + \chi(H(C'))$, this proves (c).

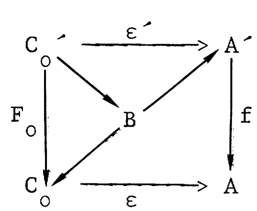
(d) Since $MC(f)_{n+1} = C_{n+1} \oplus C_n'$ we see that MC(f) is a finite complex in $\underline{\underline{C}}$ and $\chi(MC(f)) = \chi(C) - \chi(C')$. If H(f) is an isomorphism then, according to (I, 5.4), MC(f) is acyclic. Hence part (b) implies $\chi(MC(f)) = 0$, and we conclude from that $\chi(C) = \chi(C')$.

(4.2) THEOREM (Grothendieck). *Let $\underline{\underline{C}}_0 \subset \underline{\underline{C}}$ be admissible subcategories of an abelian category such that every object of $\underline{\underline{C}}$ has a finite $\underline{\underline{C}}_0$-resolution. Then the inclusion induces an isomorphism* $K_0(\underline{\underline{C}}_0) \longrightarrow K_0(\underline{\underline{C}})$.

We begin the proof with a lemma.

(4.3) LEMMA. *Given a morphism* $f\colon A' \longrightarrow A$ *in* $\underline{\underline{C}}$, *and a finite* $\underline{\underline{C}}_0$*-resolution* $C \xrightarrow{\varepsilon} A$, *we can find a finite* $\underline{\underline{C}}_0$*-resolution* $C' \xrightarrow{\varepsilon'} A'$ *and a morphism* $F\colon C' \longrightarrow C$ *covering* f.

Proof. Let $B = \mathrm{Ker}(C_0 \oplus A' \xrightarrow{(\varepsilon, -f)} A)$ be the fibre product of $C_0 \xrightarrow{\varepsilon} A \xleftarrow{f} A'$, and choose an epimorphism $C_0' \longrightarrow B$ with $C_0' \in \underline{\underline{C}}_0$. Define ε' and F_0 to make the diagram



commutative. Since ε is surjective it follows that ε' is also. Suppose now that we have constructed a commutative diagram

$$\begin{array}{ccccccccccccc}
 & & C'_{n-1} & \xrightarrow{d'_{n-1}} & .. & \longrightarrow & C'_o & \xrightarrow{\varepsilon'} & A' & \longrightarrow & 0 \\
 & & \Big\downarrow F_{n-1} & & & & \Big\downarrow F_o & & \Big\downarrow f & & \\
.. \; C_n & \xrightarrow{d_n} & C_{n-1} & \xrightarrow[d_{n-1}]{} & .. & \longrightarrow & C_o & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

with exact rows and with each $C_i' \in \underline{\underline{C}}_o$. Then, as we observed at the beginning of this section, Z_{n-1} and Z'_{n-1} (= $\mathrm{Ker}(d'_{n-1})$) are in $\underline{\underline{C}}$. Therefore we can apply the construction above to find a commutative diagram

$$\begin{array}{ccccc}
C'_n & \xrightarrow{"d_n'"} & Z'_{n-1} & \longrightarrow & 0 \\
F_n \Big\downarrow & & \Big\downarrow "F_{n-1}" & & \\
C_n & \xrightarrow["d_n"]{} & Z_{n-1} & \longrightarrow & 0
\end{array}$$

with exact rows. Eventually we reach a point where $C_n = 0$, at which time we complete C' with a finite $\underline{\underline{C}}_o$-resoltuion of Z'_{n-1}. q.e.d.

<u>Proof of Theorem</u> (4.2). Suppose $C \longrightarrow A$ and $C' \longrightarrow A$ are two finite $\underline{\underline{C}}_o$-resolutions of $A \in \underline{\underline{C}}$. Apply the lemma to the resolution $C \oplus C' \longrightarrow A \oplus A$ and the diagonal map $A \longrightarrow A \oplus A$. The result is a finite $\underline{\underline{C}}_o$-resolution $C'' \longrightarrow A$ and morphisms $C'' \longrightarrow C$ and $C'' \longrightarrow C'$ both covering 1_A. In other words we have 1_A as induced homology map $A = H(C'') \longrightarrow H(C) = A$, and similarly for $C'' \longrightarrow C'$. Therefore (4.1) (d) implies that $\chi^{\underline{\underline{C}}_o}(C) = \chi^{\underline{\underline{C}}_o}(C'') = \chi^{\underline{\underline{C}}_o}(C')$. This shows that $A \longmapsto \chi^{\underline{\underline{C}}_o}(C)$, where $C \longrightarrow A$ is a finite $\underline{\underline{C}}_o$-resolution, is a well defined map, $r\colon \mathrm{ob}\underline{\underline{C}} \longrightarrow K_o(\underline{\underline{C}}_o)$.

Let $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$ be an exact sequence in $\underline{\underline{C}}$. Let $C \longrightarrow A$ be a finite $\underline{\underline{C}}_0$-resolution, and use the lemma to fill in a commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{\ I\ } & C \\ \downarrow & & \downarrow \\ A' & \xrightarrow{\ i\ } & A \end{array}$$

where $C' \longrightarrow A'$ is a finite $\underline{\underline{C}}_0$-resolution. Consider the homology sequence (I, 5.4),

$$\ldots H_1(C) \longrightarrow H_1(MC(I)) \longrightarrow H_0(C') \longrightarrow H_0(C) \longrightarrow H_0(MC(I)) \longrightarrow \ldots$$

Since C and C' are resolutions we have $H_n(C) = 0 = H_n(C')$ for $n \neq 0$, and $(H_0(C') \longrightarrow H_0(C)) = (A' \xrightarrow{i} A)$, a monomorphism with cokernel A''. Since $MC(I)$ is a finite positive complex in $\underline{\underline{C}}_0$ the homology sequence shows that $MC(I)$ is a finite $\underline{\underline{C}}_0$-resolution of Coker $(i) = A''$. Therefore

$$rA'' = \chi^{\underline{\underline{C}}}_{\underline{\underline{C}}_0}(MC(I)) = \chi^{\underline{\underline{C}}}_{\underline{\underline{C}}_0}(C) - \chi^{\underline{\underline{C}}}_{\underline{\underline{C}}_0}(C') \qquad ((4.1)\ (d))$$

$$= rA - rA'.$$

It follows that r induces a homomorphism $r\colon K_0(\underline{\underline{C}}) \longrightarrow K_0(\underline{\underline{C}}_0)$. If $A \in \underline{\underline{C}}_0$ then the complex with only A in degree zero is a finite $\underline{\underline{C}}_0$-resolution, so $r[A]_{\underline{\underline{C}}} = [A]_{\underline{\underline{C}}_0}$. On the other hand, if $A \in \underline{\underline{C}}$ and if C is a finite $\underline{\underline{C}}_0$-resolution, then the finite acyclic complex $\ldots C_n \longrightarrow \ldots \longrightarrow C_0 \longrightarrow A \longrightarrow 0 \ldots$ shows, using (4.1) (b), that $\chi^{\underline{\underline{C}}}(C) = [A]_{\underline{\underline{C}}}$. Since $r[A]_{\underline{\underline{C}}} = \chi^{\underline{\underline{C}}}_{\underline{\underline{C}}_0}(C)$, the map $K_0(\underline{\underline{C}}_0) \longrightarrow K_0(\underline{\underline{C}})$ sends $r[A]_{\underline{\underline{C}}}$ to $\chi^{\underline{\underline{C}}}(C) = [A]_{\underline{\underline{C}}}$. This shows that r is an inverse for $K_0(\underline{\underline{C}}_0) \longrightarrow K_0(\underline{\underline{C}})$ and hence proves the theorem.

(4.4) THEOREM. _Let_ $\underline{\underline{P}} \subset \underline{\underline{C}}_0 \subset \underline{\underline{C}}$ _be admissible subcategories of an abelian category, let_ F _be an exact admissible functor on_ $\underline{\underline{C}}$, _and let_ F_0 _be its restriction to_ $\underline{\underline{C}}_0$. _Assume,_

<u>for each</u> $A \in \underline{\underline{C}}$, <u>that</u>:

(1) <u>If</u> $\alpha, \beta \in \mathrm{Aut}_{\underline{\underline{C}}}(A, F)$ (so $F\alpha = 1_{FA} = F\beta$) <u>then there is an epimorphism</u> $C_0 \longrightarrow A$ <u>with</u> $C_0 \in \underline{\underline{P}}$ <u>such that</u> α <u>and</u> β <u>lift to automorphisms in</u> $\mathrm{Aut}_{\underline{\underline{C}}_0}(C_0, F_0)$; <u>and</u>

(2) <u>If</u> $C \longrightarrow A$ <u>is a</u> $\underline{\underline{P}}$<u>-resolution then</u> $Z_n(C) \in \underline{\underline{C}}_0$ <u>for some</u> $n > 0$.

<u>Then the inclusion</u> $\underline{\underline{C}}_0 \subset \underline{\underline{C}}$ induces an isomorphism $K_1(\underline{\underline{C}}_0, F_0) \longrightarrow K_1(\underline{\underline{C}}, F)$.

<u>Proof</u>. Suppose $(A, \alpha) \in \mathrm{Ker}\, \Sigma F$. Recall that $\mathrm{Ker}\, \Sigma F$ is the full subcategory of $\Sigma\underline{\underline{C}}$ consisting of objects (A, α) such that $F\alpha = 1_{FA}$. According to (1) above we can find an exact sequence in $\Sigma\underline{\underline{A}}$,

$$0 \longrightarrow (Z_0, \beta) \longrightarrow (C_0, \gamma_0) \longrightarrow (A, \alpha) \longrightarrow 0,$$

with $C_0 \in \underline{\underline{P}}$ and $F\gamma_0 = 1_{FC_0}$, i.e. with $(C_0, \gamma_0) \in \mathrm{Ker}\, \Sigma F_0$. Since $\underline{\underline{C}}$ is admissible $Z_0 \in \underline{\underline{C}}$. Since F is exact and $F\gamma_0 = 1_{FC_0}$, we have $F\beta = F\gamma_0 \mid FZ_0 = 1_{FZ_0}$. Therefore $(Z_0, \beta) \in \mathrm{Ker}\, \Sigma F$. Hence we can iterate this construction and produce a $\mathrm{Ker}(\Sigma F_0)$-resolution, $(C, \gamma) \longrightarrow (A, \alpha)$, with $C \longrightarrow A$ a $\underline{\underline{P}}$-resolution. It need not be finite, but condition (2) above implies that we can truncate it at some stage, if necessary, to replace it by a finite one. The full strength of (1) implies that if we are given $\alpha, \alpha' \in \mathrm{Aut}_{\underline{\underline{C}}}(A, F)$ then we can, by the above procedure, find finite $\mathrm{Ker}(\Sigma F_0)$-resolutions $(C, \gamma) \longrightarrow (A, \alpha)$ and $(C, \gamma') \longrightarrow (A, \alpha')$ using the same complex C in each case.

If $(C, \gamma) \longrightarrow (A, \alpha)$ is a resolution as above set $r(A, \alpha) = \chi(C, \gamma) = \Sigma(-1)^n [C_n, \gamma_n] \in K_1(\underline{\underline{C}}_0, F_0)$. It follows from the proof of theorem (4.2) that r is additive over exact sequences. If $(C, \gamma') \longrightarrow (A, \alpha')$ is a resolution of (A, α') as above then $(C, \gamma\gamma') \longrightarrow (A, \alpha\alpha')$ is a finite

Ker ΣF_o-resolution, clearly, so $r(A, \alpha\alpha') = \chi(C, \gamma\gamma') = \Sigma(-1)^n [C_n, \gamma_n\gamma_n'] = \Sigma(-1)^n ([C_n, \gamma_n] + [C_n, \gamma_n']) = r(A, \alpha) + r(A, \alpha')$. Hence r induces a homomorphism $K_1(\underline{\underline{C}}, F) \longrightarrow K_1(\underline{\underline{C}}_o, F_o)$. Just as in the proof of Theorem 4.2 we see that this gives the required inverse for $K_1(\underline{\underline{C}}_o, F_o) \longrightarrow K_1(\underline{\underline{C}}, F)$. q.e.d.

We shall now indicate certain circumstances in which the hypothesis (1) of Theorem (4.4) can be achieved.

(4.5) PROPOSITION. Let $\underline{\underline{A}}$ be an abelian category.

(a) Let F be an additive functor on $\underline{\underline{A}}$ and let $f: P \longrightarrow A$ be an epimorphism in $\underline{\underline{A}}$ such that every $h \in End_{\underline{\underline{A}}}(A)$ for which $Fh = 0$ lifts to an $h' \in End_{\underline{\underline{A}}}(P)$ such that $Fh' = 0$. (This is automatic, for example, if P is projective and F = 0). Let $(f, 0): Q = P \oplus P \longrightarrow A$. Then every $\alpha \in Aut_{\underline{\underline{A}}}(A)$ such that $F\alpha = 1_{FA}$ lifts to an $\alpha' \in Aut_{\underline{\underline{A}}}(Q)$ such that $F\alpha' = 1_{FQ}$.

(b) Let $f_1: P_1 \longrightarrow A_1$ and $f_2: P_2 \longrightarrow A_2$ be epimorphisms in $\underline{\underline{A}}$ with $Q = P_1 \oplus P_2$ projective. Let $(f_1, 0): Q \longrightarrow A_1$ and $(0, f_2): Q \longrightarrow A_2$. Then any isomorphism $\alpha: A_1 \longrightarrow A_2$ lifts to an isomorphism $\alpha': Q \longrightarrow Q$.

Proof. (a) In $Aut_{\underline{\underline{A}}}(A \oplus A) = GL_2(End_{\underline{\underline{A}}}(A))$ we have the following formulas:

$$(1) \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \alpha^{-1}-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1-\alpha \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1-\alpha^{-1} \\ 0 & 1 \end{pmatrix};$$

and

$$(2)\qquad \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

Since $F\alpha = 1_{FA}$ we can, by hypothesis, lift $1 - \alpha$ and $1 - \alpha^{-1}$ to endomorphisms of P killed by F. Therefore, under $f \oplus f$: $Q \longrightarrow A \oplus A$ we can lift $\alpha \oplus \alpha^{-1}$ to an automorphism α' represented in $GL_2(\mathrm{End}_{\underline{\underline{A}}}(\underline{p}))$ in the form $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ h_1 & 1\end{pmatrix}$ $\begin{pmatrix}1 & h_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & -h_1\\ 0 & 1\end{pmatrix}$, where $Fh_1 = 0 = Fh_2$. Evidently then $F\alpha' = 1_{FQ}$. Under the composite, $Q \xrightarrow{f \oplus f} A \oplus A \xrightarrow{(1,\ 0)} A$, α lifts first to $\alpha \oplus \alpha^{-1}$, clearly, and from there to α', as required.

(b) Using the same device as above it suffices to find an isomorphism α' making the diagram

$$\begin{array}{ccc}
P_1 \oplus P_2 & \xrightarrow{\alpha'} & P_1 \oplus P_2 \\
{\scriptstyle f_1 \oplus f_2}\downarrow & & \downarrow{\scriptstyle f_1 \oplus f_2} \\
A_1 \oplus A_2 & \xrightarrow{\begin{pmatrix}0 & -\alpha^{-1}\\ \alpha & 0\end{pmatrix}} & A_1 \oplus A_2 \\
{\scriptstyle (1,\ 0)}\downarrow & & \downarrow{\scriptstyle (0,\ 1)} \\
A_1 & \xrightarrow{\alpha} & A_2
\end{array}$$

commute, since the bottom rectangle certainly commutes. Using the isomorphism $\alpha \oplus 1_{A_2} : A_1 \oplus A_2 \longrightarrow A_2 \oplus A_2$ the top half of the diagram is seen to be isomorphic to

$$\begin{array}{ccc} P_1 \oplus P_2 & & P_1 \oplus P_2 \\ \alpha f_1 \oplus f_2 \Big\downarrow & & \Big\downarrow \alpha f_1 \oplus f_2 \\ A_2 \oplus A_2 & \xrightarrow{\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} & A_2 \oplus A_2 \end{array} .$$

Since the P_i are projective we can lift 1_{A_2} to homomorphisms h_1 and h_2 each making

$$\begin{array}{ccc} P_1 & \overset{h_1}{\underset{h_2}{\rightleftarrows}} & P_2 \\ {\scriptstyle \alpha f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow{\;1\;} & A_2 \end{array}$$

commute. Then, using formula (2), above we can lift

$$\begin{pmatrix} 0 & -1_{A_2} \\ 1_{A_2} & 0 \end{pmatrix} \text{ to } \begin{pmatrix} 1_{P_1} & 0 \\ h_1 & 1_{P_2} \end{pmatrix}\begin{pmatrix} 1_{P_1} & -h_2 \\ 0 & 1_{P_2} \end{pmatrix}\begin{pmatrix} 1_{P_1} & 0 \\ h_1 & 1_{P_2} \end{pmatrix}.$$ This shows that we can lift $\begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix}$ also to an isomorphism and hence completes the proof of (b).

(4.6) THEOREM. *Let* $\mathbf{C}_o \subset \mathbf{C}$ *be admissible subcategories of an abelian category. Assume, for each* $A \in \mathbf{C}$, *that:*

(1) *There is an epimorphism* $f: P_o \longrightarrow A$ *with* P_o *a projective object in* $\mathbf{C}_o$; *and*

(2) *If* $P \longrightarrow A$ *is a resolution of* A *by projective objects in* $\mathbf{C}_o$ *then* $Z_n(P) \in \mathbf{C}_o$ *for some* $n > 0$. *Then the inclusion* $\mathbf{C}_o \subset \mathbf{C}$ *induces isomorphisms*, $K_i(\mathbf{C}_o) \longrightarrow K_i(\mathbf{C})$ $(i = 0, 1)$.

More generally, let F *be an admissible functor on* $\mathbf{C}$ *with restriction* F_o *to* $\mathbf{C}_o$. *Suppose in* (1) *above that* f *can be chosen so that any* $h \in \mathrm{End}_{\mathbf{C}}(A)$ *such that* $Fh = 0$ *lifts to an* $h' \in \mathrm{End}_{\mathbf{C}}(p)$ *such that* $Fh' = 0$. *Then*

$$K_1(\mathbf{C}_o, F_o) \longrightarrow K_1(\mathbf{C}, F)$$

is likewise an isomorphism.

Proof. Conditions (1) and (2) clearly imply that every $A \in \underline{\underline{C}}$ has a finite $\underline{\underline{C}}_0$-resolution, so Theorem (4.2) implies $K_0(\underline{\underline{C}}_0) \longrightarrow K_0(\underline{\underline{C}})$ is an isomorphism. To prove that $K_1(\underline{\underline{C}}_0, F_0) \longrightarrow K_1(\underline{\underline{C}}, F)$ is an isomorphism we need only verify the hypotheses of Theorem (4.4), taking $\underline{\underline{P}}$ there to be the full subcategory of projective objects in $\underline{\underline{C}}_0$. Then conditions (2) of Theorem (4.4) and of the present theorem are identical. Condition (1) of Theorem (4.4) is an immediate consequence of our hypotheses above together with part (a) of Proposition (4.5). The isomorphism $K_1(\underline{\underline{C}}_0) \longrightarrow K_1(\underline{\underline{C}})$ corresponds to the case when F is the zero functor, in which case the extra hypothesis on lifting endomorphisms killed by F is automatic. q.e.d.

We close with a similar result on the relative groups.

(4.7) PROPOSITION. *Let*

$$\begin{array}{ccc} \underline{\underline{C}} & \xrightarrow{F} & \underline{\underline{C}}' \\ \cup & & \cup \\ \underline{\underline{C}}_0 & \xrightarrow{F_0} & \underline{\underline{C}}_0' \end{array}$$

be a commutative diagram of exact admissible functors between admissible subcategories of abelian categories. Assume that we have conditions (1) *and* (2) *of Theorem* (4.6) *for* $\underline{\underline{C}}_0 \subset \underline{\underline{C}}$, *and that* F *carries projective objects of* $\underline{\underline{C}}_0$ *to projective objects of* $\underline{\underline{C}}_0'$. *Then the inclusion* $co(F_0) \subset co(F)$ *induces an isomorphism*

$$K_0'(F_0) \longrightarrow K_0'(F).$$

Proof. Suppose $(A_1, \alpha_1, A_2) \in co(F)$. Choose epimorphisms $f_i : P_i \longrightarrow A_i$ with P_i projective and in $\underline{\underline{C}}_0$ $(i = 1, 2)$. Set $Q = P_1 \oplus P_2$ and $(f_1, 0): Q \longrightarrow A_1$, $(0, f_2): Q \longrightarrow A_2$. Since F is exact and preserves projectives it follows that FP_i is projective and Ff_i is surjective $(i = 1, 2)$. Now it follows from part (b) of Proposition (4.5) that there is an isomorphism $\alpha_1{}'$ making

$$\begin{array}{ccc} FQ & \xrightarrow{\alpha_1{}'} & FQ \\ {\scriptstyle F(f_1,\ 0)}\downarrow & & \downarrow{\scriptstyle F(0,\ f_2)} \\ FA_1 & \xrightarrow[\alpha_1]{} & FA_2 \end{array}$$

commute. Therefore we have constructed an epimorphism $(Q, \alpha_1{}', Q) \longrightarrow (A_1, \alpha_1, A_2)$ with Q projective and in $\underline{\underline{C}}_o$. Do this again for the kernel, etc., and we obtain a resolution $(C, \gamma_1, C) \longrightarrow (A_1, \alpha_1, A_2)$ where C is a complex of projective objects in $\underline{\underline{C}}_o$. Condition (2) of Theorem (4.6) permits us to truncate this, if necessary, to obtain a finite $co(F_o)$-resolution of (A_1, α_1, A_2). Suppose we are also given $(A_2, \alpha_2, A_3) \in co(F)$, and let $f_3\colon P_3 \longrightarrow A_3$ be an epimorphism. Then in the construction above replace Q by $Q' = Q \oplus P_3$ and define $Q' \longrightarrow A_i$ $(i = 1, 2, 3)$ by $(f_1, 0, 0)$, $(0, f_2, 0)$, and $(0, 0, f_2)$. We can lift α_1 to $\alpha_1'' = \alpha_1{}' \oplus 1_{FP_3}$. Similarly we can lift α_2 to an automorphism $\alpha_2{}'$ of $F(P_2 \oplus P_3)$, and we lift it to $\alpha_2'' = 1_{P_1} \oplus \alpha_2{}'$ on Q'. Then we have epimorphisms $(Q', \alpha_1'', Q') \longrightarrow (A_1, \alpha_1, A_2)$ and $(Q', \alpha_2'', Q') \longrightarrow (A_2, \alpha_2, A_3)$. If we similarly modify the procedure above we can finally produce resolutions $(C, \gamma_1, C) \longrightarrow (A_1, \alpha_1, A_2)$ and $(C, \gamma_2, C) \longrightarrow (A_2, \alpha_2, A_3)$. By truncating each one at the same point we can assume further that they are finite $co(F_o)$-resolutions. Then $(C, \gamma_2\gamma_1, C)$ will be a finite $co(F_o)$-resolution of $(A_1, \alpha_2\alpha_1, A_2)$.

To prove that $K_o{}'(F_o) \longrightarrow K_o{}'(F)$ is an isomorphism we construct an inverse by setting $r(A_1, \alpha_1, A_2) = \chi^{co(F_o)}(C, \gamma_1, C) \in K_o{}'(F_o)$, where (C, γ_1, C) is a finite $co(F_o)$-resolution. The proof of Theorem (4.2) shows that r is additive over exact sequences. Given (A_2, α_2, A_3) we construct the resolutions compatibly, as above, and then we have $r(A_1, \alpha_2\alpha_1, A_3) = \chi(C, \gamma_2\gamma_1, C) = \chi(C, \gamma_1, C) + \chi(C, \gamma_2, C) = r(A_1, \alpha_1, A_2) + r(A_2, \alpha_2, A_3)$. Therefore r does indeed induce a homomorphism on $K_o'(F)$ (see (VII, 5.1)) and it is easily seen to be the required inverse (cf. proof of Theorem (4.2)).

§5. THE EXACT SEQUENCE OF A LOCALIZING FUNCTOR

Let $\overline{S}$: $\underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ be an _exact_ functor between abelian categories, and let

$$\underline{\underline{S}} = \text{"Ker S"}$$

be the full subcategory of objects $A \in \underline{\underline{A}}$ such that $SA = 0$. Since S is exact it is evident that:

(*) If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact in $\underline{\underline{A}}$ then $A \in \underline{\underline{S}} \Longleftrightarrow A', A'' \in \underline{\underline{S}}$.

In general we shall call a full subcategory $\underline{\underline{S}} \subset \underline{\underline{A}}$ a _Serre subcategory_ if it satisfies (*). The above method for producing them (as "kernels" of exact functors) is in fact only one:

(5.1) THEOREM. _Let_ $\underline{\underline{S}}$ _be a Serre subcategory of an abelain category_ $\underline{\underline{A}}$. _Then there is an abelian category_ $\underline{\underline{A}}/\underline{\underline{S}}$ _and an exact "quotient" functor_ $\overline{S}$: $\underline{\underline{A}} \longrightarrow \underline{\underline{A}}/\underline{\underline{S}}$ _such that_ $\underline{\underline{S}}$ = "Ker $\overline{S}$", _and solving the following universal problem: Given an exact functor_ T: $\underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ _such that_ $TA = 0$ _for all_ $A \in \underline{\underline{S}}$, _there is a unique functor_ U: $\underline{\underline{A}}/\underline{\underline{S}} \longrightarrow \underline{\underline{B}}$ _such that_ $T = U \circ \overline{S}$. _Moreover_ U _is exact_.

We shall not prove this theorem here, referring the reader instead to Gabriel [1, Chapter III]. Our intention is to quote a number of properties of the quotient functor $\overline{S}$ of the theorem - in fact enough to indicate how $\underline{\underline{A}}/\underline{\underline{S}}$ is constructed - and then to use these properties to prove a theorem of Heller [1] asserting that

$$K_0'(\overline{S}) \simeq K_0(\underline{\underline{S}}).$$

This is a very useful fact, and it permits us easily to compute $K_0'(S)$ in many cases of interest.

Since the reader is to be burdened with several unproved propositions we shall mention two basic examples of the situation in Theorem (5.1) which he can bear in mind. The propositions below can be checked directly in these examples. The examples also explain how the quotient construction is related to localization.

(5.2) EXAMPLE. Let S be a multiplicative set in the commutative ring R, let A be an R-algebra, and let $\underline{\underline{S}} \subset$ mod-A be the Serre subcategory of modules M such that, given $x \in M$, $xs = 0$ for some $s \in S$. Let $\overline{S}$: mod-A $\longrightarrow$ (mod-A)/$\underline{\underline{S}}$ be the quotient functor. Since the localizing functor, S^{-1}: mod-A $\longrightarrow$ mod-($S^{-1}A$), kills $\underline{\underline{S}}$, there is a functor U: (mod-A)/$\underline{\underline{S}}$ $\longrightarrow$ mod-($S^{-1}A$) such that $S^{-1} = U \circ \overline{S}$. Now the point is that U _is an equivalence_, so that, up to equivalence, the localization functor S^{-1} is a quotient functor.

(5.3) EXAMPLE. Let A be a sheaf of rings on a topological space X, and let mod-A denote the category of sheaves of A-modules. Let U be an open set, with complement F, and let $\overline{S}$: mod-A $\longrightarrow$ mod-(A|U) be the restriction functor. Then, just as in the example above, $\overline{S}$ is equivalent to a quotient functor whose "kernel" has as objects the sheaves with support in F.

For the rest of this section now we fix a quotient functor

$$\overline{S}: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}' = \underline{\underline{A}}/\underline{\underline{S}}.$$

We write I for the set of morphisms f in $\underline{\underline{A}}$ such that $\overline{S}f$ is an isomorphism. If A, B $\in \underline{\underline{A}}$, and if $\overline{f}: \overline{S}A \longrightarrow \overline{S}B$ is a morphism in $\underline{\underline{A}}'$, we shall say that the diagram $A \xleftarrow{a} A' \xrightarrow{f} B' \xleftarrow{b} B$ in $\underline{\underline{A}}$ is a _representation of_ f, or _represents_ f, if $a, b \in I$ and if $\overline{f} = (\overline{S}b)^{-1} (\overline{S}f) (\overline{S}a)^{-1}$. The two basic facts about $\overline{S}$, other than those in Theorem (5.1), are:

(1) $\overline{S}$ is bijective on objects.

(2) Every morphism in $\underline{\underline{A}}'$ has a representation.

We shall now derive some further properties.

(3) Let $A \xleftarrow{a} A' \xrightarrow{f} B' \xleftarrow{b} B$ represent $\overline{f}$. Then we can form the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f_A} & B'' \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle b'} \\ A' & \xrightarrow{f} & B' \\ {\scriptstyle a'}\uparrow & & \uparrow{\scriptstyle b} \\ A'' & \xrightarrow[f_B]{} & B \end{array} ,$$

where the top rectangle is cocartesian and the bottom is cartesian. Since a, b ε I it follows easily that a', b' ε I. For example, since $\overline{S}$ is exact it preserves cartesian and cocartesian squares, so this follows from the fact that a pullback or pushout of an isomorphism is again one. Consequently we have produced new representations,

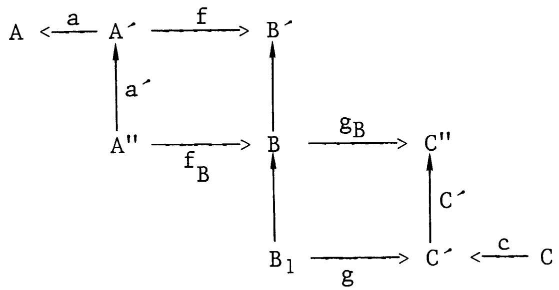
$$A = A \xrightarrow{f_A} B'' \xleftarrow{b'b} B$$

and

$$A \xleftarrow{aa'} A'' \xrightarrow{f_B} B = B,$$

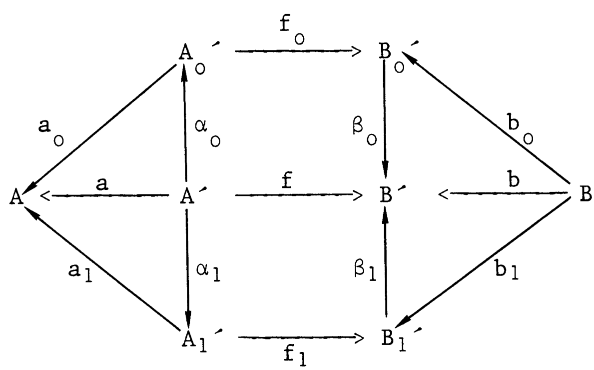
of $\overline{f}$.

(4) Suppose $\overline{S}A \xrightarrow{\overline{f}} \overline{S}B \xrightarrow{\overline{g}} \overline{S}C$ are two morphisms in $\underline{\underline{A}}'$ with representations $A \xleftarrow{a} A' \xrightarrow{f} B' \xleftarrow{b} B$ and $B \xleftarrow{b_1} B_1 \xrightarrow{g} C' \xleftarrow{c} C$, respectively. Then we construct the diagram



as in (3) and it is seen that $A \xleftarrow{aa'} A'' \xrightarrow{g_B f_B} C'' \xleftarrow{c'c} C$ represents $\overline{gf}$.

(5) If $A \xleftarrow{a_i} A_i' \xrightarrow{f_i} B_i' \xleftarrow{b_i} B$ $(i = 0, 1)$ are two representations of $\overline{f}$ then there is a commutative diagram

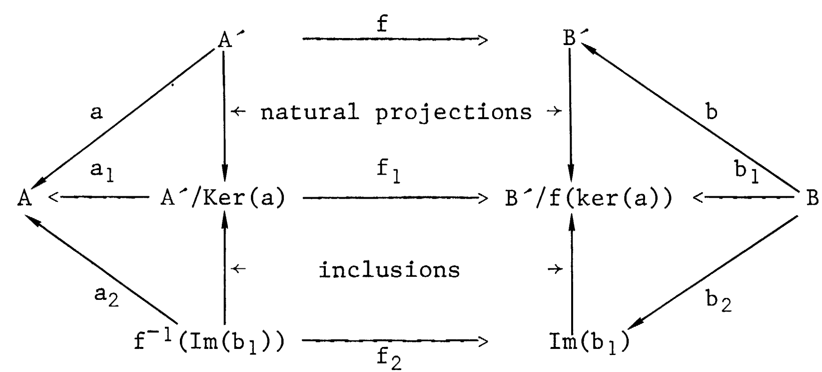


whose middle row also represents $\overline{f}$. Note that then $\alpha_i, \beta_i \in I$ $(i = 0, 1)$.

To see this we first make the A's the corners of a cartesian square and the B's the corners of a cocartesian square. Setting $f_i' = \beta_i f_i \alpha_i$ $(i = 0, 1)$ we see that $\overline{S}f_o' = \overline{S}f_1'$, so $\mathrm{Im}(f_1' - f_o') \in \underline{\underline{S}}$. Therefore if we replace B' by Coker $(f_1' - f_o')$ we can correspondingly collapse the diagram on the right so that the whole diagram now commutes, with f induced by f_o' (and f_1').

(6) Let $A \xleftarrow{a} A' \xrightarrow{f} B' \xleftarrow{b} B$ represent $\overline{f}$.

Then we can construct the commutative diagram



starting at the top left, and proceeding to the right and down. The bottom then exhibits a new representation of $\bar{f}$ for which a_2 is a monomorphism and b_2 is an epimorphism.

(7) Let

$$\begin{array}{ccccc}
\bar{S}A_2 & \xrightarrow{\bar{S}\alpha_1} & \bar{S}A_1 & \xrightarrow{\bar{S}\alpha_o} & \bar{S}A_o \\
\downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_o} \\
\bar{S}B_2 & \xrightarrow[\bar{S}\beta_1]{} & \bar{S}B_1 & \xrightarrow[\bar{S}\beta_o]{} & \bar{S}B_o
\end{array}$$

be a commutative diagram in $\underline{\underline{A}}'$. Then there is a commutative diagram as follows in $\underline{\underline{A}}$:

$$\begin{array}{ccccc}
A_2 & \xrightarrow{\alpha_1} & A_1 & \xrightarrow{\alpha_o} & A_o \\
\uparrow{\scriptstyle a_2} & & \uparrow{\scriptstyle a_1} & & \uparrow{\scriptstyle a_o} \\
A_2' & \xrightarrow{\alpha_1'} & A_1' & \xrightarrow{\alpha_o'} & A_o' \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_o} \\
B_2' & \xrightarrow{\beta_1'} & B_1' & \xrightarrow{\beta_o'} & B_o' \\
\uparrow{\scriptstyle b_2} & & \uparrow{\scriptstyle b_1} & & \uparrow{\scriptstyle b_o} \\
B_2 & \xrightarrow{\beta_1} & B_1 & \xrightarrow{\beta_o} & B_o
\end{array}$$

Here the verticals represent γ_2, γ_1, and γ_o, respectively, the a's are monomorphisms, and the b's are epimorphisms. In particular, if $\alpha_o\gamma_1$ and $\beta_o\beta_1$ are zero then so also are $\alpha_o'\alpha_1'$ and $\beta_o'\beta_1'$.

To construct such a diagram we start with vertical representations of the γ's so that the a's are monomorphisms and the b's are epimorphisms, using (6) above. In order to complete the construction we shall replace the initial choices of the A''s and B''s by "smaller" ones. For an A_i' this means a smaller subobject of A_i such that the inclusion into A_i is still in I. For the B_i' this means a smaller quotient of B_i such that the projection from B_i is still in I.

Step 1. Make A_1' and B_1' smaller so that α_o' and β_1' exist making the upper right and lower left rectangles, respectively, commute.

Step 2. Make A_2' and B_o' smaller so that α_o' and β_o' exist, making the upper left and lower right rectangles commute.

Step 3. Make B_o' still smaller so that the middle right rectangle commutes.

Step 4. Make B_1' smaller so that the middle left rectangle commutes.

It is easily seen that all of the above reductions are possible, and that each step leaves intact the conditions achieved by the previous ones.

(8) If $\overline{A} = (0 \longrightarrow \overline{A}_2 \longrightarrow \overline{A}_1 \xrightarrow{\overline{f}} \overline{A}_o \longrightarrow 0) \in \mathrm{Ex}(\underline{\underline{A}}')$, i.e. $\overline{A}$ is a short exact sequence in $\underline{\underline{A}}'$, then there is an $A \in \mathrm{Ex}(\underline{\underline{A}})$ and an isomorphism $\overline{S}A \simeq \overline{A}$.

For since $\overline{S}$ is exact it suffices to lift $\overline{f}$ to an epimorphism $f: A_1 \longrightarrow A_o$ in $\underline{\underline{A}}$. Say $\overline{A}_i = \overline{S}B_i$. Using (3) we can represent $\overline{f}$ by a diagram $B_1 = B_1 \xrightarrow{f'} B_o' \xleftarrow{b_o} B_o$. We now choose $A_1 = B_1$ and $A_o = \mathrm{Im}(f') \longrightarrow B_o'$ is the inclusion then $(1_{\overline{S}B_1}, (Sb_o)^{-1}(\overline{S}_i)): (\overline{S}A_1 \xrightarrow{\overline{S}f} \overline{S}A_o) \longrightarrow (\overline{S}B_1 \longrightarrow \overline{S}B_o)$ is the required isomorphism.

(5.4) THEOREM (Heller). Let $\overline{S}: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}' = \underline{\underline{A}}/\underline{\underline{S}}$ be a quotient functor. Let $\underline{\underline{C}}$ be an admissible subcategory of $\underline{\underline{A}}$ such that $A \in \underline{\underline{A}}$, $C \in \underline{\underline{C}}$, and $\overline{S}A \simeq \overline{S}C$ implies $A \in \underline{\underline{C}}$. Let $\underline{\underline{C}}'$ be the full subcategory of $\underline{\underline{A}}'$ with objects $\overline{S}A$ ($A \in \underline{\underline{C}}$), and let $S: \underline{\underline{C}} \longrightarrow \underline{\underline{C}}'$ be the functor induced by $\overline{S}$. Then the functor $F: \underline{\underline{S}} \longrightarrow \mathrm{co}(S)$, defined by $FA = (0, 0, A)$, induces an isomorphism

$$\phi: K_0(\underline{\underline{S}}) \longrightarrow K_0'(S).$$

Remark. Condition (d) in the definition (1.1) of admissible subcategory is not necessary here, and it will not be used in the proof.

Proof. First note that F is exact so that the homomorphism ϕ exists.

Suppose $(A, \alpha, B) \in co(S)$. Let $A \xleftarrow{a} A' \xrightarrow{f} B' \xleftarrow{b} B$ represent α. The hypothesis on $\underline{\underline{C}}$ implies that $A', B' \in \underline{\underline{C}}$. Therefore the equation $\alpha = (Sb)^{-1}(Sf)(Sa)^{-1}$ shows that $[A, \alpha, B] = [A', Sf, B'] - [A', Sa, A] - [B, Sb, B']$ in $K_0'(S)$. Moreover, we can filter (A', Sf, B') in co(S) as follows:

$$(\text{Ker } f, 0, 0) \subset (A', \bar{f}, \text{Im}(f)) \subset (A', f, B'),$$

where $\bar{f}: SA' \longrightarrow S\,\text{Im}(f) = \text{Im}(Sf)$ is induced by Sf. Since the successive quotients are in co(S) we find that

$$[A', Sf, B'] = [\text{Ker } f, 0, 0] + [\text{Coim } f, Sg, \text{Im } f] + [0, 0, \text{Coker } f].$$

Here g is the isomorphism induced by f, so we have an isomorphism $(g, 1): (\text{Coim } f, Sg, \text{Im } f) \longrightarrow (\text{Im } f, 1, \text{Im } f)$. It follows that $[\text{Coim } f, Sg, \text{Im } f] = 0$. Next observe that if $C \in \underline{\underline{S}}$ then $(C, 0, 0) \oplus (0, 0, C) = (C, 0, C) = (C, 1_{SC}, C)$, and hence $[C, 0, 0] = -[0, 0, C]$. We conclude therefore that $[A', Sf, B'] = [0, 0, \text{Coker } f] - [0, 0, \text{Ker } f] = \phi([\text{Coker } f] - [\text{Ker } f])$. Similar conclusions apply to (A', Sa, A) and (B, Sb, B').

If f is any morphism in I write

$$\chi(f) = [\text{Coker } f] - [\text{Ker } f] \in K_0(\underline{\underline{S}}).$$

The discussion above shows, in summary, that if $(A, \alpha, B) \in$

co(S), and if $A \xleftarrow{a} A' \xrightarrow{f} B' \xleftarrow{b} B$ represents α, then

$$[A, \alpha, B] = \phi(\chi(f) - \chi(a) - \chi(b)).$$

This suggests that we define

$$\psi\colon \text{ob co}(S) \longrightarrow K_0(\underline{\underline{S}})$$

by $\psi(A, \alpha, B) = \chi(f) - \chi(a) - \chi(b)$. If we show that ψ is well defined and that it induces a homomorphism $\psi\colon K_0{}'(S) \longrightarrow K_0(\underline{\underline{S}})$ then the equation above will imply that $\phi\circ\psi$ = identity. In the other direction, if $A \in \underline{\underline{S}}$, then we can represent $0 \in \underline{\underline{A}}'(S0, SA)$ by $0 = 0 \xrightarrow{f} A = A$, so $\psi(\phi[A]) = \psi[0, 0, A] = [\text{Coker } f] - [\text{Ker } f] = [A]$. Therefore the theorem will be proved once we show that:

(i) ψ is well defined.

(ii) If $(A, \alpha, B), (B, \beta, C) \in$ co(S) then

$$\psi(A, \beta\alpha, C) = \psi(A, \alpha, B) + \psi(B, \beta, C).$$

(iii) If $0 \longrightarrow (A_2, \gamma_2, B_2) \longrightarrow (A_1, \gamma_1, B_1) \longrightarrow (A_0, \gamma_0, B_0) \longrightarrow 0$ is an exact sequence in co(S) then

$$\psi(A_1, \gamma_1, B_1) = \psi(A_2, \gamma_2, B_2) + \psi(A_0, \gamma_0, B_0).$$

We begin by noting that if $f, g \in I$ and if gf is defined then $\chi(gf) = \chi(g) + \chi(f)$. This follows from the exact sequence (I, 4.7): $0 \longrightarrow \text{Ker } f \longrightarrow \text{Ker } gf \longrightarrow \text{Ker } g \longrightarrow \text{Coker } f \longrightarrow \text{Coker } gf \longrightarrow \text{Coker } g \longrightarrow 0$.

<u>Proof of (i)</u>. If $A \xleftarrow{a_i} A_i' \xrightarrow{f_i} B_i' \xleftarrow{b_i} B$ $(i = 0, 1)$ are two representations of α, where $(A, \alpha, B) \in$ co(S), then we construct a commutative diagram as in (5) above. Then $\chi(f) - \chi(a) - \chi(b) = \chi(\beta_i f_i \alpha_i) - \chi(a_i\alpha_i) - \chi(\beta_i b_i) = \chi(f_i) - \chi(a_i) - \chi(b_i)$, $(i = 0, 1)$.

<u>Proof of (ii)</u>. Using (3) above we can choose

representations $A \xleftarrow{a} A' \xrightarrow{f} B = B$ of α and $B = B \xrightarrow{g} C' \xleftarrow{c} C$ of β, whereupon $A \xleftarrow{a} A' \xrightarrow{gf} C' \xleftarrow{c} C$ represents $\beta\alpha$. Hence $\psi(A, \beta\alpha, C) = \chi(gf) - \chi(a) - \chi(c) = (\chi(f) - \chi(a)) + (\chi(g) - \chi(C)) = \psi(A, \alpha, B) + \psi(B, \beta, C)$.

Proof of (iii). Starting from the given exact sequence in co(S) we construct a commutative diagram as in (7) above. We shall view its rows as complexes (zero except in degrees 0, 1, 2) and the verticals as morphisms of complexes: $A \xleftarrow{a} A' \xrightarrow{f} B' \xleftarrow{b} B$. Here, for example, $A = (\ldots A_2 \longrightarrow A_1 \longrightarrow A_o \ldots)$, $a = (\ldots, a_2, a_1, a_o, \ldots)$, etc. We deduce several exact sequences of complexes:

$$0 \longrightarrow A' \xrightarrow{a} A \longrightarrow \mathrm{Coker}(a) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow A' \longrightarrow \mathrm{Im}(f) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}(f) \longrightarrow B' \longrightarrow \mathrm{Coker}(f) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ker}(b) \longrightarrow B \longrightarrow B' \longrightarrow 0.$$

Since Sf, Sa, and Sb are isomorphisms, the kernels and cokernels of f, a, and b are complexes in $\underline{\underline{S}}$. Moreover, since A and B are acyclic the complexes SA′ and SB′ are also, so HA′ and HB′ are graded objects in $\underline{\underline{S}}$.

If $C = (C_n)$ is a finite graded object in $\underline{\underline{S}}$ we write $\chi^{\underline{\underline{S}}}(C) = \Sigma(-1)^n [C_n] \in K_o(\underline{\underline{S}})$. With this notation the assertion of (iii) can be formulated:

(**) $$\chi^{\underline{\underline{S}}}(\mathrm{Coker}(f)) - \chi^{\underline{\underline{S}}}(\mathrm{Ker}(f)) = \chi^{\underline{\underline{S}}}(\mathrm{Ker}(b)) - \chi^{\underline{\underline{S}}}(\mathrm{Coker}(a)).$$

To prove this we first recall ((4.1) (b)) that if C is a finite complex in $\underline{\underline{S}}$ then $\chi^{\underline{\underline{S}}}(H(C)) = \chi^{\underline{\underline{S}}}(C)$, and ((4.1) (c)) that if $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ is an exact sequence of complexes whose homology is finite and in $\underline{\underline{S}}$ then $\chi^{\underline{\underline{S}}}(HC) = \chi^{\underline{\underline{S}}}(HC') + \chi^{\underline{\underline{S}}}(HC'')$. From these facts and the exact sequences above we deduce:

$$0 = \chi^{\underline{\underline{S}}}(HA') + \chi^{\underline{\underline{S}}}(\mathrm{Coker}(a))$$

$$\chi^{\underline{\underline{S}}}(HA') = \chi^{\underline{\underline{S}}}(\mathrm{Ker}(f)) + \chi^{\underline{\underline{S}}}(\mathrm{Im}(f))$$

$$\chi^{\underline{\underline{S}}}(HB') = \chi^{\underline{\underline{S}}}(\mathrm{Im}(f)) + \chi^{\underline{\underline{S}}}(\mathrm{Coker}(f))$$

$$0 = \chi^{\underline{\underline{S}}}(\mathrm{Ker}(b)) + \chi^{\underline{\underline{S}}}(HB').$$

Subtracting the second from the first we find that $\chi^{\underline{\underline{S}}}(\mathrm{Coker}(f)) - \chi^{\underline{\underline{S}}}(\mathrm{Ker}(f)) = \chi^{\underline{\underline{S}}}(HB') - \chi^{\underline{\underline{S}}}(HA')$, and subtracting from the first and last equations gives (**). This concludes the proof of Theorem (5.4).

(5.5) COROLLARY. Keep the notation of Theorem (5.4). Then the functors $\underline{\underline{S}} \subset \underline{\underline{C}} \xrightarrow{S} \underline{\underline{C}}'$ induce an exact sequence

$$K_0(\underline{\underline{S}}) \xrightarrow{d} K_0(\underline{\underline{C}}) \longrightarrow K_0(\underline{\underline{C}}') \longrightarrow 0.$$

Moreover, there is a "connecting homomorphism" $\partial: K_1(\underline{\underline{C}}') \longrightarrow K_0(\underline{\underline{S}})$ defined as follows: If $(SA, \alpha) \in \Sigma\underline{\underline{C}}'$, and if $A \xleftarrow{a} A' \xrightarrow{f} B' \xleftarrow{b} A$ represents α, then $\partial[SA, \alpha]_{\underline{\underline{C}}'} = \chi(f) - \chi(a) - \chi(b)$. Here, for a $g \in \mathrm{mor}\,\underline{\underline{C}}$ such that Sg is an isomorphism, we write $\chi(g) = [\mathrm{Coker}\, g]_{\underline{\underline{S}}} - [\mathrm{Ker}\, g]_{\underline{\underline{S}}}$. The sequence

(5.6) $$K_1(\underline{\underline{C}}) \longrightarrow K_1(\underline{\underline{C}}') \xrightarrow{\partial} K_0(\underline{\underline{S}}) \xrightarrow{d} K_0(\underline{\underline{C}}) \longrightarrow K_0(\underline{\underline{C}}') \longrightarrow 0$$

is exact, except possibly at $K_1(\underline{\underline{C}}')$, if $\underline{\underline{C}}'$ is semi-simple, and it is exact if $\underline{\underline{C}}$ is semi-simple.

Proof. According to (8) above, given $A' \in Ex(\underline{\underline{C}}')$, i.e. a short exact sequence in $\underline{\underline{C}}'$, there is an $A \in Ex(\underline{\underline{A}})$ such that $\overline{S}A \simeq A'$. Automatically then $A \in Ex(\underline{\underline{C}})$, so $Ex(S)\colon Ex(\underline{\underline{C}}) \longrightarrow Ex(\underline{\underline{C}}')$ is surjective on isomorphism classes of objects. Proposition (2.1) therefore gives us a sequence

$$K_1(\mathbf{C}) \longrightarrow K_1(\mathbf{C}') \xrightarrow{\partial'} K_o'(S) \xrightarrow{d'} K_o(\mathbf{C})$$
$$\longrightarrow K_o(\mathbf{C}') \longrightarrow 0,$$

which is exact if we remove the two left terms. Moreover Theorem (2.2) gives the remaining exactness conclusions under the appropriate semi-simplicity hypotheses. Theorem (5.4) just proved gives us an isomorphism ϕ: $K_o(\mathbf{S}) \longrightarrow K_o'(S)$ which permits us to substitute $K_o(\mathbf{S})$ for $K_o'(S)$ in the sequence above. If $A \in \mathbf{S}$ then $d'\phi[A]_{\mathbf{S}} = d'[0, 0, A]_S = -[A]_{\mathbf{C}}$, so $d = -d'\phi$ is induced by the inclusion $\mathbf{S} \subset \mathbf{C}$. Moreover the definition of $\psi(= \phi^{-1})$ in the proof of Theorem (5.4) provides the description given above for $\partial = \phi^{-1}\partial'$. q.e.d.

(5.7) COROLLARY. <u>Suppose, in the setting of</u> (5.5), <u>that $\mathbf{C}'$ is semi-simple, and assume either</u> (a) <u>or</u> (b) <u>below</u>:

(a) <u>$\mathbf{C}$ is abelian and every object of $\mathbf{A}$ has a finite $\mathbf{C}$-filtration</u>.

(b) <u>Every object of $\mathbf{A}$ has a finite $\mathbf{C}$-resolution</u>.

<u>Then the functors $\mathbf{S} \subset \mathbf{A} \longrightarrow \mathbf{A}'$ induce an exact sequence</u>

$$K_1(\mathbf{A}') \xrightarrow{\bar{\partial}} K_o(\mathbf{S}) \xrightarrow{\bar{d}} K_o(\mathbf{A}) \longrightarrow K_o(\mathbf{A}') \longrightarrow 0.$$

<u>Proof</u>. The commutative square

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\bar{S}} & \mathbf{A}' \\ \cup & & \cup \\ \mathbf{C} & \xrightarrow{S} & \mathbf{C}' \end{array}$$

leads to a commutative diagram

$$\begin{array}{ccccccccc} K_1(\mathbf{A}') & \xrightarrow{\bar{\partial}} & K_o(\mathbf{S}) & \xrightarrow{\bar{d}} & K_o(\mathbf{A}) & \longrightarrow & K_o(\mathbf{A}') & \longrightarrow & 0 \\ \uparrow & & \| & & \uparrow{\scriptstyle h} & & \uparrow & & \\ K_1(\mathbf{C}') & \xrightarrow{\partial} & K_o(\mathbf{S}) & \longrightarrow & K_o(\mathbf{C}) & \longrightarrow & K_o(\mathbf{C}') & \longrightarrow & 0. \end{array}$$

Corollary (5.5) gives the exactness of both rows, except possibly at $K_o(\underline{\underline{S}})$. Exactness there for the bottom row follows from Theorem (2.2) (b) and the semi-simplicity of $\underline{\underline{C}}'$. Since $\overline{d\partial} = 0$ the exactness of the top row at $K_o(\underline{\underline{S}})$ will follow if we know that h is a monomorphism. This follows from Theorem (3.3) in case (a) and from Theorem (4.2) in case (b). q.e.d.

We close this section now with a result which is somewhat similar in spirit to Theorem (5.4), but whose proof requires slightly different techniques.

(5.8) THEOREM. *Let*

$$\begin{array}{ccc} \underline{\underline{A}} & \xrightarrow{\overline{S}} & \underline{\underline{A}}' = \underline{\underline{A}}/\underline{\underline{S}} \\ \cup & & \cup \\ \underline{\underline{P}} & \xrightarrow{S} & \underline{\underline{P}}' \end{array}$$

be a commutative square, where $\overline{S}$ *is a quotient functor. Assume:*

(1) *The objects of* $\underline{\underline{P}}$ *and of* $\underline{\underline{P}}'$ *are projective, and* S *is cofinal.*

(2) *If* f: $P \longrightarrow Q$ *is a morphism in* $\underline{\underline{A}}$ *such that* $P \in \underline{\underline{P}}$ *and such that* $\overline{S}f$ *is a monomorphism then* f *is a monomorphism.*

(3) *If* $P \in \underline{\underline{P}}$ *and if* $Q \subset P$ *is such that* $P/Q \in \underline{\underline{S}}$ *then there exists a* $P' \subset Q$ *such that* $P' \in \underline{\underline{P}}$ *and* $P/P' \in \underline{\underline{S}}$.

Let $\underline{\underline{H}}$ *be the full subcategory of objects* $A \in \underline{\underline{A}}$ *having finite* $\underline{\underline{P}}$*-resolutions, and let* $\underline{\underline{H}}_S = \underline{\underline{H}} \cap \underline{\underline{S}}$. *Then there is an exact sequence*

$$K_1(\underline{\underline{P}}) \longrightarrow K_1(\underline{\underline{P}}') \xrightarrow{\partial} K_o(\underline{\underline{H}}_S) \xrightarrow{d} K_o(\underline{\underline{P}}) \longrightarrow K_o(\underline{\underline{P}}').$$

Here d *is the composite of* $K_o(\underline{\underline{H}}_S) \longrightarrow K_o(\underline{\underline{H}})$ *and the inverse* *of* $K_0(\underline{\underline{P}}) \longrightarrow K_0(\underline{\underline{H}})$, *induced by the inclusions* $\underline{\underline{H}}_S \subset \underline{\underline{H}} \subset \underline{\underline{P}}$. *The map* ∂ *is defined as follows: If* $(SP, \alpha) \in \Sigma\underline{\underline{P}}'$, *and if* $P \supset P' \xrightarrow{f} P$ *represents* α, *where* $P' \in \underline{\underline{P}}$, *then* $\partial[SP, \alpha]_{\underline{\underline{P}}'} = [\mathrm{Coker}(f)]_{\underline{\underline{H}}_S} - [P/P']_{\underline{\underline{H}}_S}$.

Proof. Let $\underline{\underline{H}}'$ be the full subcategory of objects $A' \in \underline{\underline{A}}'$ having finite $\underline{\underline{P}}'$-resolutions. Then we have a commutative square

$$\begin{array}{ccc} \underline{\underline{H}} & \xrightarrow{T} & \underline{\underline{H}}' \\ \cup & & \cup \\ \underline{\underline{P}} & \xrightarrow[S]{} & \underline{\underline{P}}' \end{array}$$

where T is induced by $\overline{S}$. (T is defined because $\overline{S}$ is exact). Then we have a commutative diagram

$$
\begin{array}{ccccccccc}
K_1(\underline{\underline{H}}) & \longrightarrow & K_1(\underline{\underline{H}}') & \longrightarrow & K_o'(T) & \xrightarrow{d''} & K_o(H) & \longrightarrow & K_o(\underline{\underline{H}}') \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle h} & & \uparrow & & \uparrow \\
K_1(\underline{\underline{P}}) & \longrightarrow & K_1(\underline{\underline{P}}') & \xrightarrow{\partial'} & K_o'(S) & \xrightarrow{d'} & K_o(\underline{\underline{P}}) & \longrightarrow & K_o(\underline{\underline{P}}')
\end{array}
$$

in which, thanks to Theorem (4.6) and Proposition (4.7), all the verticals are isomorphisms. Moreover, since $\underline{\underline{P}}$ is semi-simple and S is cofinal it follows from Theorem (2.2) that the bottom row is exact; hence the top is also.

We shall prove the theorem now by constructing isomorphisms ϕ and ψ such that

(Δ)
$$\begin{array}{ccc} & K(\underline{\underline{H}}_S) & \\ {\scriptstyle\psi}\nearrow & & \searrow{\scriptstyle\phi} \\ K_0{}'(S) & \xrightarrow{\ h\ } & K_0{}'(T) \end{array}$$

commutes and such that $d = -d'\psi^{-1}$ and $\partial = \psi\partial'$ admit the descriptions given in the statement of the theorem.

The functor $F: \underline{\underline{H}}_S \longrightarrow co(T)$, $FA = (0, 0, A)$, is exact, so it induces a homomorphism $\phi: K_0(\underline{\underline{H}}_S) \longrightarrow K_0{}'(T)$. Moreover $d''\phi[A]_{\underline{\underline{H}}_S} = d''[0, 0, A]_T = -[A]_H$, so $-d''\phi$ is the map induced by $\underline{\underline{H}}_S \subset \underline{\underline{H}}$.

To construct ψ suppose we are given $(P, \alpha, Q) \in co(S)$.

(i) α <u>has a representation</u> $P \supset P' \xrightarrow{f} Q$ <u>with</u> $P' \in \underline{\underline{P}}$ <u>and</u> f <u>a monomorphism</u>. For let $P \supset P' \xrightarrow{f'} Q' \xleftarrow{b} Q$ be any representation with b an epimorphism. Condition (1) says we can make P' smaller, if necessary, to achieve $P' \in \underline{\underline{P}}$. Then condition (2) implies, since Sb and Sf′ are isomorphisms, that b and f′ are monomorphisms. Therefore we can replace Q′ by Q, b by 1_Q, and f′ by $f = b^{-1}f'$.

(ii) $\psi(P, \alpha, Q) = [Q/fP']_{\underline{\underline{H}}_S} - [P/P']_{\underline{\underline{H}}_S}$ <u>is well defined</u>. For suppose that $P \supset P_i{}' \xrightarrow{f_i} Q$, $(i = 0, 1)$, are two representations of α as in (i). The sequence $0 \longrightarrow P_0{}' \cap P_1{}' \longrightarrow P \longrightarrow (P/P_0{}') \oplus (P/P_1{}')$ shows that $P/(P_0{}' \cap P_1{}') \in \underline{\underline{S}}$. Condition (1) therefore gives us a $P' \subset P_0{}' \cap P_1{}'$ such that $P' \in \underline{\underline{P}}$ and $(P/P') \in \underline{\underline{S}}$. Let $g_i: P' \longrightarrow Q$ be the morphism induced by f_i, $(i = 0, 1)$. Since $Sg_0 = Sg_1$ we have a morphism $g': Q \longrightarrow \mathrm{Coker}(g_0 - g_1)$ such that Sg' is an isomorphism. Hence g' is a monomorphism, by condition (2), so $g_0 = g_1$; call this morphism f. Then we have, in $K_0(\underline{\underline{H}}_S)$,

$$\begin{aligned}[Q/FP'] - [P/P'] &= [Q/f_iP_i'] + [f_iP_i'/f_iP'] \\ &\quad - (P/P_i'] - [P_i'/P'] \\ &= [Q/f_iP_i'] - [P/P_i'] \qquad (i = 0, 1),\end{aligned}$$

because $f_iP_i'/f_iP' \simeq P_i'/P'$.

(iii) If $(P_1, \alpha_1, Q_1) \in \mathrm{co}(S)$, $(i = 0, 1)$, <u>then</u>

$$\psi(P_0 \oplus P_1, \alpha_0 \oplus \alpha_1, Q_0 \oplus Q_1) = \psi(P_0, \alpha_0, Q_0) + \psi(P_1, \alpha_1, Q_1).$$

This is easily seen, because if $P_i \supset P_i' \xrightarrow{f_i} Q_i$ represents α_i as in (i) $(i = 0, 1)$, then $P_0 \oplus P_1 \supset P_0' \oplus P_1' \xrightarrow{f_0 \oplus f_1} Q_0 \oplus Q_1$ is such a representation of $\alpha_0 \oplus \alpha_1$.

(iv) <u>If</u> (P, α, Q), $(Q, \beta, R) \in \mathrm{co}(S)$ <u>then</u>

$$\psi(P, \beta\alpha, R) = \psi(P, \alpha, Q) + \psi(Q, \beta, R).$$

First choose a representation $Q \supset Q' \xrightarrow{g} R$ of β, as in (i). Now we seek such a representation, $P \supset P' \xrightarrow{f} Q$, of α for which $fP' \supset Q'$. If the latter is not the case already, we can make a smaller choice of P' for which it will be, as follows. Since $fP'/(fP' \cap Q') \subset Q/Q' \in \underline{\underline{S}}$ we can use condition (1) to find $P'' \subset P'$, $P'' \in \underline{\underline{P}}$, such that $fP'' \subset fP' \cap Q'$ (recall that f is a monomorphism) and such that $P'/P'' \in \underline{\underline{S}}$. We can then replace P' by P'' to achieve the condition above. This done, we have the representation $P \supset P' \xrightarrow{gf'} R$ of $\beta\alpha$, where $f': P' \longrightarrow Q'$ is induced by f. Therefore,

$$\psi(P, \beta\alpha, R) = [R/gf'P'] - [P/P']$$
$$= [R/gQ'] + [gQ'/gf'P'] - [P/P']$$
$$= [R/gQ'] - [Q/Q'] + [Q/Q'] + [Q'/fP'] - [P/P']$$

(g is a monomorphism, so $gQ'/gf'P' \simeq Q'/f'P' = Q'/fP'$)

$$= \psi(Q, \beta, R) + [Q/fP'] - [P/P']$$
$$= \psi(Q, \beta, R) + \psi(P, \alpha, Q).$$

These conclusions imply that ψ induces a homomorphism $\psi\colon K_0'(S) \longrightarrow K_0(\underline{\underline{H}}_S)$. The fact that, in (iii) above, we accounted only for direct sums rather than arbitrary exact sequences, is permitted by Theorem (2.2) (a), thanks to the semi-simplicity of $\underline{\underline{P}}$. If we recall that, for $(SP, \alpha) \in \Sigma\underline{\underline{P}}'$, $\partial'[SP, \alpha]_{\underline{\underline{P}}'} = [P, \alpha, P]_S$, then it is evident that $\partial = \psi\partial'$ admits the description given in the theorem.

To show that ϕ and ψ are isomorphisms, and finish the proof of the theorem, we will show that, in the triangle (Δ) above, we have $h = \phi\psi$ and $\psi h^{-1}\phi$ = identity on $K_0(\underline{\underline{H}}_S)$. This suffices because h is an isomorphism.

Proof that $h = \phi\psi$. If $(P, \alpha, Q) \in co(S)$ choose a representation $P \supset P' \xrightarrow{f} Q$ of α as in (i). Then in $K_0'(S)$ we have $[P, \alpha, Q] = [P', Sf, Q] - [P', Sj, P]$, where j is the inclusion of P' in P. Therefore it suffices to show that if $f\colon P \longrightarrow Q$ is a monomorphism in $\underline{\underline{P}}$ such that Sf is an isomorphism then $\phi\psi[P, Sf, Q] = h[P, Sf, Q]$. First, $\phi\psi[P, Sf, Q]_S = \phi([Q/fP]_{\underline{\underline{H}}_S}) = [0, 0, Q/fP]_T$. On the other hand the exact sequence

$$0 \longrightarrow (P, 1_{SP}, P) \xrightarrow{(1, f)} (P, Sf, Q) \longrightarrow (0, 0, Q/fP) \longrightarrow 0$$

in co(T) shows that $h[P, Sf, Q]_S = [P, Sf, Q]_T = [0, 0, Q/fP]_T$.

Proof that $\psi h^{-1}\phi$ = identity on $K_o(\underline{\underline{H}}_S)$. We begin with a lemma.

(5.9) LEMMA. Let $\underline{\underline{H}}_S{}^1 \subset \underline{\underline{H}}_S$ be the full subcategory whose objects have $\underline{\underline{P}}$-resolution of length ≤ 1. Then every object of $\underline{\underline{H}}_S$ has a finite $\underline{\underline{H}}_S{}^1$-resolution, and hence the inclusion induces an isomorphism $K_o(\underline{\underline{H}}_S{}^1) \longrightarrow K_o(\underline{\underline{H}}_S)$.

Proof. If $A \in \underline{\underline{H}}$ write d(A) for the shortest length of a $\underline{\underline{P}}$-resolution of A. If $A \in \underline{\underline{H}}_S$ we will prove by induction on d(A) that A has a finite $\underline{\underline{H}}_S{}^1$-resolution. The case $d(A) \leq 1$ is trivial so assume $d(A) > 1$, and choose an exact sequence $0 \longrightarrow B \longrightarrow P \longrightarrow A \longrightarrow 0$ with $P \in \underline{\underline{P}}$. Condition (1) of the theorem says there is a $P' \subset B$, $P' \in \underline{\underline{P}}$, such that $P/P' \in \underline{\underline{S}}$. Hence we have an exact sequence $0 \longrightarrow B/P' \longrightarrow P/P' \longrightarrow A \longrightarrow 0$ in $\underline{\underline{S}}$, and clearly $d(P/P') \leq 1$. Since $d(A) > 1$ it follows from (I, 6.8) that $d(B/P') < d(A)$. Therefore B/P' has a finite $\underline{\underline{H}}_S{}^1$-resolution, by induction, and therefore so also does A. The last assertion of the lemma now follows from Theorem (4.2). q.e.d.

Thanks to the lemma it suffices to show that $\psi h^{-1}\phi [A]_{\underline{\underline{H}}_S} = [A]_{H_S}$ when $d(A) \leq 1$. Choose a resolution $0 \longrightarrow P_1 \xrightarrow{f} P_o \longrightarrow A \longrightarrow 0$ with $P_i \in \underline{\underline{P}}$. Then we have a co(S)-resolution of $(0, 0, A) \in co(T)$:

$$0 \longrightarrow (P_1, 1_{SP_1}, P_1) \xrightarrow{(1,\ f)} (P_1, Sf, P_o) \longrightarrow (0, 0, A) \longrightarrow 0.$$

Therefore $h^{-1}\phi[A]_{\underline{\underline{H}}_S} = h^{-1}([0, 0, A]_T) = [P_1, Sf, P_o]_S$. Finally, $\psi[P_1, Sf, P_o]_S = [P_o/fP_1]_{\underline{\underline{H}}_S} = [A]_{\underline{\underline{H}}_S}$. q.e.d.

(5.10) THEOREM. In the setting of Theorem (5.8) suppose that every object of $\underline{\underline{S}}$ has a finite $\underline{\underline{P}}$-resolution, i.e. that $\underline{\underline{S}} = \underline{\underline{H}}_S$. Let $\underline{\underline{H}}'$ be the full subcategory of objects $A \in \underline{\underline{A}}'$ having finite $\underline{\underline{P}}'$-resolutions. Then:

(a) If $A \in \underline{\underline{A}}$ then $A \in \underline{\underline{H}} \iff \overline{S}A \in \underline{\underline{H}}'$.

(b) The sequence

$$K_1(\underline{\underline{P}}) \longrightarrow K_1(\underline{\underline{P}}') \xrightarrow{\partial} K_0(\underline{\underline{S}}) \longrightarrow K_0(\underline{\underline{P}}) \longrightarrow K_0(\underline{\underline{P}}') \longrightarrow 0$$

is exact.

Proof.(a) If $P \longrightarrow A$ is a finite $\underline{\underline{P}}$-resolution then $\overline{S}P \longrightarrow \overline{S}A$ is a finite $\underline{\underline{P}}'$-resolution so $\overline{S}A \in \underline{\underline{H}}'$. To prove the converse, suppose $\overline{S}A \in \underline{\underline{H}}'$ and let $d'(\overline{S}A)$ denote the minimal length of a $\underline{\underline{P}}'$-resolution of $\overline{S}A$. We shall prove, by induction on $d'(\overline{S}A)$, that $A \in \underline{\underline{H}}$. The following facts will be used repeatedly: Let $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ be an exact sequence in $\underline{\underline{A}}'$. If two of B', B, B'' are in $\underline{\underline{H}}'$ so also is the third. Moreover if $d'(B) < d'(B'')$ then $d'(B') < d'(B'')$. These, and analogous properties of $\underline{\underline{P}}$ and $\underline{\underline{H}}$, follow from (I, 6.8).

Case $d'(\overline{S}A) = 0$. Then $\overline{S}A$ is isomorphic to an object of $\underline{\underline{P}}'$. Since $S: \underline{\underline{P}} \longrightarrow \underline{\underline{P}}'$ is cofinal it follows that $\overline{S}A \oplus \overline{S}B \simeq \overline{S}P$, for some $B \in \underline{\underline{A}}$ and $P \in \underline{\underline{P}}$. Replacing A by $A \oplus B$, then, we can assume, there is an isomorphism $\alpha: \overline{S}P \longrightarrow \overline{S}A$ with $P \in \underline{\underline{P}}$. Let $P \supset P' \xrightarrow{f} A' \xleftarrow{a} A$ be a representation of α with a an epimorphism. Using condition (1) of (5.8) we can further assume $P' \in \underline{\underline{P}}$. Since Sf is an isomorphism condition (2) of (5.8) implies $0 \longrightarrow P' \xrightarrow{f} A' \longrightarrow \text{Coker}(f) \longrightarrow 0$ is exact. Since $\text{Coker}(f) \in \underline{\underline{S}} \subset \underline{\underline{H}}$ we conclude that $A' \in \underline{\underline{H}}$. Since $\text{Ker}(a) \in \underline{\underline{S}}$ the exact sequence $0 \longrightarrow \text{Ker}(a) \longrightarrow A$

$\xrightarrow{a} A' \longrightarrow 0$ shows, finally, that $A \in \underline{\underline{H}}$.

Case $d'(SA) > 0$. Any object of $\underline{\underline{P}}'$ can be lifted to an object of $\underline{\underline{A}}$ (but not necessarily in $\underline{\underline{P}}$). Therefore we can find an epimorphism $\overline{SB} \xrightarrow{\overline{f}} \overline{SA}$ for some $B \in \underline{\underline{A}}$ such that $\overline{SB} \in \underline{\underline{P}}'$. Represent $\overline{f}$ in the form $B \xleftarrow{b} B' \xrightarrow{f} A = A$. Since Sb is an isomorphism we can replace B by B′ and then assume we have a morphism $f: B \longrightarrow A$ such that $\overline{SB} \in \underline{\underline{P}}'$ and $\overline{Sf}$ is an epimorphism. Let C = Ker f. The exact sequence $0 \longrightarrow \overline{SC} \longrightarrow \overline{SB} \xrightarrow{\overline{Sf}} \overline{SA} \longrightarrow 0$ shows that $d'(\overline{SC}) < d'(\overline{SA})$. Therefore, by our induction assumption and the case $d' = 0$ we have C, B $\in \underline{\underline{H}}$. The exact sequence $0 \longrightarrow C \longrightarrow B \longrightarrow \mathrm{Im}(f) \longrightarrow 0$ shows therefore that $\mathrm{Im}(f) \in \underline{\underline{H}}$. Finally, the sequence $0 \longrightarrow \mathrm{Im}(f) \longrightarrow A \longrightarrow \mathrm{Coker}(f) \longrightarrow 0$ shows that $A \in \underline{\underline{H}}$ since $\mathrm{Coker}(f) \in \underline{\underline{S}} \subset \underline{\underline{H}}$.

(b) The exact sequence is just the exact sequence of Theorem (5.8), except for the assertion that $K_o(\underline{\underline{P}}) \longrightarrow K_o(\underline{\underline{P}}')$ is surjective. But this map is isomorphic to the corresponding one, $K_o(\underline{\underline{H}}) \longrightarrow K_o(\underline{\underline{H}}')$, and part (a) says that $\underline{\underline{H}} \longrightarrow \underline{\underline{H}}'$ is surjective on objects. q.e.d.

§6. ROBERTS' THEOREM

In this section <u>we fix an algebraically closed field</u> k, <u>and a</u> k-<u>category</u> $\underline{\underline{A}}$. Recall (cf. Chapter II) that this is an abelian category such that $\underline{\underline{A}}(A, B)$ is a k-modules for all A, B $\in \underline{\underline{A}}$, and such that composition is k-bilinear. <u>We assume further that</u> $\underline{\underline{A}}(A, B)$ <u>is always finite dimensional over</u> k.

An example of such an $\underline{\underline{A}}$ is the category of coherent sheaves of modules over the structure sheaf on a complete algebraic variety over k. It was for this example that Leslie Roberts (Harvard thesis) proved the following theorem.

(6.1) THEOREM (Roberts). <u>Let</u> k <u>be an algebraically</u>

closed field and let $\underline{\underline{A}}$ be a k-category such that $\underline{\underline{A}}(A, B)$ is finite dimensional over k for all A, B ε $\underline{\underline{A}}$. Then there is an isomorphism

$$f: K_0(\underline{\underline{A}}) \otimes_{\underline{\underline{Z}}} k^* \longrightarrow K_1(\underline{\underline{A}})$$

defined by $f([A]_{\underline{\underline{A}}} \quad a) = [A, a \cdot 1_A]_{\underline{\underline{A}}}$ for $A \in \underline{\underline{A}}$ and $a \in k^*$.

Proof. The map $(A, a) \longmapsto [A, a \cdot 1_A]_{\underline{\underline{A}}}$ from ob $\underline{\underline{A}} \times k^*$ to $K_1(\underline{\underline{A}})$ is clearly additive over exact sequences in the first variable (axiom K0 for $K_1(\underline{\underline{A}})$) and additive over products in the second variable (axiom K1 for $K_1(\underline{\underline{A}})$). Hence f is a well defined homomorphism. We propose to construct an inverse to f.

Suppose $(A, \alpha) \in \Sigma\underline{\underline{A}}$. The the subalgebra $k[\alpha] \subset \mathrm{End}_{\underline{\underline{A}}}(A)$ is finite dimensional. Therefore $k[\alpha] \simeq k[X]/(P_\alpha(X))$ where P_α is the monic polynomial of least degree such that $P_\alpha(\alpha) = 0$. Since k is algebraically closed it has a factorization, $P_\alpha(X) = \Pi(X - a_i)^{n_i}$, where the a_i are distinct, and in k^*, because α is invertible. By the Chinese Remainder Theorem (III, 2.14) we have $k[\alpha] \simeq \Pi\, k[X]/((X - a_i)^{n_i})$. Let $1 = \Sigma e_i$ be the decomposition of 1 as a sum of indecomposable idempotents $e_i \in k[\alpha]$ which corresponds to the above factorization. This induces a decomposition, $A = \amalg\, e_i A$. We can describe $e_i A$ more intrinsically as $e_i A = \mathrm{Ker}(\alpha - a_i 1_A)^{n_i} = \cup_{n>0} \mathrm{Ker}(\alpha - a_i 1_A)^n$. For any $a \in k$, $\alpha - a 1_A$ is invertible unless a is one of the a_i above. Thus

$$A_\alpha(a) = \cup_{n>0} \mathrm{Ker}(\alpha - a \cdot 1_A)^n$$

exists, and it is zero for almost all a. Moreover, we have a direct sum decomposition in $\Sigma\underline{\underline{A}}$,

$$(1) \qquad (A, \alpha) = \coprod_{a \in k^*} (A_\alpha(a), \alpha_a),$$

where α_a is the automorphism of $A_\alpha(a)$ induced by α. Since $(\alpha_a - a1_{A_\alpha(a)})^n = 0$ for some $n > 0$ it follows that $\beta = a^{-1}\alpha_a$ is unipotent. According to (3.2) (1) therefore, $[A_\alpha(a), \beta] = 0$ in $K_1(\underline{\underline{A}})$. Since $\alpha_a = a\beta$ it follows that $[A_\alpha(a), \alpha_a] = [A_\alpha(a), a1_{A_\alpha(a)}]$ in $K_1(\underline{\underline{A}})$. Referring now to (1) above we conclude from this that, for any $(A, \alpha) \in \Sigma\underline{\underline{A}}$,

$$[A, \alpha] = \Sigma_{a \in k^*} [A_\alpha(a), a \cdot 1_{A_\alpha(a)}]$$
$$= f(\Sigma_{a \in k^*} [A\ (a)] \otimes a).$$

This suggests that we construct the inverse to f by introducing

(2) $$g: \text{ob } \Sigma\underline{\underline{A}} \longrightarrow K_o(\underline{\underline{A}}) \otimes k$$
$$g(A, \alpha) = \Sigma_{a \in k^*} [A_\alpha(a)] \otimes a.$$

Suppose that this g does, indeed, induce a homomorphism $g: K_1(\underline{\underline{A}}) \longrightarrow K_o(\underline{\underline{A}}) \otimes k^*$. Then the formula above shows that $f \circ g$ = the identity on $K_1(\underline{\underline{A}})$. In the other direction we have, trivially, $g(f([A] \otimes a)) = g([A, a1_A]) = [A] \otimes a$. Thus the theorem will be proved once we show that (2) induces a homomorphism on $K_1(\underline{\underline{A}})$. We must verify:

K0. If $0 \longrightarrow (A, \alpha) \longrightarrow (B, \beta) \longrightarrow (C, \gamma) \longrightarrow 0$ is an exact sequence in $\Sigma\underline{\underline{A}}$ then $g(B, \beta) = g(A, \alpha) + g(C, \gamma)$; and

K1. If $A \in \underline{\underline{A}}$ and if $\alpha, \beta \in \text{Aut}_{\underline{\underline{A}}}(A)$ then $g(A, \alpha\beta) = g(A, \alpha) + g(A, \beta)$.

Proof of K0. Let $h: (A, \alpha) \longrightarrow (B, \beta)$ be a morphism in $\Sigma\underline{\underline{A}}$; thus $h\alpha = \beta h$. Then if $a \in k$ we have $h(\alpha - a \cdot 1_A)^n = (\beta - a \cdot 1_B)^n h$ for all $n > 0$, so $h(A_\alpha(a)) \subset B_\beta(a)$. This implies that h is the direct sum of morphisms $h_\alpha: (A_\alpha(a), \alpha_a) \longrightarrow (B_\beta(a), \beta_a)$ $(a \in k^*)$. In particular, the exact sequence in K0 above splits, in this way, into a direct sum of exact sequences

$$0 \longrightarrow (A_\alpha(a), \alpha_a) \longrightarrow (B_\beta(a), \beta_a)$$
$$\longrightarrow (C_\gamma(a), \gamma_a) \longrightarrow 0$$

($a \in k^*$). It follows that $[B_\beta(a)] = [A_\alpha(a)] + [C_\gamma(a)]$ in $K_0(\underline{\underline{A}})$ (axiom K0 for $K_0(\underline{\underline{A}})$ so condition K0 above results immediately from the definition (2) of g.

Proof of K1. This will be carried out in several steps.

(i) (cf. (III, §§1-2)). Let E be a finite dimensional k-algebra. Then rad E is nilpotent, and $\bar{E}$ = E/rad E is a finite product of full matrix rings over division algebras. Since k is algebraically closed all division algebras are trivial so that $\bar{E}$ is a finite product of algebras of the form $M_n(k)$.

Any (finite) set of orthogonal idempotents in $\bar{E}$ can be lifted to a set of orthogonal idempotents in E (see (III, 2.10)). In particular, if $e \neq 0$ is an idempotent in E then e is indecomposable $\iff$ its image $\bar{e} \in \bar{E}$ is indecomposable. (e is indecomposable if $e \neq 0$ and e is not the sum of two non zero orthogonal idempotents). In view of the structure of $\bar{E}$ it follows, in particular, that E has no non trivial idempotents (i.e. 1 is indecomposable) $\iff \bar{E} = k$. This implies that E is a (not necessarily commutative) local ring.

(ii) Let $B \neq 0$ be an indecomposable object of $\underline{\underline{A}}$ and $A = B^n$ for some $n > 0$. If $R = \mathrm{End}_{\underline{\underline{A}}}(B)$ then $E = \mathrm{End}_{\underline{\underline{A}}}(A) \simeq M_n(R)$. The remarks above show that R is a local ring with residue class field $\bar{R} = k$, so it follows that $\bar{E} \simeq M_n(k)$. This isomorphism is determined by A up to an inner automorphism. Therefore the determinant,

$$\det: \bar{E} \longrightarrow k,$$

is well defined.

Let C be an indecomposable object not isomorphic to B, and let $B \xrightarrow{h} C \xrightarrow{h'} B$ be morphisms in $\underline{\underline{A}}$. If $h'h$ were not in rad R it would be an automorphism of B, and this would imply that B is a direct summand of C, contradicting indecomposability. Thus $h'h \in$ rad R. It follows, more generally, that if $B^n \xrightarrow{h} C^m \xrightarrow{h'} B^n$ are morphisms in $\underline{\underline{A}}$ then $h'h \in \text{rad End}_{\underline{\underline{A}}}(B^n)$. This is because rad $M_n(R) = M_n$ (rad R), a fact we have used already above. (see (III, 2.6)).

(iii) Let A be any object of $\underline{\underline{A}}$. By the Krull-Schmidt theorem in A (see (I, 3.6)) we can write $A = \amalg A_j$ where each $A_j \simeq B_j^{n_j}$ and the B_j are pairwise non isomorphic indecomposable objects. Moreover any other representation of A as a direct sum of indecomposable subobjects is obtained by applying an automorphism of A to the decomposition above.

Let $E = \text{End}_{\underline{\underline{A}}}(A)$ and let $E_j = \text{End}_{\underline{\underline{A}}}(A_j) \simeq M_{n_j}(R_j)$, where $R_j = \text{End}_{\underline{\underline{A}}}(B_j)$. The decomposition of A above induces a monomorphism of k-algebras,

$$h: \Pi E_j \longrightarrow E,$$

and it depends on the choice of decomposition only up to an inner automorphism of E. We claim now that h induces an isomorphism

$$\overline{h}: \Pi \overline{E}_j \xrightarrow{\simeq} \overline{E}.$$

This amounts to saying that rad E is the sum of all rad $\underline{\underline{A}}(A_i, A_i)$ and of all $\underline{\underline{A}}(A_i, A_j)$ $(i \neq j)$. The second paragraph of (ii) above shows that this is, indeed, an ideal; call it I. It is evident that h induces an isomorphism from $\Pi \overline{E}_j$ to E/I. It remains to be seen that $I \subset$ rad E. For this it suffices to show that if $\alpha \in E$ and if $\alpha \equiv 1$ mod I then α is invertible. Write α in matrix form, $\alpha = (\alpha_{ij})$, $\alpha_{ij} \in \underline{\underline{A}}(A_i, A_j)$. Since each $\alpha_{jj} \equiv 1$ mod rad E_j, the α_{jj} are invertible. Now by elementary column operations we can transform the first row to the form $(\alpha_{11}, 0, \ldots, 0)$. This will alter the α_{jj} $(j > 1)$ by sums of morphisms $A_j \longrightarrow A_j$ which factor through some A_k, $k \neq j$. Therefore, by part (ii) above, the α_{jj} are unaltered modulo rad E_j, for each j. We can therefore pass to the smaller matrix obtained by deleting the first row and column, and continue the process. In the end we will have, by elementary operations, put the matrix in triangular form,

$$\alpha' = \begin{pmatrix} \alpha_{11}' & & 0 \\ & \ddots & \\ * & & \alpha_{nn}' \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \begin{pmatrix} \alpha_{11}' & & 0 \\ & \ddots & \\ 0 & & \alpha_{nn}' \end{pmatrix}$$

Since the α_{jj}' are invertible so also is α'. Since elementary operations are multiplications by invertible matrices, the original α is invertible.

Suppose now that $\alpha \in \mathrm{Aut}_{\underline{\underline{A}}}(A)$. Then its image $\bar{\alpha} \in \bar{E}$ can be written in the $\Pi\, \bar{E}_j$ coordinates as $\bar{\alpha} = (\bar{\alpha}_j)$, $(\bar{\alpha}_j \in \bar{E}_j)$. Recall that $\bar{E}_j \simeq M_{n_j}(k)$ so we have $\det(\bar{\alpha}_j) \in k$. Set

$$g'(A, \alpha) = \Sigma [B_j] \otimes \det(\bar{\alpha}_j) \in K_o(A) \otimes k^*.$$

A priori this definition depends on the decomposition of A chosen. However any two decompositions differ by an inner automorphism of E. This will not affect the isomorphism classes of the B_j's, and it will only change the $\bar{\alpha}_j$'s by a conjugation. Hence $[B_j] \otimes \det(\bar{\alpha}_j)$ is unaltered, for each j, by the new choice, so g' is well defined.

If also $\beta \in \mathrm{Aut}_{\underline{\underline{A}}}(A)$ then we have

$$\begin{aligned} g'(A, \alpha\beta) &= \Sigma [B_j] \otimes \det(\overline{\alpha\beta}_j) \\ &= \Sigma [B_j] \otimes \det(\bar{\alpha}_j\, \bar{\beta}_j) \\ &= \Sigma ([B_j] \otimes \det\bar{\alpha}_j) + ([B_j] \otimes \det\bar{\beta}_j) \\ &= g'(A, \alpha) + g'(A, \beta). \end{aligned}$$

(iv) In view of the last observation we can now finish the proof of K1 for g, and hence of the theorem, by showing that

$$g(A, \alpha) = g'(A, \alpha) \text{ for } (A, \alpha) \in \Sigma\underline{\underline{A}}.$$

First suppose $(B, \beta) \in \Sigma\underline{\underline{A}}$ also. We can decompose $C = A \oplus B$ into a direct sum of indecomposables by first decomposing A and B separately, and then combining these decompositions. The result will be a decomposition $C = \amalg C_j$ as in part (iii), where $C_j \simeq D_j^{n_j} \oplus D_j^{m_j}$, with D_j indecomposable, and with the first summand in A, the second in B. In computing $g'(A \oplus B, \alpha \oplus \beta)$ from this decomposition we will then have $\overline{(\alpha \oplus \beta)}_j = \begin{pmatrix} \overline{\alpha}_j & 0 \\ 0 & \overline{\beta}_j \end{pmatrix}$, in matrix form in $\mathrm{End}_{\underline{\underline{A}}}(C_j)/(\mathrm{rad}\ \mathrm{End}_{\underline{\underline{A}}}(C_j))$, and so $\det\overline{(\alpha \oplus \beta)}_j = \det(\overline{\alpha}_j)\det(\overline{\beta}_j)$. Consequently $g'(A \oplus B, \alpha \oplus \beta) = \Sigma[D_j] \otimes \det\overline{(\alpha \oplus \beta)}_j = \Sigma[D_j] \otimes \det(\overline{\alpha}_j) + \Sigma[D_j] \otimes \det(\overline{\beta}_j) = g'(A, \alpha) + g'(B, \overline{\beta})$.

Now to prove $g = g'$ we first write $(A, \alpha) = \coprod_{a \in k} (A_\alpha(a), \alpha_a)$ as in (1) above. The last paragraph shows that it suffices to prove $g(A_\alpha(a), \alpha_a) = g'(A_\alpha(a), \alpha_a)$ for each a. In other words we can reduce to the case when $(\alpha - a1_A)^n = 0$ for some $a \in k$, and hence $g(A, \alpha) = [A] \otimes a$.

Write $A = \amalg A_j$, $A_j \simeq B_j^{n_j}$ as in (iii). Since we now have $(\alpha - a \cdot 1)^n = 0$ in E $(n > 0)$ it follows that $(\overline{\alpha} - a \cdot 1)^n = 0$ in $\overline{E}$, and hence likewise for each $\overline{\alpha}_j$. It follows that $\overline{\alpha}_j \in M_{n_j}(k)$ has only a as eigenvalue so $\det(\overline{\alpha}_j) = a^{n_j}$. Therefore

$$\begin{aligned} g'(A, \alpha) &= \Sigma[B_j] \otimes a^{n_j} \\ &= \Sigma n_j[B_j] \otimes a \\ &= [\amalg B_j^{n_j}] \otimes a \\ &= [A] \otimes a \\ &= g(A, \alpha). \qquad \text{q.e.d.} \end{aligned}$$

HISTORICAL REMARKS

For the material of the chapter my main sources have been Bass-Heller-Swan [1], Heller [1], and Bass-Murthy [1]. The last two are used principally for the results in §5.

Robert's Theorem was communicated to me directly by Roberts, and I have followed his proof rather closely. Roberts has further shown that, on a projective variety over k, the coherent sheaves and the locally free sheaves have the same K_1.

Part 4

K-THEORY OF PROJECTIVE MODULES

Chapter IX

K-THEORY OF PROJECTIVE MODULES

In this chapter we apply the general theorems of Part 3 to the categories $\underline{\underline{P}}(A)$, of finitely generated projective modules over rings A. We can bring to bear all of the special features of these categories, in particular the structure theory developed in Part 2, in order to obtain information about the groups

$$K_i A = K_i \underline{\underline{P}}(A) \qquad (i = 0, 1).$$

Related categories are also treated. For example, when A is right noetherian we introduce

$$G_i A = K_i \underline{\underline{M}}(A) \qquad (i = 0, 1),$$

and when A is commutative we have the groups

$$K_i \underline{\underline{\mathrm{Pic}}}(A) = \begin{cases} \mathrm{Pic}(A) & \text{if } i = 0 \\ U(A) & \text{if } i = 1. \end{cases}$$

The first two sections establish the basic properties of K_i and G_i and record some exact sequences. In §3 we discuss, for commutative rings A, an exact sequence

$$0 \longrightarrow Rk_0(A) \longrightarrow K_0(A) \xrightarrow{\text{rank}} H_0(A) \longrightarrow 0$$

and a functor

$$\det: \underline{\underline{P}}(A) \longrightarrow \underline{\underline{Pic}}(A).$$

The stability theorems of Chapter IV are interpreted in terms of K_o, in §4. These considerations also allow us to introduce a filtration on K_o from which one can deduce that $Rk_o(A)$ is a nil ideal (when A is commutative). This is a very useful fact.

In §5 we discuss the Mayer-Vietoris sequence of a fibre product, as in Chapter VII. This, together with the exact sequence of a localization (in §6), constitute the two basic tools of the theory. The theorem which makes the Mayer-Vietoris sequence available to us is the following result of Milnor: If

$$\begin{array}{ccc} A & \longrightarrow & A_2 \\ \downarrow & & \downarrow f_2 \\ A_1 & \xrightarrow{f_1} & A' \end{array}$$

is a cartesian square of rings, then the corresponding square

$$\begin{array}{ccc} \underline{\underline{P}}(A) & \longrightarrow & \underline{\underline{P}}(A_2) \\ \downarrow & & \downarrow \\ \underline{\underline{P}}(A_1) & \longrightarrow & \underline{\underline{P}}(A') \end{array}$$

is cartesian, in the sense of Chapter VII, §3, provided f_1 or f_2 is surjective.

In §6 we apply the results of Chapter VIII, §5, to a localization $A \longrightarrow S^{-1}A$. The theorem of Heller then gives us a (G_1, G_o)-exact sequence here. A related exact sequence is also established for (K_1, K_o).

There are two appendices. In §7 we compute the groups $K_i\underline{\underline{FP}}(A)$ $(i = 0, 1)$ where $\underline{\underline{FP}}(A)$ is the category of faithfully projective modules over a commutative ring A, with product $\otimes_A$. In §8 we give a formula in K_o which relates the operations defined by exterior and symmetric powers of a module. These last two sections are used nowhere else in these notes.

§1. DEFINITIONS AND FUNCTORIALITY OF K_iA (i = 0, 1).

The objects of study in this section are the functors

$$K_iA = K_i\underline{\underline{P}}(A) \qquad (i = 0, 1).$$

Here A is a ring and $\underline{\underline{P}}(A)$, we recall, is the category of finitely generated projective right A-modules. We can view $\underline{\underline{P}}(A)$ as a category with the product, $\oplus$, as in Chapter VII, or as an admissible subcategory of the abelian category mod-A, as in Chapter VIII. The two possible definitions of $K_i\underline{\underline{P}}(A)$ arising from these two points of view in fact coincide, because $\underline{\underline{P}}(A)$ is "semi-simple" in the sense of (VIII, §2), i.e. all short exact sequences split. (See Theorem (VII, 2.2)).

A ring homomorphism f: A ⟶ B induces an additive functor $\underline{\underline{P}}(f) = (\cdot \otimes_A B): \underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}(B)$. Since the free modules are cofinal in each $\underline{\underline{P}}$, and since this functor carries free modules to free modules, we obtain an exact sequence as in (VII, 5.3). The relative term, $K_o'(\underline{\underline{P}}(f))$, appearing in that sequence, will be denoted here simply by $K_o'(f)$. We now record the exact sequence.

(1.1) THEOREM. <u>The</u> K_i (i = 0, 1) <u>are functors from rings to abelian groups. A ring homomorphism</u> f: A ⟶ B <u>induces an exact sequence</u>

$$K_1(A) \longrightarrow K_1(B) \longrightarrow K_o'(f) \longrightarrow K_o(A) \longrightarrow K_o(B).$$

Of course this sequence is natural, in an obvious sense, with respect to commutative squares of ring homomorphisms.

When f is the projection onto $B = A/\underline{q}$ for an ideal $\underline{q}$ in A we shall sometimes write $K_o(A, \underline{q})$ in place of $K_o'(f)$. We also write $K_1(A, \underline{q})$ for the group denoted $K_1(\underline{\underline{P}}(A), \underline{\underline{P}}(f))$ in Chapter VII (§2). Recall that it is a Whitehead group constructed from pairs (P, α) in $\Sigma\underline{\underline{P}}(A)$ (i.e. $P \in \underline{\underline{P}}(A)$ and $\alpha \in \mathrm{Aut}_A(P)$) such that $\alpha \otimes_A (A/\underline{q}) = \bar{1}_{P/P\underline{q}}$. In this setting we can strengthen Theorem (1.1) as follows:

(1.2) THEOREM. *Let $\underline{q}$ be a two sided ideal in* A. *Then there is an exact sequence*

$$(1) \qquad K_1(A, \underline{q}) \longrightarrow K_1(A) \longrightarrow K_1(A/\underline{q}) \longrightarrow K_0(A, \underline{q}) \longrightarrow K_0(A) \longrightarrow K_0(A/\underline{q})$$

extending the sequence in (1.1). *The three term sequence of* K_1's *here is naturally isomorphic to*

$$GL(A, \underline{q})/E(A, \underline{q}) \longrightarrow GL(A)/E(A) \longrightarrow GL(A/\underline{q})/E(A/\underline{q}).$$

If $\underline{q}'$ *is an ideal containing* $\underline{q}$ *then there is an exact sequence*

$$(2) \qquad K_1(A, \underline{q}) \longrightarrow K_1(A, \underline{q}') \longrightarrow K_1(A/\underline{q}, \underline{q}'/\underline{q}) \longrightarrow K_0(A, \underline{q}) \longrightarrow K_0(A, \underline{q}') \longrightarrow K_0(A/\underline{q}, \underline{q}'/\underline{q}).$$

Proof. Consider the commutative triangle of functors

$$\begin{array}{ccc} & \underline{\underline{P}}(A/\underline{q}) & \\ \nearrow & & \searrow \\ \underline{\underline{P}}(A) & \longrightarrow & P(A/\underline{q}') \end{array}$$

The exact sequence (1) will follow from (VII, 5.3), and (2) will follow from (VII, 5.4) and (VII, 5.5), provided we verify the "E-surjectivity" conditions in the hypotheses of those theorems. We begin by verifying the condition (10) in Proposition (VII, 5.5). This requires that, given $P \in \underline{\underline{P}}(A)$ and $\alpha \in \mathrm{Aut}_A(P)$ such that

$$\alpha \otimes (A/\underline{q}) \in [\mathrm{Aut}_{A/\underline{q}}(P/P\underline{q}), \mathrm{Aut}_{A/\underline{q}}(P/P\underline{q}, \underline{q}'/\underline{q})],$$

there exists a $Q = P \oplus P' \in \underline{\underline{P}}(A)$ and an

$$\varepsilon \in [\mathrm{Aut}_A(Q), \mathrm{Aut}_A(Q, \underline{q}')]$$

such that $\varepsilon \otimes (A/\underline{q}) = (\alpha \otimes (A/\underline{q})) \oplus 1_{P'/P'\underline{q}}$. Here $\mathrm{Aut}_A(Q, \underline{q}) = \mathrm{Ker}(\mathrm{Aut}_A(Q) \longrightarrow \mathrm{Aut}_{A/\underline{q}}(Q/Q\underline{q}))$, and $\mathrm{Aut}_{A/\underline{q}}(P/P\underline{q}, \underline{q}'/\underline{q})$ is defined similarly.

Since the free modules in $\underline{\underline{P}}$ are cofinal it suffices to establish the above condition for free modules P, P′, etc. In this case it reads: Given an $\alpha \in GL_n(A)$ whose image mod $\underline{q}$ lies in $[GL_n(A/\underline{q}), GL_n(A/\underline{q}, \underline{q}'/\underline{q})]$, we can find an $m \geq 0$ and an $\varepsilon \in [GL_{n+m}(A), GL_{n+m}(A, \underline{q})]$ such that $\varepsilon \equiv \alpha \oplus I_m$ mod $\underline{q}$. If we pass to the limit over n we see that it is sufficient to show that

$$[GL(A), GL(A, \underline{q}')] \longrightarrow [GL(A/\underline{q}), GL(A/\underline{q}, \underline{q}'/\underline{q})]$$

is surjective. According to (V, 2.1) we have $[GL(A), GL(A, \underline{q}')] = E(A, \underline{q}')$, and $E(A, \underline{q}') \longrightarrow E(A/\underline{q}, \underline{q}'/\underline{q})$ is indeed surjective.

In case $\underline{q}' = A$ the argument above shows that $\underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}(A/\underline{q})$ is E-surjective. It follows that all functors in the triangle above are E-surjective so we obtain the two exact sequences.

Since the free modules are cofinal it follows from (VII, 2.3) that the homomorphisms $GL_n(A, \underline{q}) \longrightarrow K_1(A, \underline{q})$ induce an isomorphism in the limit (over n), $GL(A, \underline{q})/[GL(A), GL(A, \underline{q})] \longrightarrow K_1(A, \underline{q})$. This gives the isomorphism $GL(A, \underline{q})/E(A, \underline{q}) \longrightarrow K_1(A, \underline{q})$ and, in case $\underline{q} = A$, the isomorphism $GL(A)/E(A) \longrightarrow K_1(A)$. These isomorphisms are clearly natural, so we have now established all assertions of the theorem.

We shall now describe the behavior of the groups $K_i(A, \underline{q})$ in some special situations.

(1.3) PROPOSITION. _Assume_ $\underline{q} \subset \mathrm{rad}\, A$.

(0) $K_0(A) \longrightarrow K_0(A/\underline{q})$ _is a monomorphism, and it is an isomorphism if_ A _is_ q-_adically complete. Moreover_ $K_0(A, \underline{q}) = 0$.

(1) _We have_ $GL_1(A, \underline{q}) = 1 + \underline{q}$, _and_ $GL_1(A, \underline{q}) \longrightarrow$

$K_1(A, \underline{q})$ *is an epimorphism. It is an isomorphism if* A *is commutative. Moreover* $K_1(A) \longrightarrow K_1(A/\underline{q})$ *is an epimorphism.*

Proof. According to (III, 2.12) $\underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}(A/\underline{q})$ is injective on isomorphism classes of objects, and bijective if A is $\underline{q}$-adically complete. The first assertion follows from this. Since $\underline{q}M_n(A) \subset \text{rad } M_n(A)$ (see (III, 2.6)) it follows that a matrix over A which is invertible mod $\underline{q}$ is invertible. In particular $GL_n(A) \longrightarrow GL_n(A/\underline{q})$ is surjective for all n, and the inclusion $GL_1(A, \underline{q}) \subset 1 + \underline{q}$ is an equality. The first assertion here implies that $K_1(A) \longrightarrow K_1(A/\underline{q})$ is surjective, so the exact sequence (1) now shows that $K_0(A, \underline{q}) = 0$. The fact that $GL_1(A, \underline{q}) \longrightarrow K_1(A, \underline{q}) = GL(A, \underline{q})/E(A, \underline{q})$ is surjective is just (V, 9.1). If A is commutative then the determinant, $GL(A, \underline{q}) \longrightarrow GL_1(A, \underline{q})$, induces its inverse.

(1.4) PROPOSITION. *Suppose that* A *is semi-local.*

(0) $K_0(A)$ *is a free abelian group of finite rank.*

(1) *For any two sided ideal* $\underline{q}$, $K_1(A) \longrightarrow K_1(A/\underline{q})$ *is surjective. Moreover* $GL_1(A, \underline{q}) \longrightarrow K_1(A, \underline{q})$ *is an epimorphism, and an isomorphism if* A *is commutative.*

Proof. (0) $K_0(A) \longrightarrow K_0(A/\text{rad } A)$ is a monomorphism, by (1.3) (0). Since A/rad A is semi-simple $K_0(A/\text{rad } A)$ is a free abelian group generated by the classes of simple modules. Since a subgroup of a free abelian group is free, and of no larger rank, this proves (0).

(1) If $\alpha \in A$ becomes a unit mod $\underline{q}$, then $\underline{q} + \alpha A = A$ so it follows from (III, 2.8) that $\underline{q} + \alpha$ contains a unit. Thus $U(A) \longrightarrow U(A/\underline{q})$ is surjective, where U denotes "units". Applying this to the matrix algebras over A we find that $GL_n(A) \longrightarrow GL_n(A/\underline{q})$ is surjective for all n, so $K_1(A) \longrightarrow K_1(A/\underline{q})$ is also surjective. According to (V, 9.1) we have $GL_n(A, \underline{q}) = GL_1(A, \underline{q})\ E_n(A, \underline{q})$ for all n. Hence $GL_1(A, \underline{q}) \longrightarrow K_1(A, \underline{q})$ is an epimorphism. If A is commutative then det: $K_1(A, \underline{q}) \longrightarrow GL_1(A, \underline{q})$ is its inverse.

(1.5) PROPOSITION. *Let $\underline{q}_1$ and $\underline{q}_2$ be two sided ideals of* A *such that* $\underline{q}_1 \cap \underline{q}_2 = 0$. *Then*

$$K_1(A, \underline{q}_1 + \underline{q}_2) = K_1(A, \underline{q}_1) \oplus K_1(A, \underline{q}_2),$$

and the map $K_1(A, \underline{q}_1) \longrightarrow K_1(A/\underline{q}_2, \underline{q}_1 + \underline{q}_2/\underline{q}_2)$ *is an isomorphism.*

Proof. Since $\underline{q}_1 \cap \underline{q}_2 = 0$ it follows that $\underline{q}_1\underline{q}_2 = 0 = \underline{q}_2\underline{q}_1$ and so $GL(A, \underline{q}_1 + \underline{q}_2) = GL(A, \underline{q}_1) \times GL(A, \underline{q}_2)$ (direct product). A similar decomposition holds for $E(A, \underline{q}_1 + \underline{q}_2)$. because the direct product $E(A, \underline{q}_1) \times E(A, \underline{q}_2)$ is normal in GL(A) and contains all $(\underline{q}_1 + \underline{q}_2)$-elementary matrices. Since $K_1(A, \underline{q}_1 + \underline{q}_2) = GL(A, \underline{q}_1 + \underline{q}_2)/E(A, \underline{q}_1 + \underline{q}_2)$, both assertions of the proposition are now clear.

Sometimes the K_i behave like contravariant functors. For example, if $f: A \longrightarrow B$ makes B a finitely generated projective right A-module, then the restriction of scalars from mod-B to mod-A induces a functor res: $\underline{\underline{P}}(B) \longrightarrow \underline{\underline{P}}(A)$. Then phenomenon occurs, more generally, as follows.

Let $\underline{\underline{H}}(A)$ denote the category of modules having finite $\underline{\underline{P}}(A)$-resolutions (see (III, §6)). According to (VII, 4.2) the inclusion $\underline{\underline{P}}(A) \subset \underline{\underline{H}}(A)$ induces isomorphisms

$$(1.6) \qquad K_i(A) = K_i(\underline{\underline{P}}(A)) \xrightarrow{\ \approx\ } K_i(\underline{\underline{H}}(A)) \qquad (i = 0, 1).$$

Now suppose above that $B \in \underline{\underline{H}}(A)$ as a right A-module. Then it follows from (I, 6.9) that restriction induces a functor res: $\underline{\underline{H}}(B) \longrightarrow \underline{\underline{H}}(A)$. Hence we can define res: $K_i(B) \longrightarrow K_i(A)$ so that the diagram

$$(1.7) \qquad \begin{array}{ccc} K_i(B) & \xrightarrow{\ res\ } & K_i(A) \\ \downarrow \approx & & \downarrow \approx \\ K_i(\underline{\underline{H}}(B)) & \xrightarrow[res]{} & K_i(\underline{\underline{H}}(A)) \end{array}$$

commutes.

Let R be a commutative ring and let A and B be R-algebras. Then $\otimes_R$ defines an additive bifunctor

$$\otimes_R : \underline{\underline{P}}(A) \times \underline{\underline{P}}(B) \longrightarrow P(A \otimes_R B).$$

Using this we can make $K_o(R)$ a commutative ring (when A = R = B) and we can then further make $K_o(A)$ and $K_1(A)$ $K_o(R)$-modules. Moreover we obtain pairings $K_i(A) \times K_j(B) \longrightarrow K_{i+j}(A \otimes_R B)$ (i = 0 and j = 0 or 1) which are $K_0(R)$-bilinear. To illustrate these structures, suppose $P \in \underline{\underline{P}}(R)$, $Q \in \underline{\underline{P}}(A)$, and $\alpha \in \mathrm{Aut}_A(Q)$. Then

$$[P]_R \, [Q]_A = [P \otimes_R Q]_A \in K_o(A)$$

$$[P]_R \, [Q, \alpha]_A = [P \otimes_R Q, 1_P \otimes \alpha]_A \in K_1(A).$$

Similarly, if $\underline{q}$ is a two sided A ideal and $\alpha \in \mathrm{Aut}_A(Q, \underline{q})$ we thus make $K_1(A, \underline{q})$ also into a $K_o(R)$-module. If f: A ⟶ B is an R-algebra homomorphism then $K_o{}'(f)$ is a $K_o(R)$-module via the action $[P]_R \, [Q_1, \alpha, Q_2]' = [P \otimes_R Q_1, 1_P \otimes_R \alpha, P \otimes_R Q_2]'$, where $Q_i \in \underline{\underline{P}}(A)$ and $\alpha: Q_1 \otimes_A B \longrightarrow Q_2 \otimes_A B$ is an isomorphism. Moreover the exact sequence of (1.1) (or of (1.3)) is then an exact sequence of $K_o(R)$-modules. The restriction homomorphism (1.7) is likewise $K_o(R)$-linear, when defined.

If R ⟶ S is a homomorphism of commutative rings then the functor $\otimes_R S: \underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}(A \otimes_R S)$ is naturally isomorphic to the functor $\otimes_A (A \otimes_R S)$. In case S is a finitely generated projective R-module then we have the restriction homomorphism res: $K_i(A \otimes_R S) \longrightarrow K_i(A)$, and the following proposition is evident.

(1.8) PROPOSITION. <u>Let</u> A <u>and</u> S <u>be</u> R-<u>algebras with</u> S <u>commutative, and a finitely generated projective</u> R-<u>module</u>. <u>Then the composite</u>

$$K_i(A) \longrightarrow K_i(A \otimes_R S) \xrightarrow{\text{res}} K_i(A)$$

is multiplication by $[S]_R \in K_o(R)$. Hence $\text{Ker}(K_i(A) \longrightarrow K_i(A \otimes_R S))$ is a $K_o(R)$-module annihilated by $[S]_R$.

§2. G_i, AND THE CARTAN HOMOMORPHISMS $K_i \longrightarrow G_i$ (i = 0, 1)

For a right noetherian ring A we introduce the groups

$$G_i(A) = K_i(\underline{\underline{M}}(A)) \qquad (i = 0, 1).$$

Here $\underline{\underline{M}}(A)$ is the abelian category of all finitely generated right A-modules. The inclusion $\underline{\underline{P}}(A) \subset \underline{\underline{M}}(A)$ induces homomorphisms

$$(1) \qquad C_i(A)\colon K_i(A) \longrightarrow G_i(A) \quad (i = 0, 1)$$

which we call the Cartan homomorphisms. Recall from (1.5) that $K_i(A) \longrightarrow K_i(\underline{\underline{H}}(A))$, is an isomorphism, where $\underline{\underline{H}}(A)$ is the category of modules with finite $\underline{\underline{P}}(A)$-resolutions. We have $\underline{\underline{P}}(A) \subset \underline{\underline{H}}(A) \subset \underline{\underline{M}}(A)$, and A is called right regular (see (III, §6)), if $\underline{\underline{H}}(A) = \underline{\underline{M}}(A)$. Thus:

(2.1) PROPOSITION. If A is right regular then the Cartan homomorphisms (1) are isomorphisms.

(2.2) COROLLARY. Let A be a right regular ring. If $P \in \underline{\underline{P}}(A)$ and if $\alpha \in \text{Aut}_A(P)$ is unipotent then $[P, \alpha] = 0$ in $K_1(A)$. If J is a nilpotent two sided ideal in A then $GL(A, J) \subset E(A)$, and $K_1(A) \longrightarrow K_1(A/J)$ is an isomorphism.

Proof. By (VIII, 3.2) $[P, \alpha]$ goes to zero in $G_1(A)$, so the first assertion follows from (2.1). Since GL(A, J) consists of unipotents it goes to zero in $K_1(A) = GL(A)/E(A)$. Thus $K_1(A) \longrightarrow K_1(A/J)$ is injective, and (1.3) implies it is surjective.

Remark. In case A is commutative the corollary implies $GL(A, J) \subset SL(A)$. But the image of GL(A, J) under det: $GL(A) \longrightarrow U(A)$ (the group of units) is $1 + J$. This shows that $J = 0$; i.e. a commutative regular ring has no non zero nilpotent ideals. Thus, of course, is well known. In fact such a ring is locally a unique factorization domain.

Let f: A ⟶ B be a homomorphism of right noetherian rings. The functor $\otimes_A B: \underline{\underline{M}}(A) \longrightarrow \underline{\underline{M}}(B)$ is not generally exact. If it is, i.e. if B is <u>flat</u> (as left A-module), then we obtain induced homomorphisms

(1) $G_i(A) \longrightarrow G_i(B) \qquad (i = 0, 1).$

More generally, if $Tor_n^A(M, B) = 0$ for all A-modules M and all sufficiently large n then one can still define the homomorphism (1) by the formulas

$$[M]_A \longmapsto \Sigma(-1)^i\ [Tor_i^A(M, B)]_B \in K_o(B),$$

$$[M, \alpha]_A \longmapsto \Sigma(-1)^i\ [Tor_i^A(M, B),\ Tor_i^A(\alpha, B)]_B$$

$$\in K_1(B).$$

In any case, when the homomorphisms (1) are defined, the Cartan homomorphisms are natural transformations, i.e. the diagrams

$$\begin{array}{ccc} K_i(A) & \longrightarrow & K_i(B) \\ {\scriptstyle C_i(A)}\downarrow & & \downarrow{\scriptstyle C_i(B)} \\ G_i(A) & \longrightarrow & G_i(B) \end{array} \qquad (i = 0, 1)$$

commute. This is easily verified.

In case B is a finitely generated right A-module we have a restriction functor res: $\underline{\underline{M}}(B) \longrightarrow \underline{\underline{M}}(A)$, and this induces

$$\text{res}: G_i(B) \longrightarrow G_i(A).$$

If B is also A-projective then res: $K_i(B) \longrightarrow K_i(A)$ is defined, and again the Cartan homomorphisms are natural.

(2.3) PROPOSITION. <u>Let</u> A <u>be a right noetherian ring and let</u> J <u>be a nilpotent two sided ideal in</u> A. <u>Then the</u>

restriction homomorphisms

$$G_i(A/J) \longrightarrow G_i(A) \qquad (i = 0, 1)$$

are isomorphisms.

Proof. If $M \in \underline{\underline{M}}(A)$ then $M \supset MJ \supset MJ^2 \supset \ldots$ is a finite and characteristic filtration of M whose successive quotient, MJ^i/MJ^{i+1}, lie in $\underline{\underline{M}}(A/J)$. The proposition therefore follows from (VIII, 3.3).

The remainder of this section is devoted to a discussion of the comportment of $G_i(A)$ when A is Artinian.

(2.4) A is semi-simple: Then $\underline{\underline{P}}(A) = \underline{\underline{M}}(A)$ and $K_o(A) = G_o(A)$ is a free abelian group with a canonical basis, $[S_1]$, ..., $[S_n]$, determined up to order, where the S_i represent the isomorphism classes of simple A-modules. A itself is isomorphic to a product of full matrix algebras over the division algebras $D_i = \text{End}_A(S_i)$, and $K_1(A)$ is the direct sum of the commutator factor groups, $D_i^*/[D_i^*, D_i^*]$ (see (VIII, 3.4)).

(2.5) A is Artinian: Write $\bar{A} = A/\text{rad } A$ and $\bar{M} = M \otimes_A \bar{A} = M/(M \cdot \text{rad } A)$ for $M \in$ mod-A. Since rad A is nilpotent it follows from (III, 2.12) that $\underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}(\bar{A})$ is bijective on isomorphism classes. Thus, there exist $P_1, \ldots, P_n \in \underline{\underline{P}}(A)$, determined uniquely up to isomorphism and order, such that $\bar{P}_1, \ldots, \bar{P}_n$ represent the isomorphism classes of simple $\bar{A}$-modules. We see thereby that $K_o(A)$ is free abelian with basis $[P_1], \ldots, [P_n]$ and that $K_o(A) \longrightarrow K_o(\bar{A})$ is an isomorphsim. Moreover $K_1(A) \longrightarrow K_1(\bar{A})$ is an epimorphism (see (1.3)). By "restriction" we can identify $\underline{\underline{M}}(\bar{A})$ with the category of semi-simple objects in $\underline{\underline{M}}(A)$. Therefore it follows from (VIII, 3.3) that res: $G_i(\bar{A}) \longrightarrow G_i(A)$ $(i = 0, 1)$ are isomorphisms. With the aid of (2.4) we can thus determine the groups $G_i(A) \simeq G_i(\bar{A})$. The Cartan homomorphism $C_o(A) : K_o(A) \longrightarrow G_o(A)$ is defined by the matrix $(C_{ij})_{1 \leq i,\ j \leq n}$, where

$$C_o(A)\ [P_i] = \Sigma_{1 \leq j \leq n}\ C_{ij}[\overline{P}_j] \qquad (1 \leq i \leq n).$$

This matrix over $\underline{\underline{Z}}$, which is determined up to conjugation by a permutation matrix (resulting from a reordering of the P_i's), is call the *Cartan matrix* of A. The coefficient C_{ij}, is just the multiplicity of $\overline{P}_j$ as a factor in a Jordan-Holder series for P_i.

(2.6) *Base change*. An Artin ring A will be called *basically commutative* if $\overline{A} = A/\text{rad } A$ is a finite product of full matrix algebras over fields (not just division rings). In this connection we quote (see Bourbaki [2]):

(2.7) THEOREM (Wedderburn). *A finite ring* A *is basically commutative*.

The structure theory for Artin rings reduces this theorem immediately to the case when A is a division ring.

Our interest in this notion is explained by the next proposition.

(2.8) PROPOSITION. *Let* A *be a finite dimensional algebra over a field* R, *and let* L *be a field extension of* R. *Then*

$$K_o(A) \longrightarrow K_o(A \otimes_R L) \text{ and } G_o(A) \longrightarrow G_o(A \otimes_R L)$$

are monomorphisms. If A *is basically commutative and if* L *is separable over* R *then they are split monomorphisms*.

Proof. The vertical arrows in

$$\begin{array}{ccc} K_o(A) & \longrightarrow & K_o(A \otimes_R L) \\ \big\downarrow & & \big\downarrow \\ K_o(\overline{A}) & \longrightarrow & K_o(\overline{A} \otimes_R L) \end{array} \quad \text{and} \quad \begin{array}{ccc} G_o(A) & \longrightarrow & G_o(A \otimes_R L) \\ \big\uparrow & & \big\uparrow \\ G_o(\overline{A}) & \longrightarrow & G_o(\overline{A} \otimes_R L) \end{array}$$

are isomorphisms, thanks to (1.3) and (2.3), respectively. Hence we can replace A by $\bar{A}$ and assume A is semi-simple. Decomposing A into a product we can then further reduce to the case when A is simple, say $A = M_n(D)$ where D is a division algebra. In this case $K_o(A) = G_o(A) \simeq \underline{\underline{Z}}$, and both homomorphisms in question are non zero with values in free abelian groups. Hence they are monomorphisms.

If A is basically commutative then D above is a finite field extension of R. The separability of L over R implies that $D \otimes_R L = \Pi D_i$, a finite product of fields D_i. Then $A \otimes_R L = M_n(D \otimes_R L) = \Pi M_n(D_i)$ is semi-simple. If S is the simple A-module then $A \simeq S^n$ so $A \otimes_R L \simeq (S \otimes_R L)^n$. It follows that $S \otimes_R L$ is the direct sum of the simple $A \otimes_R L$-modules, each with multiplicity one. Relative to the bases given by simple modules, $K_o(A) \longrightarrow K_o(A \otimes_R L)$ is therefore represented by the matrix (1, 1,..., 1). This clearly represents a split monomorphism. Since $G_o = K_o$ in this case the proof is complete.

The situation for K_1 and G_1 is more complicated. In the first place, even though $G_1(\bar{A}) \longrightarrow G_1(A)$ is still an isomorphism, $K_1(A) \longrightarrow K_1(\bar{A})$ need not be one. More serious, however, is the fact that matters remain unclear even when A is semi-simple (so that K_1 and G_1 coincide). The problem here quickly reduces to the case of a division algebra D. If then follows from Dieudonne's Theorem (see (V, §9)) that $K_1(D) \simeq D^*/[D^*, D^*]$. Suppose, for simplicity, that R is the center of D. It is then known that we can choose a finite (even galois) extension L of R such that $D \otimes_R L \simeq M_n(L)$, where $n^2 = [D:R]$. Then the determinant defines an isomorphism $K_1(D \otimes_R L) \xrightarrow{\det} L^*$. The homomorphism $K_1(D) \longrightarrow K_1(D \otimes_R L)$ then corresponds to a homomorphism $D^* \longrightarrow L^*$ which we discussed in (III, §8); it is called the <u>reduced norm</u>. In fact, its image lies in $R^* \subset L^*$, and the resulting homomorphism $D^* \longrightarrow R^*$ is independent of L. Thus in order that $K_1(D) \longrightarrow K_1(D \otimes_R L)$ be a monomorphism in the case above it is necessary that the kernel of the reduced norm be exactly $[D^*, D^*]$. When R is a number field this is the case, according to a theorem of Wang (see (V, 9.7)). Of course it will also be true if D is commutative. From the latter one can easily deduce the following general result: Let A be a basically commutative R-algebra. Then $G_1(A) \longrightarrow G_1(A \otimes_R L)$ is a monomorphism. If, further, A is right regular, then $K_1(A) \longrightarrow K_1(A \otimes_R L)$ is an isomorphism. We leave the proof of this as an excercise.

(2.9) EXAMPLE. If A is a local Artin ring (i.e. $\bar{A}$ is a division ring) then the Cartan matrix of A is the one by one matrix $(\ell_A(A))$, where $\ell_A(M)$ is the length (of a Jordan-Holder series) of an A-module M. If A is a product of local rings then the Cartan matrix is diagonal with positive diagonal entries. In particular it has positive determinant. The latter case covers all commutative Artin rings A.

(2.10) EXAMPLE. If A is a regular Artin ring then (2.1) implies the Cartan homomorphisms are isomorphisms. It follows that the Cartan matrix has determinant ± 1 in this case. Moreover (2.2) implies that $K_1(A) \longrightarrow K_1(\bar{A})$ is an isomorphism.

As an exercise, let R be a field and show that $A = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in R\}$ is regular with Cartan matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

§3. RANK: $K_0 \longrightarrow H_0$ AND DET: $\underline{P} \longrightarrow \underline{\underline{Pic}}$.

In this section all rings will be commutative.

Let A be a commutative ring and let X = spec(A). From (III, §3) we know that X is quasi-compact and that its lattice of open and closed subsets is isomorphic, via $e \mapsto \operatorname{supp}(eA)$, to the lattice of idempotents $e \in A$ (III, 3.14). We now introduce

$$H_0(A) = \{\text{continuous functions: } \operatorname{spec}(A) \longrightarrow \underline{\underline{Z}}\},$$

where $\underline{\underline{Z}}$ is given the discrete topology. If $r \in H_0(A)$ it follows from quasi-compactness that r is bounded, i.e. takes only finitely many values. Thus, the $X_n = r^{-1}\{n\}$ are disjoint open sets, almost all empty, whose union is X. If e_n is the indempotent such that $X_n = \operatorname{supp}(e_n A)$ then the e_n are orthogonal and almost all zero, and $1 = \Sigma e_n$.

A ring homomorphism f: A ⟶ B induces a continuous map ${}^a f\colon \operatorname{spec}(B) \longrightarrow \operatorname{spec}(A)$, ${}^a f(\underline{p}) = f^{-1}(\underline{p})$, and hence a ring homomorphism $H_0(f)\colon H_0(A) \longrightarrow H_0(B)$. It is easily deduced from the remarks above that:

(3.1) LEMMA. $H_0(f)\colon H_0(A) \longrightarrow H_0(B)$ <u>is injective if and only if</u> Ker(f) <u>contains no non zero idempotents. It is surjective if and only if every set of orthogonal idempotents in</u> B <u>lifts to such a set in</u> A.

According to (III, 7.1) we have $[P\colon A] \in H_0(A)$ for $P \in \underline{\underline{P}}(A)$, and (III, 7.2) implies that this induces a ring homomorphism, which we call <u>rank</u>,

$$rk\colon K_0(A) \longrightarrow H_0(A).$$

We shall write

$$Rk_0(A) = \operatorname{Ker}(rk).$$

Its elements are of the form [P] - [Q] where [P:A] = [Q:A]. It follows from (III, 7.3) that rk is a natural transformation and hence that $Rk_0(A)$ is a covariant functor of A.

We shall now construct a (natural) right inverse, $\varepsilon\colon H_0(A) \longrightarrow K_0(A)$, for rk. If $e^2 = e$ in A write r_e for the characteristic function of supp(eA). These functions additively generate $H_0(A)$. We propose to define

$$\varepsilon(\Sigma\, n_i r_{e_i}) = \Sigma\, n_i [e_i A].$$

The argument used in integration theory to show that a similar definition of the integral of step functions is well defined, shows that this ε is well defined, and is a ring homomorphism. Moreover $rk[eA] = r_e$ so ε is a right inverse for rk. If $f: A \longrightarrow B$ is a homomorphism then $H_0(f)$ carries r_e to $r_{f(e)}$, and $K_0(A) \longrightarrow K_0(B)$ sends $[eA]_A$ to $[f(e)B]_B$; thus ε is natural. We summarize:

(3.2) PROPOSITION. _The exact sequence_

$$0 \longrightarrow Rk_0(A) \longrightarrow K_0(A) \xrightarrow{rk} H_0(A) \longrightarrow 0$$

is natural with respect to A. _It is split by a ring homomorphism_, $\varepsilon: H_0(A) \longrightarrow K_0(A)$, _which is also natural, and whose image is additively generated by all_ [eA], e _an idempotent in_ A.

Next we treat $\underline{\underline{Pic}}(A)$ as a category with product $(\otimes_A)$ in the sense of Chapter VII. Evidently

$$K_0\underline{\underline{Pic}}(A) = Pic(A).$$

If $P \in \underline{\underline{Pic}}(A)$ then $End_A(P) = A$, and hence $Aut_A(P) = U(A)$, an abelian group. The single object A is cofinal in $\underline{\underline{Pic}}(A)$ so (VII, 2.2) inplies

$$K_1\underline{\underline{Pic}}(A) = U(A).$$

If $f: A \longrightarrow B$ is a ring homomorphism the functor $\otimes_A B$: $\underline{\underline{Pic}}(A) \longrightarrow \underline{\underline{Pic}}(B)$ is product preserving and cofinal. Moreover, it is E-surjective, since this condition involves liftability of automorphisms in commutator subgroups, and all automorphism groups in $\underline{\underline{Pic}}$ are abelian. Similary, if $q \subset q'$ are ideals in A then, for the same reason, the diagram

$$\begin{array}{ccc} & \underline{\underline{\mathrm{Pic}}}(A/\underline{q}) & \\ \nearrow & & \searrow \\ \underline{\underline{\mathrm{Pic}}}(A) & \longrightarrow & \underline{\underline{\mathrm{Pic}}}(A/\underline{q}') \end{array}$$

satisfies the hypotheses of (VII, 5.4) and (VII, 5.5). We shall denote the 0^{th} relative group of f: A ⟶ B by Pic(f), or Pic(A, $\underline{q}$) if f is the projection modulo an ideal $\underline{q}$. If $F = \otimes_A B$: $\underline{\underline{\mathrm{Pic}}}(A) \longrightarrow \underline{\underline{\mathrm{Pic}}}(B)$ then it is easy to see that the relative group denoted $K_1(\underline{\underline{\mathrm{Pic}}}(A), F)$ is just

$$\begin{aligned} U(A, \underline{q}) &= \mathrm{Ker}(U(A) \longrightarrow U(B)) \\ &= \mathrm{Ker}(U(A) \longrightarrow U(A/\underline{q})), \end{aligned}$$

where $\underline{q}$ = Ker(f). With this notation we now summarize the results of (VII, §5) alluded to above.

(3.3) THEOREM. *A ring homomorphism* f: A ⟶ B *with kernel* $\underline{q}$ *induces an exact sequence*

(1) $$0 \longrightarrow U(A, \underline{q}) \longrightarrow U(A) \longrightarrow U(B) \longrightarrow \mathrm{Pic}(f) \longrightarrow \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(B).$$

(We write Pic(A, $\underline{q}$) for Pic(f) if f is surjective). *If* $\underline{q}'$ *is an ideal containing* $\underline{q}$ *then we have an exact sequence*

(2) $$0 \longrightarrow U(A, \underline{q}) \longrightarrow U(A, \underline{q}') \longrightarrow U(A/\underline{q}, \underline{q}'/\underline{q}) \longrightarrow \mathrm{Pic}(A, \underline{q}) \longrightarrow \mathrm{Pic}(A, \underline{q}') \longrightarrow \mathrm{Pic}(A/\underline{q}, \underline{q}'/\underline{q}).$$

(3.4) PROPOSITION. *Assume* $\underline{q} \subset \mathrm{rad}\, A$. *Then* Pic(A) ⟶ Pic(A/$\underline{q}$) *is a monomorphism, and an isomorphism if* A *is* $\underline{q}$*-adically complete. Moreover* U(A) ⟶ U(A/$\underline{q}$) *is an epimorphism so* Pic(A, $\underline{q}$) = 0.

Proof. This follows exactly as in the proof of its K-analogue, (1.3) above.

(3.5) PROPOSITION. *If* A *is semi-local then, for all ideals* $\underline{q}$, Pic(A) = 0 = Pic(A, $\underline{q}$), *and* U(A) ⟶ U(A/$\underline{q}$) *is surjective*.

Proof. The vanishing of Pic(A) follows, for example, from Serre's Theorem (IV, 2.7), and the surjectivity of $U(A) \longrightarrow U(A/\underline{q})$ follows from (IV, 2.9). The exact sequence (1) now implies $\text{Pic}(A, \underline{q}) = 0$.

The next objective is to construct a product preserving functor

$$\det: \underline{\underline{P}}(A) \longrightarrow \underline{\underline{\text{Pic}}}(A).$$

If $[P:A] = r$ then $\det(P) = \Lambda^r P$, the r^{th} exterior power. However we must make some preliminary remarks to explain this when r is not a constant function on spec(A).

Recall that the exterior algebra of $M \in$ mod-A is a graded anti-commutative algebra,

$$\Lambda M = \Lambda^0 M \oplus \Lambda^1 M \oplus \Lambda^2 M \oplus \dots,$$

over $A = \Lambda^0 M$. Moreover $M = \Lambda^1 M \subset \Lambda M$ is universal among A-linear maps $h: M \longrightarrow \Lambda'$, where Λ' is an A-algebra and $h(x)^2 = 0$ for all $x \in M$. From this universal mapping property it is easy to establish a natural isomorphism

$$\Lambda(M \oplus N) \simeq \Lambda(M) \otimes_A \Lambda(N),$$

where the right side is a tensor product in the sense of graded algebras. In particular, for each $r \geq 0$, there is a natural isomorphism of A-modules,

$$\Lambda^r(M \oplus N) \simeq \coprod_{0 \leq i \leq r} \Lambda^i(M) \otimes_A \Lambda^{r-i}(N). \tag{3}$$

Moreover $\Lambda^i(A) = 0$ if $i > 1$. Thus (3) implies

$$\Lambda^r(A^n) \simeq A^{c_{n,r}},$$

where $c_{n,r}$ is the (binomial) coefficient of t^r in $(1 + t)^n$. In particular

$$\Lambda^n(A^n) \simeq A, \text{ and } \Lambda^r(A^n) = 0 \text{ if } r > n. \tag{4}$$

Suppose $1 = \Sigma e_i$ where the e_i are orthogonal idempotents in A. Then for each $M \in$ mod-A we have a canonical identification, $M = \coprod Me_i$. In particular, for any integer $r \geq 0$ we have $\Lambda^r M = \coprod \Lambda^r(M)e_i$. (We do not write $\Lambda^r(Me_i)$, which differs from $\Lambda^r(M)e_i$ when $r = 0$).

Next suppose r is any continuous function from spec(A) to $\underline{\underline{Z}}$ taking only non-negative values. Then we can write $1 = \Sigma e_i$ as above so that r is constant, say with value r_i, on $\mathrm{supp}(e_i A)$, for each i. We then set

$$\Lambda^r M = \Lambda^{r_i}(M)e_i.$$

If we replace $1 = \Sigma e_i$ by a finer decomposition then the preceding paragraph shows that the new definition of $\Lambda^r M$ so obtained can be canonically identified with that above. Since any two decompositions have a common refinement we see that $\Lambda^r M$ is well defined, and it is a functor (non additive, of course) of M.

If $f: A \longrightarrow B$ is a ring homomorphism then there is a natural isomorphism, $\Lambda_A(M) \otimes_A B \simeq \Lambda_B(M \otimes_A B)$, i.e. Λ commutes with base change. It follows that there is a natural isomorphism $\Lambda^r_B(M) \otimes_A B \simeq \Lambda^{r'}_B(M \otimes_A B)$, where $r' \in H_o(A)$ is the image of $r \in H_o(A)$. In particular, we have the following compatibility with localization.

$$(5) \qquad \Lambda^r(M)_{\underline{p}} \simeq \Lambda^{r(\underline{p})}(M_{\underline{p}}) \qquad (\underline{p} \in \mathrm{spec}(A)).$$

Finally, we propose to define

$$\det: \underline{\underline{P}}(A) \longrightarrow \underline{\underline{\mathrm{Pic}}}(A), \qquad \det(P) = \Lambda^{[P:A]}(P).$$

Localizing, with the aid of (5), and using (4), we see indeed that $\det(P) \in \underline{\underline{\mathrm{Pic}}}(A)$. If $[P:A] = r$ and $[Q:A] = s$ are constant then $\Lambda^i(P) = 0 = \Lambda^j(Q)$ for $i > r$ and $j > s$. Therefore the isomorphism (3) for Λ^{r+s} in this case becomes

$$(6) \qquad \det(P \oplus Q) \simeq \det(P) \otimes_A \det(Q).$$

There is, in fact, such a natural isomorphism in general. By virtue of the manner in which det is defined, we can choose a decomposition $1 = \Sigma e_i$ so that $[P:A]$ and $[Q:A]$ are both constant on $\mathrm{supp}(e_i A)$ for each i, and then one can easily reduce the construction of (6) to the case of constant rank.

If we restrict the morphisms in $\underline{\underline{P}}(A)$ to isomorphism (a restriction that does not affect the groups $K_i(A) = K_i(\underline{\underline{P}}(A))$ and their associated exact sequences) then det is a functor. Moreover, it is natural in the sense that, if $f: A \longrightarrow B$ is a ring homomorphism, the square

$$\begin{array}{ccc} \underline{\underline{P}}(A) & \xrightarrow{\otimes_A B} & \underline{\underline{P}}(B) \\ {\scriptstyle \det(A)}\downarrow & & \downarrow{\scriptstyle \det(B)} \\ \underline{\underline{\mathrm{Pic}}}(A) & \xrightarrow[\otimes_A B]{} & \underline{\underline{\mathrm{Pic}}}(B) \end{array}$$

commutes up to natural isomorphism. After a partial localization this reduces to the case of modules a constant rank, whereupon it follows from the commutativity of Λ with base change. From this we deduce a morphism of exact sequence

$$
(7) \qquad
\begin{array}{ccccccccc}
K_1(A) & \longrightarrow & K_1(B) & \longrightarrow & K_o{}'(f) & \longrightarrow & K_o(A) & \longrightarrow & K_o(B) \\
\downarrow{\scriptstyle det_1(A)} & & \downarrow{\scriptstyle det_1(B)} & & \downarrow{\scriptstyle det_o(f)} & & \downarrow{\scriptstyle det_o(A)} & & \downarrow{\scriptstyle det_o(B)} \\
U(A) & \longrightarrow & U(B) & \longrightarrow & Pic(f) & \longrightarrow & Pic(A) & \longrightarrow & Pic(B)
\end{array}
$$

If $P \in \underline{\underline{Pic}}(A)$ then $\det(P) = \Lambda^1 P = P$, and if $u \in \mathrm{Aut}_A(P)$, $\det_1(u) = u$. Moreover if $[P, \alpha, Q]_{\underline{\underline{P}}} \in K_o'(f)$ with $P, Q \in \underline{\underline{Pic}}(A)$ then $\det_o(f)\ [P, \alpha, Q]_{\underline{\underline{P}}} = [P, \alpha, Q]_{\underline{\underline{Pic}}}$. It follows that the verticals in (7) are epimorphisms. (They do not split because the inclusion $\underline{\underline{Pic}} \subset \underline{\underline{P}}$ is not product preserving, so it does not induce a homomorphism).

Recall that $K_o(A) \simeq H_o(A) \oplus Rk_o(A)$, where the first term is spanned by all $[eA]$ with $e^2 = e$. Since $[eA:A]$ is zero on $\mathrm{supp}((1 - e)A)$ and one on $\mathrm{supp}(eA)$ we have $\det(eA) = (1 - e)\ (\Lambda^o eA) \oplus e(\Lambda^1 eA) = (1 - e)A \oplus e(eA) = A$. Hence $\det_o(A)$ is trivial on the first term, $H_o(A)$, and we are left with an <u>epimorphism</u>

$$\det_o(A)\colon Rk_o(A) \longrightarrow Pic(A).$$

If $\mathrm{Im}(K_o'(f) \longrightarrow K_o(A)) \subset Rk_o(A)$, i.e. if $H_o(A) \longrightarrow H_o(B)$ is a monomorphism, then we can replace the K_o's by Rk_o's in (7) and preserve exactness. According to (3.1) this happens when Ker(f) contains no non zero indempotents.

(3.6) PROPOSITION. <u>If</u> $f\colon A \longrightarrow B$ <u>is a homomorphism whose kernel contains no non zero idempotents then there is an epimorphism of exact sequences</u>,

$$
\begin{array}{ccccccccc}
K_1(A) & \longrightarrow & K_1(B) & \longrightarrow & K_o'(f) & \longrightarrow & Rk_o(A) & \longrightarrow & Rk_o(B) \\
\downarrow{\scriptstyle det_1(A)} & & \downarrow{\scriptstyle det_1(B)} & & \downarrow{\scriptstyle det_o(f)} & & \downarrow{\scriptstyle det_o(A)} & & \downarrow{\scriptstyle det_o(B)} \\
U(A) & \longrightarrow & U(B) & \longrightarrow & Pic(f) & \longrightarrow & Pic(A) & \longrightarrow & Pic(B)\ .
\end{array}
$$

We shall now interprete what it means for $\det_o$ to be an isomorphism. Recall that A-modules M and N are called *stably isomorphic* if $M \oplus A^n \simeq N \oplus A^n$ for some $n \geq 0$.

(3.7) PROPOSITION. *The following conditions are equivalent*:

(a) $\det_o(A)\colon Rk_o(A) \longrightarrow Pic(A)$ *is an isomorphism*.

(b) *If* $P \in \underline{\underline{P}}(A)$ *has constant rank* $r > 0$ *then* P *is stably isomorphic to* $\det P \oplus A^{r-1}$.

(c) (i) *A projective module of constant rank is stably isomorphic to a direct sum of invertible modules; and*

(ii) *If* P Q $\in \underline{\underline{Pic}}(A)$ *then* $P \oplus Q$ *is stably isomorphic to* $(P \otimes_A Q) \oplus A$.

Proof. (a) $\Rightarrow$ (b). $([P] - [A^r]) - ([\det P] - [A]) \in Rk_o(A)$ has trivial determinant, so (a) implies $[P \oplus A] = [\det P \oplus A^r]$. This clearly implies (b).

(b) $\Rightarrow$ (c). (i) is clear. Part (ii) also follows immediately once we note that $\det (P \oplus Q) = P \otimes_A Q$.

(c) $\Rightarrow$ (a). Condition (ii) implies that $[P]_{\underline{\underline{Pic}}} \longmapsto [P]_{\underline{\underline{P}}} - [A]_{\underline{\underline{P}}}$ defines a homomorphism $h\colon Pic(A) \longrightarrow Rk_o(A)$, and clearly $\det_o(A) \circ h = 1_{Pic(A)}$. Condition (i) implies that h is surjective, and this shows that $\det_o(A)$ is an isomorphism.

(3.8) COROLLARY. *If* max(A) *is a noetherian space of dimension* ≤ 1 *then* $\det_o(A)\colon Rk_o(A) \longrightarrow Pic(A)$ *is an isomorphism*.

Proof. Serre's Theorem (IV, 2.7) implies that a P as in condition (b) above is isomorphic to $L \oplus A^{r-1}$ for some module L, necessarily of rank 1. It follows that det P =

$\det(L \oplus A^{r-1}) \simeq \det L \otimes \det(A^{r-1}) \simeq L \otimes_A A = L$. q.e.d.

We close this section with some remarks about $\det_1$. Under the isomorphism $K_1(A) \simeq GL(A)/E(A)$ we see, from the definition of the usual determinant (see Bourbaki [ ]) that $\det_1(A)$ is induced by det: $GL(A) \longrightarrow U(A)$. The map $U(A) = GL_1(A) \longrightarrow GL(A)$ splits this determinant, so we obtain a canonically split short exact sequence

$$0 \longrightarrow SK_1(A) \longrightarrow K_1(A) \xrightarrow{\det_1(A)} U(A) \longrightarrow 0,$$

where $SK_1(A) = SL(A)/E(A)$. Similarly, for any ideal $\underline{q} \subset A$ we obtain a canonical decomposition

$$K_1(A, \underline{q}) = U(A, \underline{q}) \oplus SK_1(A, \underline{q})$$

From Theorem (1.2) we therefore deduce:

(3.9) PROPOSITION. _If_ $\underline{q} \subset \underline{q}'$ _are ideals in_ A _then there are exact sequences_ $SK_1(A, \underline{q}) \longrightarrow SK_1(A) \longrightarrow SK_1(A/\underline{q})$ _and_ $SK_1(A, \underline{q}) \longrightarrow SK_1(A, \underline{q}') \longrightarrow SK_1(A/\underline{q}, \underline{q}'/\underline{q})$.

(3.10) PROPOSITION. _Let_ $\underline{q}$ _be an ideal in_ A _and let_ $\underline{a} = \mathrm{ann}_A(\underline{q})$. _If either_ $\underline{q} \subset \mathrm{rad}\, A$ _or_ $A/\underline{a}$ _is semi-local then_ $SK_1(A, \underline{q}) = 0$.

Proof. The vanishing of $SK_1(A, \underline{q})$ when $\underline{q} \subset \mathrm{rad}\, A$ follows from (1.3) (1), and its vanishing when A is semi-local follows from (1.4).

Set $\underline{q}_o = \underline{q} \cap \underline{a}$; since $\underline{q} \cdot \underline{a} = 0$ we have $\underline{q}_o^2 = 0$ and hence $\underline{q}_o \subset \mathrm{rad}\, A$. From the exact sequence $0 = SK_1(A, \underline{q}_o) \longrightarrow SK_1(A, \underline{q}) \longrightarrow SK_1(A/\underline{q}_o, \underline{q}/\underline{q}_o)$ we see that it suffices to show that $SK_1(A', \underline{q}') = 0$, where $A' = A/\underline{q}_o$, $\underline{q}' = \underline{q}/\underline{q}_o$. If we set $\underline{a}' = \underline{a}/\underline{q}_o$ then we have $\underline{q}' \cap \underline{a}' = 0$. It follows therefore from (1.5) that $SK_1(A', \underline{q}') \longrightarrow SK_1(A'/\underline{a}', \underline{q}' + \underline{a}'/\underline{a}')$ is a monomorphism (even an isomorphism). Since $A'/\mathfrak{a}' = A/\mathfrak{a}$ is semi-local the last group vanishes. q.e.d.

(3.11) COROLLARY. *Let $\mathfrak{q} \subset \mathfrak{q}'$ be ideals and let* $\mathfrak{a} = \operatorname{ann}_A(\mathfrak{q}'/\mathfrak{q})$. *If* $A/\mathfrak{a}$ *is semi-local then* $SK_1(A, \mathfrak{q}) \longrightarrow SK_1(A, \mathfrak{q}')$ *is an epimorphism.*

Proof. We apply (3.10) to kill $SK_1(A/\mathfrak{q}, \mathfrak{q}'/\mathfrak{q})$ in the exact sequence of (3.9).

§4. THE STABILITY THEOREMS

Throughout this section *we fix a commutative ring* R, *and we shall write*

$$X = \max(R).$$

The support of an R-module M refers here to its support in X: $\operatorname{supp}(M) = \{\mathfrak{m} \in X \mid M_{\mathfrak{m}} \neq 0\}$.

(4.1) PROPOSITION. *Suppose* X *is a noetherian space which is a union of a finite number of subspaces of dimensions* $\leq d$.

(a) *If* $u \in K_0(R)$ *has rank* $\geq d$ *then* $u = [P]$ *for some* $P \in \underline{\underline{P}}(R)$.

(b) *If* $P, Q \in \underline{\underline{P}}(R)$ *and if* $[P: R] > d$ *then* $[P] = [Q] \Rightarrow P \simeq Q$.

Proof. (a) We can write $u = [Q] - [R^n]$ for some $Q \in \underline{\underline{P}}(R)$. Since $[Q: R] = n + \operatorname{rank}(u) \geq n + d$ it follows from Serre's Theorem (IV, 2.7) that $Q \simeq P \oplus R^n$ for some P, so $u = [P]$.

(b) If $[P] = [Q]$ then $P \oplus R^n \simeq Q \oplus R^n$ for some n. If $[P: R] > d$ then the Cancellation Theorem (IV, 3.5) further implies that $P \simeq Q$.

We have a similar result without finiteness assumptions.

(4.2) PROPOSITION. (a) _If_ $u \in K_o(R)$ _has non negative rank then_ $nu = [P]$ _for some_ $n > 0$ _and some_ $P \in \underline{\underline{P}}(R)$. (b). _If_ $P, Q \in \underline{\underline{P}}(R)$ _and if_ $[P] = [Q]$ _then_ $P^n \simeq Q^n$ _for some_ $n > 0$.

Proof. (a) If we restrict to a direct factor of R we can, without loss, assume u has everywhere positive rank. Write $u = [P] - [R^m]$. Then P is defined over a finitely generated subring, $R_o \subset R$, and R_o is noetherian. Moreover u is the image of $u_o = [P_o] - [R_o^m]$ where $P_o \in \underline{\underline{P}}(R_o)$ is such that $P_o \otimes_{R_o} R \simeq P$. For large enough n the rank of nu_o will exceed dim max(R_o) so (4.1) (a) implies $nu_o = [Q_o]$ for some Q_o. Thus $nu = [Q_o \otimes_{R_o} R]$.

(b) Suppose $[P] = [Q]$. We can restrict to a direct factor of R and assume that P is faithful. Moreover there is an isomorphism $h: P \oplus R^m \longrightarrow Q \oplus R^m$ for some m. We can now choose an $R_o \subset R$ large enough so that there exist $P_o, Q_o \in \underline{\underline{P}}(R_o)$ and $h_o: P_o \oplus R_o^m \longrightarrow Q_o \oplus R_o^m$ such that $P = P_o \otimes_{R_o} R$, $Q = Q_o \otimes_{R_o} R$, and $h = h_o \otimes_{R_o} R$. In particular $[P_o] = [Q_o]$ in $K_o(R_o)$, so $[P_o^n] = [Q_o^n]$ for all $n > 0$. With n large enough so that $[P_o^n : R_o] >$ dim max(R_o) we can apply (4.1) (b) to conclude that $P_o \simeq Q_o$, and hence $P \simeq Q$. q.e.d.

Let A be an R-algebra. We propose to introduce a filtration on $K_o(A)$ from whose properties several useful conclusions can be drawn.

Let $C = (\cdots C_n \xrightarrow{d_n} C_{n-1} \cdots)$ and $C' = (\cdots C_n' \xrightarrow{d_n'} C_{n-1}' \cdots)$ be complexes in mod-A and mod-B, respectively, where A and B are R-algebras. Then we can define a complex

$C \otimes_R C'$ in mod-$(A \otimes_R B)$ as follows: $(C \otimes_R C')_n = \coprod_{i+j=n} C_i \otimes_R C_j'$. If $a \in C_i$ and $a' \in C_j'$ then $D(a \otimes a') = da \otimes a' + (-1)^i a \otimes d'a'$. Suppose C is contractible, i.e. there is a morphism $s: C \longrightarrow C$ of degree one such that $sd + ds = 1_C$. Then if $S = s \otimes 1_{C'}$ we have $DS + SD = 1_{C \otimes_R C'}$, so $C \otimes_R C'$ is also contractible. For let $a \otimes a'$ be as above. Then $DS(a \otimes a') = D(sa \otimes a') = dsa \otimes a' + (-1)^{i+1} sa \otimes d'a' = (a - sda) \otimes a' + (-1)^{i+1} sa \otimes d'a' = a \otimes a' - S(da \otimes a' + (-1)^i a \otimes d'a') = a \otimes a' - SD(a \otimes a')$.

If P is a finite complex in $\underline{\underline{P}}(A)$ we shall write

$$\chi(P) = \chi^A(P) = \Sigma(-1)^n [P_n] \in K_0(A).$$

If P is acyclic then $\chi(P) = 0$ (see (VII, 4.1 (b))). If $A \longrightarrow B$ is an algebra homomorphism then $K_0(A) \longrightarrow K_0(B)$ carries $\chi^A(P)$ to $\chi^B(P \otimes_A B)$. If Q is a finite complex in $\underline{\underline{P}}(B)$ then the pairing $K_0(A) \otimes_{K_0(R)} K_0(B) \longrightarrow K_0(A \otimes_R B)$ carries $\chi^A(P) \otimes \chi^B(Q)$ to $\chi^{A \otimes_R B}(P \otimes_R Q)$.

Recall from (III, 4.7) that the homology, H(P), of a finite complex P in $\underline{\underline{P}}(A)$ has closed support,

$$\text{supp}_m(H(P)) \subset X = \max(R).$$

Since localization is exact it commutes with homology. Moreover, if a finite complex in $\underline{\underline{P}}(A)$ is acyclic then it is contractible (see (I, 6.6)). Therefore we can write

$$\begin{aligned}\text{supp}_m(H(P)) &= \{\underline{m} \in X \mid H(P)_{\underline{m}} \neq 0\} \\ &= \{\underline{m} \in X \mid H(P_{\underline{m}}) \neq 0\} \\ &= \{\underline{m} \in X \mid P_{\underline{m}} \text{ is not acyclic}\} \\ &= \{\underline{m} \in X \mid P_{\underline{m}} \text{ is not contractible}\}.\end{aligned}$$

If Q is a finite complex in $\underline{\underline{P}}(B)$ as above then $(P \otimes_R Q)_{\underline{m}} = P_{\underline{m}} \otimes_{R_{\underline{m}}} Q_{\underline{m}}$, and we have seen above that the tensor product is contractible if either factor is. Therefore we conclude that

$$\text{supp}_m(H(P \otimes_R Q)) \subset \text{supp}_m(H(P)) \cap \text{supp}_m(H(Q)).$$

(4.3) DEFINITION OF $F^iK_o(A)$. For $i \geq 0$, $F^iK_o(A)$ is the set of $u \in K_o(A)$ satisfying the following condition: Given a closed set $Y \subset X$, there is a finite complex P in $\underline{\underline{P}}(A)$ such that $\chi(P) = u$ and such that

$$\text{codim}_Y(Y \cap \text{supp}_m(H(P))) \geq i.$$

(4.4) PROPOSITION. (1) *The* $F^iK_o(A)$ *are a descending chain of subgroups with* $F^oK_o(A) = K_o(A)$ *and with* $F^iK_o(A) = 0$ *if* $i > \dim X$.

(2) *If* B *is another* R-*algebra the natural pairing* $K_o(A) \otimes_{K_o(R)} K_o(B) \longrightarrow K_o(A \otimes_R B)$ *induces homomorphisms* $F^iK_o(A) \otimes F^jK_o(B) \longrightarrow F^{i+j}K_o(A \otimes_R B)$ *for all* $i, j \geq 0$. *In particular* $K_o(R)$ *is thus made into a filtered ring, and* $K_o(A)$ *into a filtered* $K_o(R)$-*module*.

(3) *An* R-*algebra homomorphism* $A \longrightarrow B$ *induces homomorphisms* $F^iK_o(A) \longrightarrow F^iK_o(B)$ *for all* $i \geq 0$.

(4) *If* A *is a finite* R-*algebra and if* X *is a noetherian space then*

$$F^1K_o(A) = \bigcap_{\underline{m} \in X} \text{Ker}(K_o(A) \longrightarrow K_o(A_{\underline{m}})).$$

In particular $F^1K_o(R) = Rk_o(R)$ *and* $Rk_o(R)^{d+1} = 0$, *where* $d = \dim X$.

Proof. (1) Let $u, v \in F^iK_o(A)$. To show that $u + v \in$

$F^iK_o(A)$ suppose we are given Y closed in X. If P and Q are finite complexes in $\underline{\underline{P}}(A)$ serving as in definition (4.3) for u and v, respectively, then $P \oplus Q$ serves for u + v, and P(-1) (where $P(-1)_n = P_{n-1}$) serves for -u. The latter is clear because H(P)-1)) = H(P) (-1). The former follows from the fact that $supp(H(P) \oplus H(Q)) = supp(H(P)) \cup supp(H(Q))$ and the fact that the codimension of a union of two closed sets is the minimum of the two codimensions. It is evident that the filtration is decreasing. If $u \in F^iK_o$ and if i > dim X then we can choose P so that $u = \chi(P)$ and $codim_X(X \cap supp(H(P))) \geq i >$ dim X. This implies H(P) = 0, i.e. P is acyclic. Therefore $u = \chi(P) = 0$.

(2) Suppose $u \in F^iK_o(A)$ and $v \in F^jK_o(B)$, and let w be the image in $K_o(A \otimes_R B)$ of $u \otimes v$. We must show that $w \in F^{i+j}K_o(A \otimes_R B)$, so suppose we are given a closed $Y \subset X$. Choose a finite complex P in $\underline{\underline{P}}(A)$ such that $\chi^A(P) = u$ and $codim_Y(Z) \geq i$, where $Z = Y \cap supp_m(H(P))$. Now choose a finite complex Q in $\underline{\underline{P}}(B)$ such that $\chi^B(Q) = v$ and $codim_Z(Z \cap supp(H(Q))) \geq j$. We now contend that the finite complex $P \otimes_R Q$ in $\underline{\underline{P}}(A \otimes_R B)$ satisfies the requirements of (4.3) for w and Y. Evidently $\chi^{A \otimes_R B}(P \otimes_R Q)$ is the image of $\chi^A(P) \otimes \chi^B(Q) = u \otimes v$, and hence equals w. Moreover we have seen above that $supp_m(H(P \otimes_R Q)) \subset supp_m(H(P)) \cap supp_m(H(Q))$. Hence

$$\begin{aligned} & codim_Y(Y \cap supp_m(H(P \otimes_R Q))) \\ & \geq codim_Y(Y \cap supp_m(H(P)) \cap supp_m(H(Q))) \\ & = codim_Y(Z \cap supp_m(H(Q))) \\ & \geq codim_Y(Z) + codim_Z(Z \cap supp_m(H(Q))) \\ & \geq i + j. \end{aligned}$$

(3) Let $A \longrightarrow B$ be an R-algebra homomorphism and let $u \in F^iK_0(A)$. Given Y closed in X choose a complex P for u as in (4.3). Then $\mathrm{supp}_m(H(P \otimes_A B)) \subset \mathrm{supp}_m(H(P))$. This follows, as above, by localizing and using the contractibility of $P_{\underline{m}}$ for $\underline{m} \in \mathrm{supp}_m H(P)$. The image, v, of u in $K_0(B)$ is clearly $\chi^B(P \otimes_A B)$, so $v \in F^iK_0(B)$.

(4) Suppose $u \in F^1K_0(A)$ and $\underline{m} \in X$. Then we can find a complex P such that $\chi(P) = u$ and such that $\mathrm{codim}_{\{\underline{m}\}}(\{\underline{m}\} \cap \mathrm{supp}_m(H(P))) \geq 1$. The last condition implies $\underline{m} \notin \mathrm{supp}(H(P))$, so $P_{\underline{m}}$ is acyclic and hence u goes to zero in $K_0(A_{\underline{m}})$.

Conversely, suppose $u \in \bigcap_{\underline{m} \in X} \mathrm{Ker}(K_0(A) \longrightarrow K_0(A_{\underline{m}}))$. We claim $u \in F^1K_0(A)$, so suppose we are given a closed Y in X. Since S is (now assumed to be) noetherian we can write Y as an irredundant union of irreducible closed subsets, say $Y_1, \ldots, Y_n$. Choose distinct $\underline{m}_i \in Y_i$ $(1 \leq i \leq n)$. If we write $u = [P] - [Q]$ then, by assumption, $[P_{\underline{m}_i}] = [Q_{\underline{m}_i}]$ in $K_0(A_{\underline{m}_i})$ for each i. After adding a large free module to both P and Q we can even assume $P_{\underline{m}_i} \simeq Q_{\underline{m}_i}$ for each i. Since $\mathrm{Hom}_{A_{\underline{m}_i}}(P_{\underline{m}_i}, Q_{\underline{m}_i}) = \mathrm{Hom}_A(P, Q)_{\underline{m}_i}$ we can find $h^i: P \longrightarrow Q$ such that $h^i_{\underline{m}_i}$ is an isomorphism $(1 \leq i \leq n)$. Since the $\underline{m}_i$ are comaximal the Chinese Remainder Theorem gives us an $h \in \mathrm{Hom}_A(P, Q)$ such that $h - h^i \in \underline{m}_i \cdot \mathrm{Hom}_A(P, Q)$, $(1 \leq i \leq n)$. Locally we have $\underline{m}_i A_{\underline{m}_i} \subset \mathrm{rad}\, A_{\underline{m}_i}$ because A is a finite R-algebra. Hence it follows from Nakayama's lemma (see (III, 2.12)), since $h_{\underline{m}_i}$ is congruent to the isomorphism $h^i_{\underline{m}_i}$ mod $\mathrm{rad}\, A_{\underline{m}_i}$, that $h_{\underline{m}_i}$ itself is an isomorphism. Therefore the complex $C = (\cdots 0 \longrightarrow P \xrightarrow{h} Q \longrightarrow 0 \cdots)$, where $Q = C_0$, is acyclic at each $\underline{m}_i$, and clearly $\chi(C) = u$. Since $Z = Y \cap \text{supp}(H(C))$ misses at least one point, $\underline{m}_i$, in each irreducible component Y_i of Y we conclude finally that $\text{codim}_Y(Z) \geq 1$, as required.

It is clear that $Rk_0(R) = \bigcap_{\underline{m} \in X} \text{Ker}(K_0(R) \longrightarrow K_0(R_{\underline{m}}))$. Consequently (1) and (2) imply now that $Rk_0(R)^n \subset F^n K_0(R) = 0$ if $n > \dim X$. q.e.d.

(4.5) COROLLARY. *Suppose* X *is a noetherian space of dimension* $\leq d$. *Let* $P \in \underline{\underline{P}}(R)$ *be faithful, and let* n *be the least common multiple of its local ranks. Then there is a* $Q \in \underline{\underline{P}}(R)$ *such that* $P \otimes_R Q \simeq R^{n^{d+1}}$.

Proof. Clearly we can find $r \in H_0(R)$ such that $r[P: R] = n$. In the decomposition $K_0(R) = H_0(R) \oplus Rk_0(R)$ write $[P] = [P: R] - t$. Then $r[P] = n - rt$. Therefore, modulo the principal ideal $[P]\,K_0(R)$, we have $n \equiv rt$, and $(rt)^{d+1} = 0$ by part (4) of (4.4) above. It follows that $n^{d+1} \in [P]\,K_0(R)$; say $n^{d+1} = [P]u$. Then the rank of u is $\geq n^d \geq d$ so (4.1) (c) implies $u = [Q]$ for some Q. Since $[P \otimes_R Q] = n^{d+1} = R^{n^{d+1}}]$, and since $n^{d+1} > d$, it follows from (4.1) (b) that $P \otimes_R Q \simeq R^{n^{d+1}}$.

Without finiteness assumptions we have:

(4.6) PROPOSITION. $Rk_0(R)$ *is a nil ideal. If* $P \in \underline{\underline{P}}(R)$ *the following conditions are equivalent*:

(1) P *is faithful (and hence faithfully projective in the sense of* (II, §1)).

(2) $[P: R]$ *is everywhere positive*.

(3) *Every* $K_0(R)$*-module annihilated by* $[P]$ *is torsion*.

(4) *There is a* $Q \in \underline{\underline{P}}(R)$ *and an* $n > 0$ *such that*

$P \otimes Q \simeq R^n$.

<u>Proof</u>. Since $K_0(R)$ is the direct limit of the $K_0(R')$, where R' ranges over finitely generated, and hence noetherian, subrings R' of R, the fact that $Rk_0(R)$ is nil follows from the corresponding property of each $Rk_0(R')$ (see (4.4) (4)). The equivalence of (1) and (2) follows from (III, 7.2). If $[P : R]$ is everywhere positive we can solve $r[P : R] = n > 0$ in $H_0(R)$, just as in the proof of (4.5) above. Then $r[P] = n - s$ for an $s \in Rk_0(R)$. Since $s^d = 0$ for some $d > 0$ it follows that $n^d \in [P]\, K_0(R)$, and this implies (3).

Further, (3) implies $t[P] = n$ for some $n > 0$ (apply (3) to $K_0(R)/[P]\, K_0(R)$). Choosing n larger, if necessary, we can force t to have large rank, so that a multiple of t is of the form $[Q]$, by (4.2) (a). Thus, with a further enlargement of n we can solve $[Q]\,[P] = n$ for some $Q \in \underline{\underline{P}}(R)$. Since $[Q \otimes_R P] = [R^n]$ it follows from (4.2) (b) that $(Q \otimes_R P)^m = Q^m \otimes_R P \simeq (R^n)^m = R^{nm}$ for some $m > 0$. This proves (4).

The implication (4) $\Rightarrow$ (1) is trivial, so the proposition is now proved.

(4.7) COROLLARY ("Torsion Criterion"). <u>Let</u> $R \longrightarrow L$ <u>be a monomorphism of commutative rings such that</u> $L \in \underline{\underline{P}}(R)$. <u>Let</u> A <u>be an</u> R-<u>algebra. Then</u>

$$\mathrm{Ker}(K_i(A) \longrightarrow K_i(L \otimes_R A)) \qquad (i = 0, 1)$$

and

$$\mathrm{Ker}(G_i(A) \longrightarrow G_i(L \otimes_R A)) \qquad (i = 0, 1)$$

<u>are torsion groups. If</u> $\max(R)$ <u>is a noetherian space of dimension</u> $\leq d$ <u>then these kernels are annihilated by</u> n^{d+1}, <u>where</u> n <u>is the least common multiple of the local ranks of</u> L <u>over</u> R.

Proof. According to (1.8) there is a homomorphism $K_i(L \otimes_R A) \longrightarrow K_i(A)$ whose composite with $K_i(A) \longrightarrow K_i(L \otimes_R A)$ is multiplication by $[L]_R \in K_o(R)$ on $K_i(A)$ $(i = 0, 1)$. Similarly we have this also for the functors G_i $(i = 0, 1)$. It follows that all the kernels in question are annihilated by $[L]_R$. The assertions of the corollary therefore follow from (4.6) (3) and from (4.5), respectively.

§5. FIBRE PRODUCTS; MILNOR'S THEOREM

Let

$$(1)\qquad \begin{array}{ccc} A & \xrightarrow{h_2} & A_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[f_1]{} & A' \end{array}$$

be a cartesian square of ring homomorphisms. Thus $A = \{(a_1, a_2) \in A_1 \times A_2 \mid f_1a_1 = f_2a_2\}$, and the h_i are induced by the coordinate projections. Writing $\underline{\underline{P}}' = \underline{\underline{P}}(A')$ and $\underline{\underline{P}}_i = \underline{\underline{P}}(A_i)$ $(i = 1, 2)$, we obtain a square of functors

$$(2)\qquad \begin{array}{ccc} \underline{\underline{P}}(A) & \xrightarrow{H_2} & \underline{\underline{P}}_2 \\ {\scriptstyle H_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \underline{\underline{P}}_1 & \xrightarrow[F_1]{} & \underline{\underline{P}}' \end{array} \qquad \beta: F_1H_1 \longrightarrow F_2H_2$$

where $F_i = \otimes_{A_i} A'$ and $H_i = \otimes_A A_i$ $(i = 1, 2)$, and β is the natural isomorphism arising from the isomorphisms $(P \otimes_A A_i) \otimes_{A_i} A' \simeq P \otimes_A A'$ $(i = 1, 2)$.

We also have the fibre product category $\underline{\underline{P}} = \underline{\underline{P}}_1 \times_{\underline{\underline{P}}'} \underline{\underline{P}}_2$ (see (VII, §3)) and the cartesian square

$$(3)\qquad \begin{array}{ccc} \underline{\underline{P}} & \xrightarrow{G_2} & \underline{\underline{P}}_2 \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \underline{\underline{P}}_1 & \xrightarrow[F_1]{} & \underline{\underline{P}}' \end{array}, \quad \alpha\colon F_1G_1 \longrightarrow F_2G_2 .$$

The universal property of (3) implies there is a unique functor

$$T\colon \underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}},$$

$$T(P) = (H_1P,\ \beta_P,\ H_2P),$$

such that $H_i = G_iT$ ($i = 1, 2$) and such that $\beta = \alpha T$.

(5.1) THEOREM (Milnor). _If_ f_1 _or_ f_2 _is surjective then the functor_

$$T\colon \underline{\underline{P}}(A_1 \times_{A'} A_2) \longrightarrow P(A_1) \times_{\underline{\underline{P}}(A')} \underline{\underline{P}}(A_2)$$

is an equivalence.

Proof. Write $\underline{\underline{M}}' = \text{mod-}A'$, $\underline{\underline{M}}_i = \text{mod-}A_i$ ($i = 1, 2$), and $\underline{\underline{M}} = M_1 \times_{\underline{\underline{M}}'} \underline{\underline{M}}_2$. These contain the corresponding categories above, and the terms of diagrams (2) and (3) above can be embedded in the corresponding terms of diagrams

$$\begin{array}{ccc} \text{mod-}A & \xrightarrow{H_2} & \underline{\underline{M}}_2 \\ {\scriptstyle H_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \underline{\underline{M}}_1 & \xrightarrow[F_1]{} & \underline{\underline{M}}' \end{array} \quad \text{and} \quad \begin{array}{ccc} \underline{\underline{M}} & \xrightarrow{G_2} & \underline{\underline{M}}_2 \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \underline{\underline{M}}_1 & \xrightarrow[F_1]{} & \underline{\underline{M}}' \end{array} .$$

We confuse the functors F, G, and H here with the functors they induce on the smaller categories. As above we obtain a functor $T\colon \text{mod-}A \longrightarrow \underline{\underline{M}}$ which induces the one above. We shall now construct an adjoint, $S\colon \underline{\underline{M}} \longrightarrow \text{mod-}A$, for T. If $M = (M_1, \alpha_M, M_2) \in \underline{\underline{M}}$ we form the cartesian rectangle

$$\begin{array}{ccccc} SM & & \longrightarrow & & M_2 \\ \downarrow & & & & \downarrow \\ M_1 & \longrightarrow & F_1M_1 & \xrightarrow[\alpha_M]{} & F_2M_2 \end{array} .$$

Explicitly,

$$SM = \{(x_1, x_2) \in M_1 \times M_2 \mid \alpha_M (x_1 \otimes 1) = x_2 \otimes 1\}.$$

Now it is clear that SM has naturally the structure of a right A-module, and that $M \longmapsto SM$ defines an additive functor from $\underline{\underline{M}}$ to mod-A. To show that S is adjoint for T we must exhibit a natural identification,

$$\mathrm{Hom}_A(N, SM) = \mathrm{Hom}_{\underline{\underline{M}}}(TN, M),$$

for $N \in$ mod-A and $M \in \underline{\underline{M}}$. By the very construction of SM as a fibre product we have

$$\mathrm{Hom}_A(N, SM) = \mathrm{Hom}_A(N, M_1) \times_{\mathrm{Hom}_A(N, F_2M_2)} \mathrm{Hom}_A(N, M_2).$$

Now there is a standard identification $\mathrm{Hom}_A(N, M_i) = \mathrm{Hom}_{A_i}(N \otimes_A A_i, M_i) = \mathrm{Hom}_{A_i}(H_iN, M_i)$, etc., so we then can write

$$\begin{aligned} \mathrm{Hom}_A(N, SM) &= \mathrm{Hom}_{A_1}(H_1N, M_1) \times_{\mathrm{Hom}_{A'}(F_1H_1N, F_2M_2)} \mathrm{Hom}_A(H_2N, M_2) \\ &= \{(h_1, h_2) \mid h_i \in \mathrm{Hom}_{A_i}(H_iN, M_i) \ (i = 1, 2), \text{ and } \alpha_M(h_1 \otimes A') = (h_2 A')\} \\ &= \mathrm{Hom}_{\underline{\underline{M}}}(TN, M). \text{ q.e.d.} \end{aligned}$$

The natural transformation $\phi_N: N \longrightarrow STN$ is clearly an isomorphism when $N = A$. By additivity, therefore, it follows that ϕ_P *is an isomorphism for all* $P \in \underline{\underline{P}}(A)$.

So far we have made no special assumptions. We shall now show that if f_1 (or f_2) is surjective then $T: \underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}$ is cofinal (with respect to $\oplus$). This means that, given $U \in \underline{\underline{P}}$, there is a $V \in \underline{\underline{P}}$ and a $P \in \underline{\underline{P}}(A)$ such that $U \oplus V \simeq TP$. It will then follow that $SU \oplus SV \simeq STP \simeq P$ (via ϕ_P) so that $SU \in \underline{\underline{P}}(A)$. Thus S then induces an adjoint $S: \underline{\underline{P}} \longrightarrow \underline{\underline{P}}(A)$ to $T: \underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}$. Moreover it will follow from (I, 7.4) that these functors are inverse equivalences. Thus the theorem will be proved once we show that T is cofinal.

If f_1 is surjective then we have seen in the proof of (1.2) above that the functor $F_1: \underline{\underline{P}}_1 \longrightarrow \underline{\underline{P}}'$ is E-surjective, and hence the diagram (3) is E-surjective in the sense of (VII, §3). It follows therefore from (VII, 3.4 (b)) that T is cofinal. q.e.d.

(5.2) *Remark*. This theorem says that a cartesian square (1) in which f_1 or f_2 is surjective leads to a square (2) which, up to equivalence, is also cartesian. If all the rings that intervene are commutative then we can deduce other such equivalences. For example the squares analogous to (2) with $\underline{\underline{Pic}}$, $\underline{\underline{Quad}}$, or $\underline{\underline{Az}}$ replacing $\underline{\underline{P}}$ (cf. (VII, 1.1)) are also essentially cartesian. The same applies to various other categories of "structures on projective modules". In each case the basic equivalence can be deducted easily from that of Milnor's Theorem. The importance of this observation is that essentially all of the results which we shall now deduce for $\underline{\underline{P}}$ have valid analogues for these other categories.

(5.3) THEOREM (Milnor). *Let*

$$\begin{array}{ccc} A & \xrightarrow{h_2} & A_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[f_1]{} & A' \end{array}$$

be a cartesian square of ring homomorphisms in which f_1 *or* f_2 *is surjective. Then there is an exact Mayer-Vietoris* *sequence*

(4) $$K_1(A) \longrightarrow K_1(A_1) \oplus K_1(A_2) \longrightarrow K_1(A') \longrightarrow K_o(A) \longrightarrow K_o(A_1) \oplus K_o(A_2) \longrightarrow K_o(A')$$

If these rings are all commutative there is also an exact Mayer-Vietoris sequence

(5) $$0 \longrightarrow U(A) \longrightarrow U(A_1) \oplus U(A_2) \longrightarrow U(A') \longrightarrow \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(A_1) \oplus \mathrm{Pic}(A_2) \longrightarrow \mathrm{Pic}(A'),$$

and an epimorphism of exact sequences, det: (4) $\longrightarrow$ (5).

Proof. The Mayer-Vietoris sequences are just those of (VII, §4). They apply here thanks to Milnor's Theorem and to the fact that the cartesian square (2) is E-surjective. The morphism of cartesian squares,

$$\text{det:}\quad \begin{array}{ccc} \underline{\underline{P}}(A) & \longrightarrow & \underline{\underline{P}}(A_2) \\ \downarrow & & \downarrow \\ \underline{\underline{P}}(A_1) & \longrightarrow & \underline{\underline{P}}(A') \end{array} \longrightarrow \begin{array}{ccc} \underline{\underline{\mathrm{Pic}}}(A) & \longrightarrow & \underline{\underline{\mathrm{Pic}}}(A_2) \\ \downarrow & & \downarrow \\ \underline{\underline{\mathrm{Pic}}}(A_1) & \longrightarrow & \underline{\underline{\mathrm{Pic}}}(A') \end{array}$$

when the rings are commutative, induces a morphism of Mayer-Vietoris sequences, and we know from §3 that the latter is surjective. Finally, the fact that $U(A) \longrightarrow U(A_1) \oplus U(A_2)$ is injective is clear. q.e.d.

(5.4) THEOREM. *In the setting of Theorem* (5.3) *the natural homomorphisms*

$$K_o'(h_2) \longrightarrow K_o'(f_1) \text{ and } K_o'(h_1) \longrightarrow K_o'(f_2)$$

are isomorphisms. If the rings are commutative then the corresponding homomorphisms

$$\mathrm{Pic}(h_2) \longrightarrow \mathrm{Pic}(f_1) \text{ and } \mathrm{Pic}(h_1) \longrightarrow \mathrm{Pic}(f_2)$$

are also isomorphisms.

Proof. These are just the excision isomorphisms of (VII, §6).

The Mayer-Vietoris sequences are useful mainly for getting information about $K_0(A)$, and about Pic(A) when A is commutative. It is therefore convenient to show how cartesian squares arise starting from A.

(5.5) EXAMPLE. Both f_1 and f_2 are surjective. We start with two sided ideals $\underline{q}_1$ and $\underline{q}_2$ in A such that $\underline{q}_1 \cap \underline{q}_2 = 0$. Then the square

$$\begin{array}{ccc} A & \longrightarrow & A/\underline{q}_2 \\ \downarrow & & \downarrow \\ A/\underline{q}_1 & \longrightarrow & A/\underline{q}_1 + \underline{q}_2 \end{array}$$

is cartesian. Excision implies that $K_0(A, \underline{q}_1) \simeq K_0(A/\underline{q}_2, \underline{q}_1 + \underline{q}_2/\underline{q}_2)$, and similarly for Pic in the commutative case. Note that the K_1 analogue of this was already proved in (1.5). Examples of this type arise in (XI, §5).

(5.6) EXAMPLE. f_1 is injective and f_2 is surjective. Let A be a subring of B and let $\underline{c}$ be a two sided B-ideal contained in A. Then we obtain a cartesian square

$$\begin{array}{ccc} A & \xrightarrow{j} & B \\ \downarrow & & \downarrow \\ A/\underline{c} & \xrightarrow[j']{} & B/\underline{c} \end{array}$$

where j and j´ are the inclusions. We shall call this a "conductor situation" because it arises frequently when $\underline{c}$ is the conductor from an integral domain A to its integral closure, B, and in similar situations. Examples of this type occur in Chapters X and XI. In this case the excision isomorphisms are

$$K_0'(j) \xrightarrow{\simeq} K_0'(j') \text{ and } K_0(A, \underline{c}) \xrightarrow{\simeq} K_0(B, \underline{c}).$$

Similary, in the commutative case we have isomorphisms

$$\mathrm{Pic}(j) \xrightarrow{\simeq} \mathrm{Pic}(j') \text{ and } \mathrm{Pic}(A, \underline{c}) \xrightarrow{\simeq} \mathrm{Pic}(B, \underline{c}).$$

(5.7) EXAMPLE. Both f_1 and f_2 are injective. A diagram of ring inclusions

$$\begin{array}{ccc} A & \subset & A_2 \\ \cap & & \cap \\ A_1 & \subset & A' \end{array}$$

is cartesian if $A = A_1 \cap A_2$. The theorems above do not apply here except in the trivial case, $A' = A_1$ or A_2. Nevertheless there is a Mayer-Vietoris sequence for the cartesian square (3) where we put $\underline{\underline{P}}(A_1) \times_{\underline{\underline{P}}(A')} \underline{\underline{P}}(A_2)$ in place of $\underline{\underline{P}}(A)$. This sequence will be used in (XII, §9) where we study K_0 of the projective line over A.

There is a special case of an excision isomorphism for K_1 which we shall have occasion to use.

(5.8) PROPOSITION. *Let* $B = \Pi_{1 \leq i \leq n} B_i$ *be a product of rings and let* $A \subset B$ *be a subring whose projection into each* B_i *is surjective. Let* $\underline{c}$ *be a two sided ideal of* B *which is contained in* A. *Then the natural homomorphism* $K_1(A, \underline{c}) \longrightarrow K_1(B, \underline{c})$ *is an isomorphism*.

Proof. $GL(B, \underline{c})$ consists of all matrices α in $GL(B)$ such that $\alpha - I$ and $\alpha^{-1} - I$ have coordinates in $\underline{c}$. It follows that α and α^{-1} have coordinates in A so we see that

$$GL(A, \underline{c}) = GL(B, \underline{c}).$$

Since $(K_1(A, \underline{c}) \longrightarrow K_1(B, \underline{c})) = (GL(A, \underline{c})/E(A, \underline{c}) \longrightarrow GL(B, \underline{c})/E(B, \underline{c}))$ the proposition will be proved once we show that the inclusion $E(A, \underline{c}) \subset E(B, \underline{c})$ is an equality.

Let S denote the set of elementary matrices which are $\equiv I \bmod \underline{c}$. Then $E(A, \underline{c})$ (resp., $E(B, \underline{c})$) is the *normal* subgroup of $E(A)$ (resp., of $E(B)$) generated by S. (See (VI, §1).)

Since $\underline{c}$ is a B-ideal it is the direct sum of ideals $\underline{c}_i$ such that $\underline{c}_i$ projects monomorphically into B_i $(1 \leq i \leq n)$, and to zero in B_j for $j \neq i$. Let S' denote the set of $\varepsilon \in S$

such that $\varepsilon \equiv I \bmod \underline{c}_i$ for some i. The group generated by S' evidently contains S, so $E(B, \underline{c})$ is the group generated by all $\beta \varepsilon \beta^{-1}$ with $\beta \in E(B)$ and $\varepsilon \in S'$. It therefore suffices to show that each such $\beta \varepsilon \beta^{-1} \in E(A, \underline{c})$. In the (ΠB_i)-coordinates we can write $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and, say, $\varepsilon = (\varepsilon_1, I, \ldots, I)$, assuming $\varepsilon \equiv I \bmod \underline{c}_1$. Since $A \longrightarrow B_1$ is surjective, by assumption, it follows that $E(A) \longrightarrow E(B_1)$ is surjective (see (V, 1.1)). Therefore, since $\beta_1 \in E(B_1)$, we can find $\alpha = (\beta_1, \alpha_2, \ldots, \alpha_n) \in E(A)$. Then $\beta \varepsilon \beta^{-1} = (\beta_1 \varepsilon_1 \beta_1^{-1}, I, \ldots, I) = \alpha \varepsilon \alpha^{-1} \in E(A, \underline{c})$. q.e.d.

We shall next establish a weak Mayer-Vietoris type proposition for the functors G_i.

(5.9) PROPOSITION. *Let*

$$\begin{array}{ccc} A & \xrightarrow{h_2} & A_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[f_1]{} & A' \end{array}$$

be a cartesian square of right noetherian rings, all of which are finitely generated right A-modules. Assume that f_1 or f_2 is surjective. Then the restriction homomorphisms induce epimorphisms

$$G_i(A_1) \oplus G_i(A_2) \longrightarrow G_i(A) \qquad (i = 0, 1).$$

Proof. The homomorphisms above are induced by the "restriction" functor from $\underline{\underline{M}}(A_1 \times A_2)$ to $\underline{\underline{M}}(A)$. According to (VIII, 3.3) it suffices to show that every $M \in \underline{\underline{M}}(A)$ has a characteristic finite filtration whose successive quotients are (restrictions of) $(A_1 \times A_2)$-modules.

Let $\underline{c}_i = \mathrm{Ker}(h_i)$ $(i = 1, 2)$, and assume, say, that f_2 is surjective. Then h_1 is also surjective, and $\underline{c}_1 \cap \underline{c}_2 = 0$

because $A \longrightarrow A_1 \times A_2$ is a monomorphism. Now we claim that $0 \subset M\underline{c}_1 \subset M$ is the required type of filtration. It is certainly characteristic, and $M/M\underline{c}_1$ is a module over $A/\underline{c}_1 = A_1$, and hence over $A_1 \times A_2$. We conclude the proof by showing that $M\underline{c}_1$ is an $(A_1 \times A_2)$-module. For this it suffices to show that $\underline{c}_1$ itself is an A_2-module. But $\underline{c}_1 = 0 \times \underline{c}$ where $\underline{c} = \mathrm{Ker}(f_2)$, so $\underline{c}_1$ is an ideal in $A_1 \times A_2$. q.e.d.

(5.10) COROLLARY. *Let* $B = \Pi_{1 \le i \le n} B_i$ *be a product of rings and let* $A \subset B$ *be a subring that projects onto each factor* B_i. *Then the homomorphisms* $G_i(B) = \amalg_j G_i(B_j) \longrightarrow G_i(A)$ $(i = 0, 1)$ *are surjective*.

Proof. Let $A_1{}'$ be the projection of A into $B_2 \times \ldots \times B_n$. Then there is a fibre product diagram,

$$\begin{array}{ccc} A & \longrightarrow & A_1{}' \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & A'' \end{array}$$

to which we may apply (5.9) and conclude that $G_i(B_1) \oplus G_i(A_1{}') \longrightarrow G_i(A)$ is surjective $(i = 0,1)$. By induction on n we conclude further that $G_i(B_2) \oplus \ldots \oplus G_i(B_n) \longrightarrow G_i(A_1{}')$ is surjective. q.e.d.

We close this section now by describing the behavior of H_o on a fibre product. This information is required in certain calculations to be made in Chapter XII.

(5.11) PROPOSITION. *Let*

$$(1) \qquad \begin{array}{ccc} A & \xrightarrow{h_2} & A_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[f_1]{} & A' \end{array}$$

be a cartesian square of homomorphisms of commutative rings in which f_1 *or* f_2 *is surjective.*

(a) *The square,*

$$(6)\qquad \begin{array}{ccc} \operatorname{spec}(A) & \xleftarrow{\;a_{h_2}\;} & \operatorname{spec}(A_2) \\ \Big\uparrow{\scriptstyle a_{h_1}} & & \Big\uparrow{\scriptstyle a_{f_2}} \\ \operatorname{spec}(A_1) & \xleftarrow[\;a_{f_1}\;]{} & \operatorname{spec}(A') \end{array},$$

is cocartesian in the category of topological spaces.

(b) *The sequence*

$$(7)\qquad 0 \longrightarrow H_o(A) \xrightarrow{\begin{pmatrix} H_o(h_1) \\ -H_o(h_2) \end{pmatrix}} H_o(A_1) \oplus H_o(A_2) \xrightarrow{(H_o(f_1),\, H_o(f_2))} H_o(A')$$

is exact and $\operatorname{Coker}(H_o(f_1), H_o(f_2))$ *is a torsion free abelian group.*

Proof. (a) Say f_2 is surjective. Then we can factor f_1 into an epimorphism followed by a monomorphism. In any category, if the two squares of a rectangle

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

are (co)cartesian then so also is the rectangle. Hence it suffices to treat separately the cases

(i) f_1 is also surjective; and

(ii) f_1 is injective.

(i) In this case we can write $A_i = A/\underline{a}_i$ (i = 1, 2), and $A' = A/(\underline{a}_1 + \underline{a}_2)$, where $\underline{a}_1$ and $\underline{a}_2$ are ideals such that $\underline{a}_1 \cap \underline{a}_2 = 0$. (See example (5.5).) Then all spec's in question can be identified with closed subsets of spec(A), via the inclusions in diagram (6). With this identification we have

$$\begin{aligned} \mathrm{spec}(A_1) \cup \mathrm{spec}(A_2) &= V(\underline{a}_1) \cup V(\underline{a}_2) = V(\underline{a}_1 \cap \underline{a}_2) \\ &= \mathrm{spec}(A) \\ \mathrm{spec}(A_1) \cap \mathrm{spec}(A_2) &= V(\underline{a}_1) \cap V(\underline{a}_2) = V(\underline{a}_1 + \underline{a}_2) \\ &= \mathrm{spec}(A'). \end{aligned}$$

These relations show that (6) is cocartesian.

(ii) In this case we can identify A with a subring of A_2, and we have $A_1 = A/\underline{c}$ and $A' = A_2/\underline{c}$ for some A_2-ideal $\underline{c} \subset A_1$. (See example (5.6).) Then we can identify $\mathrm{spec}(A_1) = V_A(\underline{c}) \subset \mathrm{spec}(A)$ and $\mathrm{spec}(A') = V_{A_2}(\underline{c}) \subset \mathrm{spec}(A_2)$. Moreover $\mathrm{spec}(A_2) \longrightarrow \mathrm{spec}(A)$ sends $\underline{p}$ to $\underline{p} \cap A$. We must show that spec(A) is the union of $V_A(\underline{c})$ and of the image of $\mathrm{spec}(A_2)$, and that if $P \in \mathrm{spec}(A_2)$ is such that $\underline{p} \cap A \in V_A(\underline{c})$ then $\underline{p} \in V_{A_2}(\underline{c})$. The latter is just the implication: "$\underline{p} \cap A \supset \underline{c} \Rightarrow \underline{p} \supset \underline{c}$", which is trivial. It remains to be shown that if $\underline{p} \in \mathrm{spec}(A)$ and $\underline{p} \not\supset \underline{c}$ then $\underline{p}$ is the restriction of a prime in A_2. Choose $t \in \underline{c}$, $t \notin \underline{p}$. Then t is a unit in $A_{\underline{p}}$. On the other hand $tA_2 \subset \underline{c} \subset A$, so we conclude that $A_{\underline{p}} = (A_2)_{\underline{p}}$. Let $\underline{q} \in \mathrm{spec}(A_2)$ correspond to the maximal ideal of $(A_2)_{\underline{p}}$. Then $\underline{q} \cap A = \underline{p}$; q.e.d.

(b) Since (6) is cocartesian it follows, by definition, that Cont. maps((6), G) is a cartesian square of sets, for any topological space G. If G is an abelian group with the discrete topology then Cont. maps((6), G) is a diagram of abelian groups, and, being cartesian as a diagram of sets, it is also cartesian as a diagram of abelian groups.

Taking $G = \mathbb{Z}$, therefore, we deduce the exact sequence (7) of H_0's. Because of the quasi-compactness of spec(A) we have Cont. maps(spec(A), G) = $H_0(A) \otimes G$ for any discrete abelian group G. Therefore (7) is an exact sequence of groups of the form $(7) = (0 \longrightarrow M_0 \longrightarrow M_1 \xrightarrow{h} M_2)$ such that $(7) \otimes G$ remain exact for all abelian groups G. Taking $G = \mathbb{Z}/n\mathbb{Z}$ we deduce easily that $nM_2 \cap \mathrm{Im}(h) = n \cdot \mathrm{Im}(h)$. In our example $M_2 = H_0(A')$ is torsion free, so the fact that $nM_2 \cap \mathrm{Im}(h) = n \cdot \mathrm{Im}(h)$ for all $n \in \mathbb{Z}$ implies that Coker(h) is torsion free. q.e.d.

(5.12) COROLLARY. *In the setting of* (5.11) *we have a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & SK_1(A) & \longrightarrow & SK_1(A_1) \oplus SK_1(A_2) & \longrightarrow & SK_1(A') & \longrightarrow & SK_o(A) & \longrightarrow & SK_o(A_1) \oplus SK_o(A_2) & \longrightarrow & SK_o(A') \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & K_1(A) & \longrightarrow & K_1(A_1) \oplus K_1(A_2) & \longrightarrow & K_1(A') & \longrightarrow & Rk_o(A) & \longrightarrow & Rk_o(A_1) \oplus Rk_o(A_2) & \longrightarrow & Rk_o(A') \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & U(A) & \longrightarrow & U(A_1) \oplus U(A_2) & \longrightarrow & U(A') & \longrightarrow & Pic(A) & \longrightarrow & Pic(A_1) \oplus Pic(A_2) & \longrightarrow & Pic(A') \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

The maps from the middle to bottom row are determinants, and we have introduced the notation $SK_0(C) = \mathrm{Ker}(\det_0(C))$ *for a commutative ring* C.

Proof. If we replace Rk_0 by K_0 then the middle row becomes the Mayer-Vietoris K-sequence of the cartesian square (1) in (5.11). The exact sequence (7) in (5.11) shows that the image of the connecting homomorphism, $K_1(A') \longrightarrow K_0(A)$, in the Mayer-Vietoris sequence actually lies in $Rk_0(A)$, and that the resulting sequence above, with Rk_0's replacing the corresponding K_0's, is exact. The bottom row is the Mayer-Vietoris Pic-sequence, and the top row is the kernel of the determinant homomorphism from the middle row to the bottom. The exact homology sequence now implies that the top row is exact. q.e.d.

(5.13) COROLLARY. *Suppose, in the setting of* (5.11), *that* $\det_0(A_1)$, $\det_0(A_2)$, *and* $\det_1(A')$ *are isomorphisms (i.e. that* $SK_0(A_i) = 0 = SK_1(A')$ $(i = 1, 2)$). *Then* $\det_0(A)$ *is an isomorphism also*.

§6. THE EXACT SEQUENCES OF A LOCALIZATION

In this section *we fix a commutative ring* R *and a multiplicative set* S *in* R. If A is an R-algebra then we have the localization,

$$S^{-1}\colon \text{mod-}A \longrightarrow \text{mod-}S^{-1}A.$$

Up to equivalence, we can view this as a quotient functor in the sense of (VIII, §5) (see example (VIII, 5.2)). Consequently we can apply the results of (VIII, §5), and that is the purpose of this section.

We begin by studying the functors G_i, so assume first that A *is right noetherian*. Then we can treat

$$\text{(1)} \qquad \underline{\underline{M}}_S(A) \subset \underline{\underline{M}}(A) \xrightarrow{S^{-1}} \underline{\underline{M}}(S^{-1}A)$$

as a quotient functor, where $\underline{\underline{M}}_S(A)$ is the full subcategory of S-*torsion modules* M (i.e. $S^{-1}M = 0$). Moreover we shall write

$$G_i(A, S) = K_i\underline{\underline{M}}_S(A) \qquad (i = 0, 1).$$

In the special case when $S = \{t^n \mid n \geq 0\}$ consists of powers of a single element then we have $\underline{\underline{M}}(A/tA) \subset \underline{\underline{M}}_S(A)$ as the subcategory of modules killed by t. If $M \in \underline{\underline{M}}_S(A)$ then $Mt^n = 0$ for some n so $0 = Mt^n \subset Mt^{n-1} \subset \ldots \subset Mt^0 = M$ is a characteristic finite filtration with quotients in $\underline{\underline{M}}(A/tA)$. It follows therefore from (VIII, 3.3) that

$$G_i(A, \{t^n\}) \simeq G_i(A/tA) \qquad (i = 0, 1).$$

The next proposition summarizes part of (VII, 5.5).

(6.1) PROPOSITION. *The sequence*

$$G_0(A, S) \longrightarrow G_0(A) \longrightarrow G_0(S^{-1}A) \longrightarrow 0$$

induced by (1) *is exact. Moreover there is a unique homomorphism* $\partial: G_1(S^{-1}A) \longrightarrow G_0(A, S)$ *such that* $\partial[S^{-1}M, S^{-1}\alpha] = [\mathrm{Coker}\ \alpha] - [\mathrm{Ker}\ \alpha]$ *whenever* $M \in \underline{\underline{M}}(A)$ *and* $\alpha \in \mathrm{End}_A(M)$ *is such that* $S^{-1}\alpha$ *is an automorphism.*

(6.2) THEOREM. (Heller-Reiner [1]) *Let* A *be a right noetherian* R-*algebra as above. Assume there is a nilpotent ideal* $J \subset S^{-1}A$ *such that* $B = (S^{-1}A)/J$ *is right regular. (This is the case, for example, if* $S^{-1}A$ *is a right Artinian ring.) Then the sequence*

$$G_1(S^{-1}A) \xrightarrow{\partial} G_0(A, S) \longrightarrow G_0(A) \longrightarrow G_0(S^{-1}A) \longrightarrow 0$$

is exact.

Proof. We begin by observing that, under the functor $S^{-1}: \underline{\underline{M}}(A) \longrightarrow M(S^{-1}A)$, finite filtrations and resolutions of objects can be lifted. Specifically:

(i) If $M \in \mathbf{M}(A)$ and if $0 = N_o \subset N_1 \subset \ldots \subset N_n = S^{-1}M$ is a finite filtration in $\mathbf{M}(S^{-1}A)$, then it is the localization of a filtration $0 = M_o \subset M_1 \subset \ldots \subset M_n = M$ in $\mathbf{M}(A)$.

(ii) If $0 \longrightarrow N_n \longrightarrow \ldots \longrightarrow N_o \longrightarrow S^{-1}M \longrightarrow 0$ is an exact sequence in $\mathbf{M}(S^{-1}A)$ then it is (isomorphic to) the localization of an exact sequence

$$0 \longrightarrow M_n \longrightarrow \ldots \longrightarrow M_o \longrightarrow M \longrightarrow 0 \text{ in } \mathbf{M}(A).$$

These facts follow from (III, 4.6)

Now we consider the subcategories

$$\mathbf{C}_o' = \mathbf{P}(B) \subset \mathbf{C}' = \mathbf{M}(B) \subset \mathbf{M}(S^{-1}A),$$

where the second inclusion is the identification of B-modules with $S^{-1}A$-modules killed by J. Next we introduce

$$\mathbf{C}_o \subset \mathbf{C} \subset \mathbf{M}(A)$$

where $\mathbf{C}$ (resp., $\mathbf{C}_o$) is the full subcategory whose objects are those M such that $S^{-1}M \in \mathbf{C}'$ (resp., such that $S^{-1} M \in \mathbf{C}_o'$). If $N \in \mathbf{M}(S^{-1}A)$ then $N \supset NJ \supset NJ^2 \supset \ldots$ gives a finite and characteristic $\mathbf{C}'$-filtration, since J is nilpotent. It follows from (i) above that every object of $\mathbf{M}(A)$ has a finite $\mathbf{C}$-filtration as well. Therefore we can apply (VIII, 3.3) to conclude that the verticles in the commutative diagram

$$\begin{array}{ccccccccc}
G_1(S^{-1}A) & \longrightarrow & G_o(A, S) & \longrightarrow & G_o(A) & \longrightarrow & G_o(S^{-1}A) & \longrightarrow & 0 \\
\uparrow & & \| & & \uparrow & & \uparrow & & \\
K_1(\mathbf{C}') & \longrightarrow & G_o(A, S) & \longrightarrow & K_o(\mathbf{C}) & \longrightarrow & K_o(\mathbf{C}') & &
\end{array}$$

are isomorphisms. It therefore suffices to show that the bottom row is exact at $G_o(A, S)$, the exactness at the other points being covered by (6.1) above.

The regularity hypothesis on B means that every object of $\underline{\underline{C}}' = \underline{\underline{M}}(B)$ has a finite resolution in $\underline{\underline{C}}_o' = \underline{\underline{P}}(B)$. Property (ii) above now further implies that every object of $\underline{\underline{C}}$ has a finite $\underline{\underline{C}}_o$-resolution. Thus we can apply (VII, 4.2) and (VII, 4.6) to conclude that the verticals in the commutative diagram

$$\begin{array}{ccccccc} K_1(\underline{\underline{C}}') & \longrightarrow & G_o(A, S) & \longrightarrow & K_o(\underline{\underline{C}}) & \longrightarrow & K_o(\underline{\underline{C}}') \\ \uparrow & & \| & & \uparrow & & \uparrow \\ K_1(\underline{\underline{C}}_o') & \longrightarrow & G_o(A, S) & \longrightarrow & K_o(\underline{\underline{C}}_o) & \longrightarrow & K_o(\underline{\underline{C}}_o') \end{array}$$

are isomorphisms. Thus we are reduced to proving exactness of the bottom row. But this follows now from (VII, 5.5) because the category $\underline{\underline{C}}_o' = \underline{\underline{P}}(B)$ is semi-simple. q.e.d.

In considering the functors K_i now *we no longer assume that* A *is right noetherian.* We shall write

(2) $$\underline{\underline{H}}_S(A) \subset \underline{\underline{H}}(A) \xrightarrow{S^{-1}} \underline{\underline{H}}(S^{-1}A)$$

where $\underline{\underline{H}}_S(A)$ is the full subcategory whose objects are the S-torsion modules in $\underline{\underline{H}}(A)$ (cf. (III, §6)). Moreover we write

$$K_i(A, S) = K_i\underline{\underline{H}}_S(A) \qquad (i = 0, 1)$$

(6.3) THEOREM. *Let* A *be an* R-*algebra on which multiplication by any* $s \in S$ *is injective. Then there is a unique homomorphism* $\partial\colon K_1(S^{-1}A) \longrightarrow K_o(A, S)$ *such that* $\partial[S^{-1}P, S^{-1}\alpha] = [\mathrm{Coker}(\alpha)]$ *whenever* $P \in \underline{\underline{P}}(A)$ *and* $\alpha \in \mathrm{End}_A(P)$ *is such that* $S^{-1}\alpha$ *is an automorphism. The sequence*

$$K_1(A) \longrightarrow K_1(S^{-1}A) \xrightarrow{\partial} K_o(A, S) \longrightarrow K_o(A) \longrightarrow K_o(S^{-1}A),$$

resulting from this and (2) *above, is exact.*

Proof. The conclusions of this theorem follow directly from those of (VIII, 5.8), so we need only verify the three hypotheses of that theorem. The first one is clear. The second requires that if $f: P \longrightarrow Q$ in $\underline{\underline{P}}(A)$ is such that $S^{-1}f$ is a monomorphism then f is already a monomorphism. This follows from the commutative square

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & Q \\ {\scriptstyle h_P}\downarrow & & \downarrow{\scriptstyle h_Q} \\ S^{-1}P & \xrightarrow[S^{-1}f]{} & S^{-1}Q \end{array}$$

and the fact that h_P is a monomorphism. The latter condition on h_P follows, in turn, from the fact that P is projective and that the $s \in S$ are not devisors of zero on A. The third hypothesis of (VIII, 5.8) requires that if $Q \subset P \in \underline{\underline{P}}(A)$ and if $S^{-1}(P/Q) = 0$, then there is a $P' \subset Q$ such that $P' \in \underline{\underline{P}}(A)$ and $S^{-1}(P/P') = 0$. Since P is finitely generated there is an $s \in S$ such that $(P/Q)s = 0$. Therefore $P' = Ps \simeq P$ fills our needs. q.e.d.

In the setting of Theorem (6.3), if A is also right noetherian, then we have a "Cartan homomorphism" between the two sequences:

$$
\begin{array}{ccccccccc}
K_1(A) & \longrightarrow & K_1(S^{-1}A) & \longrightarrow & K_o(A,\ S) & \longrightarrow & K_o(A) & \longrightarrow & K_o(S^{-1}A) & & \\
 & & \downarrow{\scriptstyle C_1(S^{-1}A)} & & \downarrow{\scriptstyle C_o(A,\ S)} & & \downarrow{\scriptstyle C_o(A)} & & \downarrow{\scriptstyle C_o(S^{-1}A)} & & \\
 & & G_1(S^{-1}A) & \longrightarrow & G_o(A,\ S) & \longrightarrow & G_o(A) & \longrightarrow & G_o(S^{-1}A) & \longrightarrow & 0
\end{array}
$$

If A is right regular then so also is $S^{-1}A$ and the verticals are isomorphisms. Thus we can splice the two sequences in this case. We record this:

(6.4) COROLLARY. In the setting of Theorem (6.3) suppose that A is right regular. Then there is an isomorphism of exact sequences

$$\begin{array}{ccccccccccc}
K_1(A) & \longrightarrow & K_1(S^{-1}A) & \longrightarrow & K_o(A, S) & \longrightarrow & K_o(A) & \longrightarrow & K_o(S^{-1}A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
G_1(A) & \longrightarrow & G_1(S^{-1}A) & \longrightarrow & G_o(A, S) & \longrightarrow & G_o(A) & \longrightarrow & G_o(S^{-1}A) & \longrightarrow & 0 .
\end{array}$$

It would be of interest to be able to extend the exact sequence above to $K_1(A, S)$ on the left. We can do this now only in certain special cases, and then only with rather delicate techniques (cf. Chapter XIII).

Let A be a right noetherian R-algebra. Recall (III, §6) that S _is regular for_ A if the inclusion $\underline{\underline{H}}_S(A) \subset \underline{\underline{M}}_S(A)$ is an equality. In other words if every finitely generated right A-module M such that $S^{-1}M = 0$ has finite homological dimension. Of course we then have

$$K_i(A, S) = G_i(A, S) \qquad (i = 0, 1).$$

Moreover we deduce the following theorem immediately from (VIII, 5.10).

(6.5) THEOREM. _Let_ A _be a right noetherian_ R-_algebra_. _Assume_ S _is regular for_ A _and that multiplication by each_ $s \in S$ _on_ A _is injective. Then_:

(a) _If_ $M \in \underline{\underline{M}}(A)$, $M \in \underline{\underline{H}}(A) \iff S^{-1}M \in \underline{\underline{H}}(S^{-1}A)$.

(b) _The sequence_

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\quad G_0(A, S) \\ \qquad\qquad\qquad\qquad\qquad\quad \| \\ K_1(A) \longrightarrow K_0(S^{-1}A) \longrightarrow K_0(A, S) \longrightarrow K_0(A) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow K_0(S^{-1}A) \longrightarrow 0 \end{array}$$

is exact.

Let A be an integral domain with field of fractions $L = S^{-1}A$ $(S = A - \{0\})$. If $M \in \underline{\underline{M}}(A)$ we define its _rank_ to be

$$\operatorname{rank}_A(M) = [M \otimes_A L : L].$$

This is clearly an additive function, inducing the composite homomorphism

$$G_0(A) \longrightarrow G_0(L) \simeq \underline{\underline{Z}}.$$

Note that this terminology is consistent with our use of the term rank for projective modules $P \in \underline{\underline{P}}(A)$. We shall write

$$\tilde{G}_o(A) = \mathrm{Ker}(G_o(A) \xrightarrow{\text{rank}} \underline{\underline{Z}}).$$

Since rank (A) = 1 we obtain a decomposition

$$G_o(A) = \underline{\underline{Z}} \cdot [A] \oplus \tilde{G}_o(A).$$

Now assume that A is a Krull ring (see (III, §7)). If $M \in$ mod-A and if $\underline{p} \in Ht_1(A)$ write $\ell_{\underline{p}}(M)$ for the length (possibly infinite) of the $A_{\underline{p}}$-module $M_{\underline{p}}$. We define the full subcategory $\underline{\underline{C}}$ of all $M \in$ mod-A such that (i) $\ell_{\underline{p}}(M)$ is finite for all $\underline{p} \in Ht_1(A)$, and (ii) $\ell_{\underline{p}}(M) = 0$ for all but finitely many $\underline{p} \in Ht_1(A)$. Then for $M \in \underline{\underline{C}}$ we can define

$$\chi(M) = \Sigma_{\underline{p} \in Ht_1(A)} \ell_{\underline{p}}(M)\underline{p} \in D(A) \quad \text{(divisor group)}.$$

Since localization is exact we see that $\underline{\underline{C}}$ is an abelian category and that χ is an additive function on $\underline{\underline{C}}$, therefore inducing

$$\chi\colon K_o(\underline{\underline{C}}) \longrightarrow D(A).$$

The category $\underline{\underline{M}}_S(A)$ of finitely generated torsion A-modules is clearly contained in $\underline{\underline{C}}$. From the inclusions $\underline{\underline{H}}_S(A) \subset \underline{\underline{M}}_S(A) \subset \underline{\underline{C}}$ we therefore obtain homomorphisms, also denoted by χ,

$$\chi\colon G_o(A, S) = K_o(\underline{\underline{M}}_S(A)) \longrightarrow D(A),$$

and

$$\chi\colon K_o(A, S) = K_o(\underline{\underline{H}}_S(A)) \longrightarrow D(A).$$

(6.6) PROPOSITION. _Let_ A _be a commutative ring, let_ $\alpha \in \mathrm{End}_A(A^n)$, _and let_ $M = \mathrm{Coker}(\alpha)$.

(a) $M \cdot \det(\alpha) = 0$.

(b) _Suppose_ A _is a Krull ring and_ $\det(\alpha) \neq 0$. _Then_

$$\chi(M) = \mathrm{div}(\det(\alpha)).$$

(c) *Let* A *be a noetherian Krull ring with field of fractions* $L = S^{-1}A$ $(S = A - \{0\})$. *Then there is a unique homomorphism* $c\ell: G_o(A) \longrightarrow C(A)$ (*the divisor class group*) *such that* $c\ell[A] = 0$ *and such that the diagram,*

$$\begin{array}{ccccccccc}
G_1(L) & \xrightarrow{\partial} & G_o(A, S) & \longrightarrow & G_o(A) & \longrightarrow & G_o(L) & \longrightarrow & 0 \\
{\scriptstyle \det}\downarrow{\scriptstyle (\approx)} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle c\ell} & & \downarrow & & \\
U(L) & \longrightarrow & D(A) & \longrightarrow & C(A) & \longrightarrow & 0, & &
\end{array}$$

commutes. The top row here is the exact sequence of (6.2), *and the bottom is the exact sequence of divisors and divisor classes* (III, §7, (1)). *Moreover the verticals are epimorphisms*.

Proof. (a) It is well known (Cramer's Rule) that there is a $\beta \in \mathrm{End}_A(A^n)$ such that $\alpha\beta = \beta\alpha = \det(\alpha)\, 1_{A^n}$. Therefore $A^n \cdot \det(\alpha) \subset \mathrm{Im}(\alpha)$, thus proving (a).

(b) We will show that

$$\begin{array}{ccc}
K_1(L) & \xrightarrow{\partial} & K_o(A, S) \\
{\scriptstyle \det}\downarrow{\scriptstyle (\approx)} & & \downarrow{\scriptstyle \chi} \\
U(L) & \longrightarrow & D(A)
\end{array}$$

commutes, where ∂ is map in (6.3). If we consider α as lying in $GL_n(L)$, it defines a class, $[\alpha] \in K_1(L)$. According to (6.3) $\partial[\alpha] = [\mathrm{Coker}(\alpha)]$. Hence (b) will follow from the commutativity of the square. Since det above is an isomorphism it suffices to show that $\chi(\partial[u]) = \mathrm{div}(u)$ for $u \in U(L)$, where $[u] = [A, u \cdot 1_A] \in K_1(L)$. Writing $u = a/b$, $a, b \neq 0$ in A, we are reduced to the case $u = a \in A$. In this case, as we saw above, $\chi(\partial[a]) = \chi(A/aA)$. If $\underline{p} \in Ht_1(A)$ and

$aA_{\underline{p}} = (\underline{p}\,A_{\underline{p}})^n$ then clearly $(A/aA)_{\underline{p}}$ has length n as an $A_{\underline{p}}$-module, because $A_{\underline{p}}$ is a DVR. Thus $\chi(A/aA) = \mathrm{div}(a)$. q.e.d.

(c) We have $G_0(A) = \underline{\underline{Z}} \cdot [A] \oplus \widetilde{G}_0(A)$ where $G_0(A) = \mathrm{Ker}(G_0(A) \longrightarrow G_0(L)) = \mathrm{Im}(G_0(A, S) \longrightarrow G_0(A)) = \mathrm{Coker}(\partial)$. It follows from part (b) that

$$\begin{array}{ccc} G_1(L) & \longrightarrow & G_0(A, S) \\ \Big\downarrow \text{det} & & \Big\downarrow \chi \\ U(L) & \longrightarrow & D(A) \end{array}$$

commutes, so there is an induced homomorphism $c\ell\colon G_0(A) \longrightarrow C(A)$ on the cokernels. Thus $c\ell$ is defined, and uniquely so, by the commutativity of the diagram and the fact that $c\ell[A] = 0$.

We have noted already that det is an isomorphism. Since $\chi[A/\underline{p}] = \underline{p}$ for $\underline{p} \in Ht_1(A)$ it follows that χ is an epimorphism. The diagram then implies $c\ell$ is likewise an epimorphism. q.e.d.

(6.7) PROPOSITION. Let A be as in (6.6) (c) and let T be a multiplicative set ($0 \notin T$) such that $B = T^{-1}A$ is regular. Then there is an epimorphism of exact sequences

$$\begin{array}{ccccccccc} G_1(B) & \longrightarrow & G_0(A, T) & \longrightarrow & G_0(A) & \longrightarrow & G_0(B) & \longrightarrow & 0 \\ \Big\downarrow \text{det} & & \Big\downarrow \chi_T & & \Big\downarrow c\ell & & \Big\downarrow c\ell & & \\ U(B) & \longrightarrow & D(A, T) & \longrightarrow & C(A) & \longrightarrow & C(B) & \longrightarrow & 0 \end{array}$$

Proof. The top row comes from (6.2). The map χ_T here is determined by the commutative square

$$\begin{array}{ccc} G_0(A,\ T) & \longrightarrow & G_0(A,\ S) \\ \Big\downarrow{\scriptstyle \chi_T} & & \Big\downarrow{\scriptstyle \chi} \\ D(A,\ T) & & D(A) \end{array}$$

where $S = A - \{0\}$ and the top is induced by $\underline{\underline{M}}_T(A) \subset \underline{\underline{M}}_S(A)$. We need only note that if $M \in \underline{\underline{M}}_T(A)$ then $\chi(M) \in D(A, T)$. But $Mt = 0$ for some $t \in T$. Therefore if $M_{\underline{p}} \neq 0$ for $\underline{p} \in Ht_1(A)$ we have $\underline{p} \supset \mathrm{ann}_A(M)$ and hence $\underline{p} \cap T \neq \phi$. Since $D(A, T)$ is generated by the $\underline{p} \in Ht_1(A)$ that meet T this shows that χ_T above exists. The exact sequence on the bottom is (III, §7, diagram (2)). From the way the maps above are defined the commutativity of the above diagram follows immediately from that of (6.6) (c).

Next we consider the localization sequence for Pic. Let A be commutative, and let $f: A \longrightarrow S^{-1}A$ be a localization. Then we have the exact sequence

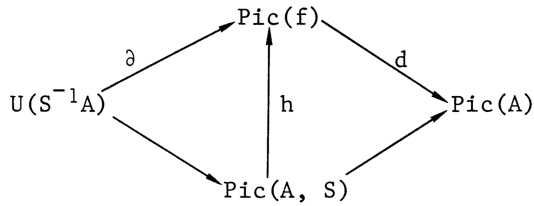
$$(3) \qquad U(A) \longrightarrow U(S^{-1}A) \xrightarrow{\partial} \mathrm{Pic}(f) \xrightarrow{d} \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(S^{-1}A)$$

of (3.3). If S consists of non divisors of zero in A then we also have the group Pic(A, S) (see (III, §7)) of invertible ideals $\underline{a} \subset S^{-1}A$ such that $S^{-1}\underline{a} = S^{-1}A$, as well as an exact sequence (III, 7.10)

$$(4) \qquad U(A) \longrightarrow U(S^{-1}A) \longrightarrow \mathrm{Pic}(A,\ S) \longrightarrow \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(S^{-1}A).$$

We shall identify these two sequences. By the 5-lemma it suffices to construct a homomorphism $h: \mathrm{Pic}(A, S) \longrightarrow \mathrm{Pic}(f)$ making the resulting diagram (4) $\longrightarrow$ (3) commute. We define $h(\underline{a}) = [\underline{a}, \alpha, A]$, where $\alpha: S^{-1}\underline{a} \longrightarrow S^{-1}A$ is the isomorphism induced by $\underline{a} \subset S^{-1}A$. It is easily checked that this is a homomorphism, thanks to the fact that $\underline{a} \otimes_A \underline{b} \longrightarrow \underline{ab}$ is an isomorphism for $\underline{a}, \underline{b} \in \mathrm{Pic}(A, S)$. Moreover

$d[\underline{a}, \alpha, A] = [\underline{a}] - [A] = [\underline{a}]$ in Pic(A). If $a \in U(S^{-1}A)$ then $\partial(a) = [A, a \cdot 1_{S^{-1}A}, A] = [aA, 1_{S^{-1}A}, A] = h(aA)$. Thus



commutes, as required. Henceforth we shall use h to identify Pic(f) with Pic(A, S). More generally, we shall write Pic(A, S) for Pic(f) for any multiplicative set, not necessarily consisting of non divisors of zero. With this notation we can now write the "determinant" homomorphisms as an epimorphism of exact sequences

$$\begin{array}{ccccccccc}
K_1(A) & \longrightarrow & K_1(S^{-1}A) & \longrightarrow & K_o(A, S) & \longrightarrow & K_o(A) & \longrightarrow & K_o(S^{-1}A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
U(A) & \longrightarrow & U(S^{-1}A) & \longrightarrow & Pic(A, S) & \longrightarrow & Pic(A) & \longrightarrow & Pic(S^{-1}A)
\end{array} \qquad (5)$$

If $A \longrightarrow S^{-1}A$ kills no non zero idempotents, e.g. if S consists of non divisors of zero, then we can replace K_0 by Rk_0 above.

(6.8) THEOREM. *Let* A *be a commutative noetherian ring, and let* S *be a multiplicative set of non divisors of zero which is regular for* A. *Then there is an epimorphism of exact sequences,*

$$
\begin{array}{ccccccccccc}
 & & & & G_o(A,\ S) & & & & & & \\
 & & & & \| & & & & & & \\
K_1(A) & \longrightarrow & K_1(S^{-1}A) & \longrightarrow & K_o(A,\ S) & \longrightarrow & Rk_o(A) & \longrightarrow & Rk_o(S^{-1}A) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \det_1(A)} & & \downarrow{\scriptstyle \det_1(S^{-1}A)} & & \downarrow{\scriptstyle \det_o(A,\ S)} & & \downarrow{\scriptstyle \det_o(A)} & & \downarrow{\scriptstyle \det_o(S^{-1}A)} & & \\
U(A) & \longrightarrow & U(S^{-1}A) & \longrightarrow & Pic(A,\ S) & \longrightarrow & Pic(A) & \longrightarrow & Pic(S^{-1}A) & \longrightarrow & 0,
\end{array}
$$

in which Pic(A, S) is a free abelian group with the primes of height one meeting S as a basis.

Proof. The diagram is that of (3.6), except for the zeros on the right and the term $G_0(A, S)$. The extra terms in the top row come from (6.7). According to (III, 7.21) S is factorial for A, so the indicated properties of the bottom row follow from (III, 7.17).

We shall now apply some of these results to algebras over Dedekind rings.

(6.9) PROPOSITION. Let R be a Dedekind ring with field of fractions $L = S^{-1}R$ $(S = R - \{0\})$, and write $X = \max(R)$. Let A be a right noetherian R-algebra which is torsion free as an R-module. Set $B = A \otimes_R L = S^{-1}A$ and assume B satisfies the conditions of (6.2). Then there is a natural isomorphism

$$\coprod_{\underline{p} \in X} G_i(A/\underline{p} A) \longrightarrow G_i(A, S) \qquad (i = 0, 1), \tag{6}$$

and hence an exact sequence

$$G_1(B) \longrightarrow \coprod_{\underline{p} \in X} G_0(A/\underline{p} A) \longrightarrow G_0(A) \longrightarrow G_0(B) \longrightarrow 0.$$

Moreover, if A is right regular then there is an exact sequence

$$K_1(A) \longrightarrow K_1(B) \longrightarrow \coprod_{\underline{p} \in X} G_0(A/\underline{p} A) \longrightarrow K_0(A) \longrightarrow K_0(B) \longrightarrow 0$$

Proof. Once the isomorphism (6) is established the exact sequences here follow from those of (6.2) and (6.7), respectively, using (6) to substitute $\coprod_{\underline{p} \in X} G_0(A/\underline{p} A)$ for $G_0(A, S)$ in the latter.

If $\underline{p} \in X$ write $\underline{\underline{M}}_{\underline{p}}(A)$ for the category of $M \in \underline{\underline{M}}(A)$

which are annihilated by a power of $\underline{p}$. If $M \in \underline{\underline{M}}_S(A)$ then $M\underline{a} = 0$ for some $\underline{a} \neq 0$ in R. If $\underline{a} = \underline{p}_1^{n_1} \cdots \underline{p}_r^{n_r}$ is the prime factorization of $\underline{a}$ in R then the Chinese Remainder Theorem implies $R/\underline{a} \simeq \Pi\, R/\underline{p}_i^{n_i}$. Thus M decomposes canonically as $M = M_1 \oplus \cdots \oplus M_n$, where M_i consists of the elements of M killed by some power of $\underline{p}_i$. It follows easily from this decomposition that $G_i(A, S) = K_i(\underline{\underline{M}}_S(A)) = \coprod_{\underline{p} \in X} K_i(\underline{\underline{M}}_{\underline{p}}(A))$. Next observe that $\underline{\underline{M}}(A/\underline{p}\, A) \subset \underline{\underline{M}}_{\underline{p}}(A)$. If $M \in \underline{\underline{M}}_{\underline{p}}(A)$ then $M \supset M\underline{p} \supset M\underline{p}^2 \supset \cdots$ is a finite characteristic filtration with successive factors in $\underline{\underline{M}}(A/\underline{p}\, A)$. Hence it follows from (VIII, 3.2) that $G_i(A/\underline{p}\, A) = K_i(\underline{\underline{M}}(A/\underline{p}\, A)) \longrightarrow K_i(\underline{\underline{M}}_{\underline{p}}(A))$ is an isomorphism ($i = 0, 1$). This completes the proof.

(6.10) PROPOSITION. Let R, $L = S^{-1}R$, and X be as in (6.9) and let $D(R) = \underline{\underline{Z}}^{(X)}$ be the divisor group of R. Let A be a commutative regular integral domain containing R such that $\underline{p}\, A$ is prime for all $\underline{p} \in X$. Set $B = A \otimes_R L$. Then there is a commutative diagram with exact rows and columns,

$$
\begin{array}{ccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & SK_1(A) & \longrightarrow & SK_1(B) & \longrightarrow & \coprod_{\underline{p} \in X} \tilde{G}_o(A/\underline{p}A) & \longrightarrow & \widetilde{Rk}_o(A) & \longrightarrow & \widetilde{Rk}_o(B) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & K_1(A) & \longrightarrow & K_1(B) & \longrightarrow & \coprod_{\underline{p} \in X} G_o(A/\underline{p}A) & \longrightarrow & Rk_o(A) & \longrightarrow & Rk_o(B) & \longrightarrow 0 \\
 & \downarrow{\scriptstyle det_1(A)} & & \downarrow{\scriptstyle det_1(B)} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle det_o(A)} & & \downarrow{\scriptstyle det_o(B)} & \\
 & U(A) & \longrightarrow & U(B) & \longrightarrow & D(R) & \longrightarrow & Pic(A) & \longrightarrow & Pic(B) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

If $\underline{p} \in X$ *and if* $M \in \underline{\underline{M}}(A/\underline{p}\, A)$ *then* $\delta[M] = rk(M)\underline{p}$, *where* $rk(M)$ *is the rank of* M *over the integral domain* $A/\underline{p}\, A$.

Proof. The proposition is trivial if $R = L$, so assume not. Then $X = Ht_1(R)$. If $\underline{p} \in X$ then $\underline{p} \subset \underline{p}\, A \cap R \subset R$. Since $\underline{p}\, A$ is prime and $\underline{p}$ is maximal we must have $\underline{p} = \underline{p}\, A \cap R$. Thus we have the hypothesis of (III, 7.18), thanks to the fact that S is factorial in R and in A, because of the regularity of R and A (cf. (III, 7.21)). It follows therefore from (III, 7.18) and from (III, 7.17) that $D(R) = Pic(R, S) \simeq Pic(A, S) \simeq D(A, S)$.

To define the lower two thirds of the diagram we start with the epimorphism of exact sequences in (6.8). We replace $G_0(A, S) = K_0(A, S) \xrightarrow{det_0(A,\, S)} Pic(A, S)$ in that sequence using the isomorphism $D(R) \simeq Pic(A, S)$ derived above, and the isomorphism $\coprod_{\underline{p} \in X} G_0(A/\underline{p}\, A) \longrightarrow G_0(A, S)$ of (6.9). Since A is regular we can use the Cartan homomorphisms to identify $det_0(A, S)$ above with $\chi_S\colon G_0(A, S) \longrightarrow D(A, S)$ (see (6.4)). Recall that for $M \in \underline{\underline{M}}_S(A)$ $\chi_S(M) = \Sigma\, \ell_{A_{\underline{p}}}(M_{\underline{p}})$ ($\underline{p} \in Ht_1(A)$, $\underline{p} \cap S \neq \phi$). According to (III, 7.18), quoted already above, $\underline{p} \longmapsto \underline{p}\, A$ is a bijection from X to $\{\underline{p} \in Ht_1(A) \mid \underline{p} \cap S \neq \phi\}$. In particular, if $\underline{p} \in X$ and if $M \in \underline{\underline{M}}(A/\underline{p}\, A)$ then $M_{\underline{q}A} = 0$ for $\underline{q} \neq \underline{p}$ in X, so $\chi_S(M) = \ell_{A_{\underline{p}A}}(M_{\underline{p}A})(\underline{p}\, A)$. Since $M\underline{p} = 0$, $M_{\underline{p}A}$ is a vector space over the field of fractions of $A/\underline{p}\, A$ (= the residue class field of $A_{\underline{p}A}$) so $\ell_{A_{\underline{p}A}}(M_{\underline{p}A})$ is just the rank (cf. proof of (6.3)) of $M_{\underline{p}A}$ as a module over the integral domain $A/\underline{p}\, A$. This establishes the alleged description of δ. Now the top row is just the kernel of the morphism from the middle to the bottom (by definition in the case of $\widetilde{Rk}_0$). The top row is exact because of the long homology sequence, plus the fact that the epimorphisms det_1 split. q.e.d.

(6.11) COROLLARY. *In the setting of* (6.10) *assume that* B *and each* $A/\underline{p}\, A$ ($\underline{p} \in X$) *is a Dedekind ring. Then* there is an exact sequence

$$SK_1(A) \longrightarrow SK_1(B) \longrightarrow \coprod_{\underline{p} \in X} \mathrm{Pic}(A/\underline{p}\, A)$$

$$\longrightarrow Rk_0(A) \xrightarrow{\det_0(A)} \mathrm{Pic}(A) \longrightarrow 0$$

Proof. In the diagram of (6.10) we can identify $G_0(A/\underline{p}\, A)$ with $K_0(A/\underline{p}\, A)$, and then $G_0(A/\underline{p}\, A) = \mathrm{Ker}(G_0(A) \xrightarrow{rk} \underline{\underline{Z}})$ is identified with $Rk_0(A/\underline{p}\, A)$. Moreover (3.8) implies $\det_0 : Rk_0 \longrightarrow \mathrm{Pic}$ is an isomorphism for Dedekind rings. Therefore $Rk_0(B) = 0$, and we deduce from (6.10) a diagram

$$SK_1(A) \longrightarrow SK_1(B) \longrightarrow \prod_{\underline{p} \in X} Pic(A/\underline{p}A) \longrightarrow \tilde{R}k_o(A) \longrightarrow 0$$

$$0 \longrightarrow \tilde{R}k_o(A) \longrightarrow Rk_o(A) \xrightarrow{det_o(A)} Pic(A) \longrightarrow 0$$

with exact row and column. The corollary follows immediately from this.

§7. APPENDIX: THE CATEGORY FP

We fix a commutative ring A. The category

$$\underline{\underline{FP}} = \underline{\underline{FP}}(A)$$

has, as objects, the faithfully projective A-modules. Its morphisms are the A-isomorphisms between such modules, and it is equipped with the product, $\otimes_A$, in the sense of Chapter VII. Note, therefore, that the inclusion

$$\underline{\underline{Pic}}(A) \subset \underline{\underline{FP}}(A)$$

is a product preserving functor (though not cofinal). According to (4.6), a module $P \in$ mod-A belongs to $\underline{\underline{FP}}$ if and only if $P \otimes_A Q \simeq A^n$ for some $Q \in$ mod-A and some $n > 0$. In particular, therefore, the free modules are cofinal in $\underline{\underline{FP}}$. It follows from this also that a homomorphism $A \longrightarrow B$ of commutative rings induces a cofinal, product preserving functor, $\otimes_A B\colon \underline{\underline{FP}}(A) \longrightarrow \underline{\underline{FP}}(B)$.

The purpose of this section is to calculate the groups $K_i\underline{\underline{FP}}$ in terms of the groups $K_i\underline{\underline{P}}$.

Recall that we have a (split) exact sequence

$$0 \longrightarrow Rk_0(A) \longrightarrow K_0(A) \xrightarrow{rk} H_0(A) \longrightarrow 0,$$

and this induces

$$0 \longrightarrow \underline{\underline{Q}} \otimes Rk_0(A) \longrightarrow \underline{\underline{Q}} \otimes K_0(A) \xrightarrow{rk} \longrightarrow \underline{\underline{Q}} \otimes H_0(A) \longrightarrow 0,$$

where the tensors here will be all understood to be over $\underline{\underline{Z}}$. Recall that $H_0(A)$, the ring of continuous functions from spec(A) to $\underline{\underline{Z}}$, is additively generated by the characteristic functions of supp(eA), $e = e^2$ in A. (This followed from quasi-compactness.) Consequently we can identify $\underline{\underline{Q}} \otimes H_0(A)$

with the ring of continuous functions from spec(A) to $\underline{\underline{Q}}$. Thus we can define

$$U^+(\underline{\underline{Q}} \otimes K_0(A))$$

to be the set of $x \in \underline{\underline{Q}} \otimes K_0(A)$ such that rk(x) takes only strictly positive values. Writing $U^+(\underline{\underline{Q}} \otimes H_0(A))$ for the functions from spec(A) to the positive rationals we see that $U^+(\underline{\underline{Q}} \otimes H_0(A))$ is a subgroup of the group of units of $\underline{\underline{Q}} \otimes H_0(A)$. Since $Rk_0(A)$ is a nil ideal (see (4.6)) it follows that $\underline{\underline{Q}} \otimes Rk_0(A)$ is also nil, and hence in rad($\underline{\underline{Q}} \otimes K_0(A)$). Therefore an element of $\underline{\underline{Q}} \otimes K_0(A)$ is invertible if and only if its rank is. Thus we have a split exact sequence of groups (of units),

(1) $$0 \longrightarrow 1 + (\underline{\underline{Q}} \otimes Rk_0(A)) \longrightarrow U^+(\underline{\underline{Q}} \otimes K_0(A)) \longrightarrow U^+(\underline{\underline{Q}} \otimes H_0(A)) \longrightarrow 0.$$

If $x \in \underline{\underline{Q}} \otimes Rk_0(A)$ then x is nilpotent, so we have the <u>polynomials</u>

$$\exp(x) = \sum_{n \geq 0} x^n/n!$$

and

$$\log(1 + x) = - \sum_{n>0} (-x)^n/n.$$

These are inverse group isomorphisms

(2) $$\underline{\underline{Q}} \otimes Rk_0(A) \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 + (\underline{\underline{Q}} \otimes Rk_0(A)).$$

Combining (1) and (2) we have

(3) $$U^+(\underline{\underline{Q}} \otimes K_0(A)) \simeq U^+(\underline{\underline{Q}} \otimes H_0(A)) \oplus (\underline{\underline{Q}} \otimes Rk_0(A)).$$

The relevance of this to our present interest is:

(7.1) THEOREM. *The map* $P \longmapsto 1 \otimes [P]_{\underline{\underline{P}}}$ *from* ob$\underline{\underline{FP}}$(A) *to* $\underline{\underline{Q}} \otimes K_0(A)$ *induces an isomorphism*

$$K_o\underline{\underline{FP}}(A) \longrightarrow U^+(Q \otimes K_o(A)).$$

Proof. If $P \in \underline{\underline{FP}}$ then $[P: A]$ is everywhere positive, by (4.6), so $hP = 1 \otimes [P]_{\underline{\underline{P}}} \in U^+(\underline{\underline{Q}} \otimes K_o(A))$. Evidently $h(P \otimes_A Q) = h(P)\, h(Q)$ for $P, Q \in \underline{\underline{FP}}$, so h induces a homomorphism

$$h\colon K_o\underline{\underline{FP}}(A) \longrightarrow U^+(\underline{\underline{Q}} \otimes K_o(A)).$$

Suppose $[P]_{\underline{\underline{FP}}} - [Q]_{\underline{\underline{FP}}}$ is in Ker(h), i.e. $1 \otimes [P]_{\underline{\underline{P}}} = 1 \otimes [Q]_{\underline{\underline{P}}}$. Then $[P]_{\underline{\underline{P}}} - [Q]_{\underline{\underline{P}}}$ has finite additive order in $K_o(A)$; say $n[P]_{\underline{\underline{P}}} = n[Q]_{\underline{\underline{P}}}$ for some $n > 0$. This means that $A^n \otimes_A P$ and $A^n \otimes_A Q$ are stably isomorphic. After multiplying n by a large factor, if necessary, we can arrange (see (4.2)) that $A^n \otimes P \simeq A^n \otimes Q$. But then $[P]_{\underline{\underline{FP}}} = [Q]_{\underline{\underline{FP}}}$. Thus h is injective.

Finally, suppose $1/n \otimes x \in U^+(\underline{\underline{Q}} \otimes K_o(A))$ where $n > 0$ and $x \in K_o(A)$. Then rk(x) is an everywhere positive function on spec(A), so (4.2) implies there is an $m > 0$ such that $mx = [P]_{\underline{\underline{P}}}$ for some $P \in \underline{\underline{P}}$. Since $[P: A]$ is everywhere positive we have $P \in \underline{\underline{FP}}$. Therefore $1/n \otimes x = 1/nm \otimes mx = (1 \otimes nm)^{-1}\ (1 \otimes [P]_{\underline{\underline{P}}}) = h(A^{nm})^{-1}\ h(P)$. This shows that h is surjective, and hence completes the proof.

In order to compute $K_1\underline{\underline{FP}}$ we shall require a lemma on direct limits.

Let $L = (W_n;\ f_{n,nm}\colon W_n \longrightarrow W_{nm})_{n,\ m \in \underline{\underline{N}}}$ be a direct system of abelian groups indexed by the positive integers $\underline{\underline{N}}$, ordered by divisibility. We then define a new direct system

$$L' = (W_n;\ f'_{n,nm}\colon W_n \longrightarrow W_{nm})$$

by $f'_{n,nm} = m\, f_{n,nm}$, and a morphism,

$$(n1_{W_n})_{n \in \mathbb{N}} : L \longrightarrow L',$$

of direct systems. The required commutativity conditions are easily seen:

$$\begin{array}{ccc} W_n & \xrightarrow{\ n1_{W_n}\ } & W_n \\ {\scriptstyle f_{n,nm}}\downarrow & & \downarrow{\scriptstyle f'_{n,nm} \,=\, m\, f_{n,nm}} \\ W_{nm} & \xrightarrow[\ nm1_{W_{nm}}\ ]{} & W_{nm} \end{array}$$

L' is a functor of L, and $L \longrightarrow L'$ is a natural transformation whose cokernel we denote by $L'' = (W_n/nW_n ; f''_{n,nm})$:

$$L \longrightarrow L' \longrightarrow L'' \longrightarrow 0.$$

(7.2) LEMMA. *Let* L *be a direct system as above. Then the sequences*

$$\underset{\rightarrow}{L} \longrightarrow \underset{\rightarrow}{L}' \longrightarrow \underset{\rightarrow}{L}'' \longrightarrow 0$$

and

$$\underset{\rightarrow}{L} \otimes (\mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0)$$

are naturally isomorphic.

Proof. Let $E = (Z_n ; e_{n,nm})$ be the system with $Z_n = \mathbb{Z}$ and $e_{n,nm} = 1_{\mathbb{Z}}$ for all $n, m \in \mathbb{N}$. Evidently the exact sequence of direct systems

$$L \longrightarrow L' \longrightarrow L'' \longrightarrow 0$$

and

$$L \otimes (E \longrightarrow E' \longrightarrow E'' \longrightarrow 0)$$

are isomorphic. (Here $L \otimes E = (W_n \otimes Z_n ; f_{n,nm} \otimes e_{n,nm})$, etc.) Since $L \longmapsto \underset{\rightarrow}{L}$ is an exact functor the lemma will follow if we show that $\underset{\rightarrow}{E} \longrightarrow \underset{\rightarrow}{E}'$ is naturally isomorphic to $\underline{\underline{Z}} \longrightarrow \underline{\underline{Q}}$. Since all $e_{n,nm}$ are isomorphisms $\underset{\rightarrow}{E} = \underline{\underline{Z}}$. Moreover $E' \longrightarrow \underline{\underline{Q}}$ E' is a monomorphism and the morphisms in the system $\underline{\underline{Q}} \otimes E'$ are isomorphisms, so $\underrightarrow{(\underline{\underline{Q}} \otimes E')} = \underline{\underline{Q}}$. However $\underset{\rightarrow}{E}'$ is clearly divisible, so $\underset{\rightarrow}{E}' = \underline{\underline{Q}}$. q.e.d.

Since the free modules are cofinal in $\underline{\underline{FP}}(A)$ it follows from (VII, 2.3) that we can compute $K_1\underline{\underline{FP}}(A)$ as the direct limit of the commutator factor groups, W_n, of $\mathrm{Aut}_A(A^n) = GL_n(A)$:

$$W_n = GL_n(A)/[GL_n(A), GL_n(A)].$$

Of course the limit is taken with respect to the homomorphisms

$$g_{n,nm}: W_n \longrightarrow W_{nm}$$

induced by

$$\alpha \longmapsto \alpha \otimes I_m = \begin{pmatrix} \alpha & & & 0 \\ & \alpha & & \\ & & \ddots & \\ 0 & & & \alpha \end{pmatrix} \quad (\alpha \in GL_n(A)).$$

Consider also the homomorphism

$$f_{n,nm}: W_n \longrightarrow W_{nm}$$

induced by

$$\alpha \longmapsto \alpha \oplus I_{n(m-1)} = \begin{pmatrix} \alpha & & & 0 \\ & I_n & & \\ & & \ddots & \\ 0 & & & I_n \end{pmatrix} (\alpha \in GL_n(A)).$$

According to the Whitehead lemma (V, 1.7) we have

$$\begin{pmatrix} \alpha & & & 0 \\ & \alpha & & \\ & & \ddots & \\ 0 & & & \alpha \end{pmatrix} \equiv \begin{pmatrix} \alpha^m & & & 0 \\ & I_n & & \\ & & \ddots & \\ 0 & & & I_n \end{pmatrix} \bmod E_{nm}(A),$$

and (V, 1.5) implies that $E_n(A) \subset [GL_n(A), GL_n(A)]$ for $n \geq 3$. If n, m > 1 then nm > 3 so we conclude then that

$$g_{n,nm} = m\, f_{n,nm},$$

i.e. that $g_{n,nm} = f'_{n,nm}$ in the notation of (7.2). Since $\underrightarrow{\lim}\ (W_n; f_{n,nm}) = K_1(A)$, clearly, we conclude from (7.2) that:

(7.3) THEOREM. <u>There is a natural isomorphism</u>

$$K_1\underline{\underline{FP}}(A) \simeq \underline{\underline{Q}} \otimes K_1(A) \simeq (\underline{\underline{Q}} \otimes U(A)) \oplus (\underline{\underline{Q}} \otimes SK_1(A)).$$

We can even pass to the limit

$$GL_\otimes(A) = \underrightarrow{\lim}\ (GL_n(A); \alpha \longmapsto \alpha \otimes I_m)_{n,\ m \in \underline{\underline{N}}},$$

and, by (VII, 2.3), write

$$K_1\underline{\underline{FP}}(A) = GL_\otimes(A)/[GL_\otimes(A), GL_\otimes(A)].$$

The elements of $GL_\otimes(A)$ can be represented as infinite matrices of the form

$$(4) \qquad \bar{\alpha} = \begin{pmatrix} \alpha & & & & 0 \\ & \alpha & & & \\ & & \ddots & & \\ & & & \alpha & \\ 0 & & & & \ddots \end{pmatrix} \qquad (\alpha \in GL_n(A) \text{ for some } n > 0).$$

If we write $\det(\bar{\alpha}) = 1/n \otimes \det(\alpha) \in \underline{\underline{Q}} \otimes U(A)$ then it is easy to see that this does not depend on the choice of α to represent $\bar{\alpha}$ (note, for example, that $\overline{\alpha \otimes I_m} = \bar{\alpha}$ for all

$m > 0$.) The resulting homomorphism

$$\det: GL_{\otimes}(A) \longrightarrow \underline{\underline{Q}} \otimes U(A)$$

is just the projection on the first summand in (7.3).

The inclusion $\underline{\underline{Pic}}(A) \subset \underline{\underline{FP}}(A)$ induces homomorphisms

(5) $\quad Pic(A) \longrightarrow K_0\underline{\underline{FP}}(A)$

and

(6) $\quad U(A) \longrightarrow K_1\underline{\underline{FP}}(A)$.

(7.4) PROPOSITION. (0) _The kernel of_ (5) _is the torsion subgroup of_ Pic(A), _and its image lies in the subgroup corresponding to_ $\underline{\underline{Q}} \otimes Rk_0(A)$ in (7.1).

(1) _The kernel of_ (6) _is the torsion subgroup of_ U(A), _and its image corresponds to the subgroup_ $1 \otimes U(A) \subset \underline{\underline{Q}} \otimes U(A)$ _in_ (7.3). _Thus the cokernel of_ (6) _is_

$$(\underline{\underline{Q}}/\underline{\underline{Z}} \otimes U(A)) \oplus (\underline{\underline{Q}} \otimes SK_1(A)).$$

Proof. (0) If $P \in \underline{\underline{Pic}}(A)$ then $[P: A] = 1$ so $[P]_{\underline{\underline{P}}} \in 1 + Rk_0(A)$; the last assertion follows from this because $\underline{\underline{Q}} \otimes Rk_0(A)$ corresponds to the subgroup $1 + (\underline{\underline{Q}} \otimes Rk_0(A)) \subset U^+(\underline{\underline{Q}} \otimes K_0(A))$ in (7.1).

If $[P]_{\underline{\underline{FP}}} = [A]_{\underline{\underline{FP}}}$ then $P \otimes_A Q \simeq A \otimes_A Q$ for some $Q \in \underline{\underline{FP}}$, and we can even take $Q = A^n$ for some $n > 0$, since the free modules are cofinal. Then $P^n \simeq A^n$ so $A = \Lambda^n(A^n) \simeq \Lambda^n(P^n) \simeq P^n$. Hence $[P]_{\underline{\underline{Pic}}}$ has order n.

Conversely, suppose $[P]_{\underline{\underline{Pic}}}$ has finite order $n > 0$, so $P^n \simeq A$. Set $Q = A \oplus P \oplus P^2 \oplus \ldots \oplus P^{(n-1)}$. Evidently, $P \otimes_A Q \simeq Q \simeq A \otimes_A Q$, and $Q \in \underline{\underline{FP}}$. Hence $[P]_{\underline{\underline{FP}}} = [A]_{\underline{\underline{FP}}}$. q.e.d.

(1) If $u \in U(A) = GL_1(A)$ then we have

$$\bar{u} = \begin{pmatrix} u & & & & & 0 \\ & u & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & u & \\ & & & & & \cdot \\ C & & & & & & \cdot \end{pmatrix} \quad \varepsilon\ GL_{\otimes}(A),$$

and the homomorphism (6): $U(A) \longrightarrow K_1(\underline{\underline{FP}}(A))$ is induced by this inclusion

$$U(A) \subset GL_{\otimes}(A).$$

In fact, we see that <u>this identifies</u> U(A) <u>with the center of</u> $GL_{\otimes}(A)$. Since the decomposition $K_1(A) = U(A) \oplus SK_1(A)$ is induced by the splitting $U(A) = GL_1(A) \subset GL_n(A) \xrightarrow{\det} U(A)$ it follows from the way in which the isomorphism in (7.3) is constructed that (6) corresponds to the map

$$U(A) \longrightarrow (\underline{\underline{Q}} \otimes U(A)) \oplus (\underline{\underline{Q}} \otimes SK_1(A))$$

$$u \longmapsto (1 \otimes u) = \det(\bar{u}).$$

The assertions of (1) follow immediately from this. q.e.d.

If we write

$$PGL(A) = GL_{\otimes}(A)/U(A) = \underrightarrow{\lim}\ (PGL_n(A),\ h_{n,nm}),$$

where $PGL_n(A) = GL_n(A)/(\text{scalars})$, and where $h_{n,nm}$ is induced by $\alpha \longmapsto \alpha \otimes I_m$ ($\alpha\ \varepsilon\ GL_n(A)$), then we conclude from (7.4) (1) that

$$PGL(A)/[PGL(A),\ PGL(A)] \simeq (\underline{\underline{Q}}/\underline{\underline{Z}} \otimes U(A)) \oplus (\underline{\underline{Q}} \otimes SK_1(A)),$$

where the projection on the first factor is the map induced by the determinant (on $GL_{\otimes}(A)$).

§8. APPENDIX: THE SYMMETRIC ALGEBRA IS INVERSE TO THE EXTERIOR ALGEBRA

If A is a ring and if t is an indeterminate we shall write, for $M \in A$-mod,

$$M[[t]] = \{\text{formal power series } \sum_{i \geq 0} m_i t^i \ (m_i \in M)\}.$$

This is a left module over the power series ring $A[[t]]$. The "constant term" is a homomorphism $A[[t]] \longrightarrow A$, and we write

$$U_1(A[[t]]) = \mathrm{Ker}(U(A[[t]]) \longrightarrow U(A)).$$

If $F \in A[[t]]$ then $1 - tF$ is invertible, with inverse $\Sigma_{n \geq 0} t^n F^n$. It follows that

$$U_1(A[[t]]) = 1 + tA[[t]].$$

Henceforth we fix a <u>commutative ring</u> A. Assume that we are given a (non additive) functor $L = L_A$ from A-modules to graded A-modules,

$$P \longmapsto L(P) = (L^n(P))_{n \geq 0},$$

which satisfies the following conditions:

(i) L° is the constant functor, $P \longmapsto A$. If $P \in \underline{\underline{P}}(A)$ then $L^n(P) \in \underline{\underline{P}}(A)$ for all $n \geq 0$.

(ii) There is a natural isomorphism

$$L(P \oplus Q) \simeq L(P) \otimes_A L(Q).$$

(This is a tensor product of graded modules, so the isomorphism consists of isomorphisms

$$L^n(P \oplus Q) \simeq \coprod_{i+j=n} L^i(P) \otimes_A L^j(Q)$$

for each $n \geq 0$.)

(iii) If $A \longrightarrow B$ is a homomorphism of commutative rings then there is a natural isomorphism of graded B-modules,

$$L_A(P) \otimes_A B \simeq L_B(P \otimes_A B).$$

I.e. "L commutes with base change."

With this L at hand we can define

$$L\colon K_o(A) \longrightarrow U_1(K_o(A)[[t]])$$

by

$$L[P] = \sum_{n\geq 0} [L^n P]t^n \qquad (P \in \underline{\underline{P}}(A)).$$

Property (i) shows that the right side lies in $U_1(K_0(A)[[t]])$, and property (ii) shows that it is an additive function from $\text{ob}\underline{\underline{P}}(A)$ to an abelian group. Hence L is a well defined homomorphism:

(1) $\qquad L(x + y) = L(x)\, L(y)$

Moreover property (iii) shows that it is natural in the sense that

$$\begin{array}{ccc} K_o(A) & \longrightarrow & K_o(B) \\ \downarrow L_A & & \downarrow L_B \\ U_1(K_o(A)[[t]]) & \longrightarrow & U_1(K_o(B)[[t]]) \end{array}$$

commutes, where $A \longrightarrow B$ is as in (iii).

Next suppose $(P, \alpha) \in \Sigma\underline{\underline{P}}(A)$, i.e. $P \in \underline{\underline{P}}(A)$ and $\alpha \in \text{Aut}_A(P)$. Then we can define

$$L\colon \text{ob}\Sigma\underline{\underline{P}}(A) \longrightarrow K_1(A)[[t]]$$

by

$$L(P, \alpha) = \Sigma_{n\geq 0}\, [L^n P, L^n\alpha]t^n.$$

(Note that $[L^\circ P, L^\circ\alpha] = [A, 1_A] = 0$ according to (i).) If also $\beta \in \text{Aut}_A(P)$ then $L^n(\alpha\beta) = L^n(\alpha)\, L^n(\beta)$ (L is a functor) so we have

(2) $\qquad L(P, \alpha\beta) = L(P, \alpha) + L(P, \beta) \qquad (P \in \underline{\underline{P}}(A);$
$\alpha, \beta \in \text{Aut}_A(P)).$

If (P, α) and (Q, β) are two objects of $\Sigma\underline{\underline{P}}(A)$ then

$$L(P \oplus Q, \alpha \oplus \beta) = \Sigma_{n\geq 0} [\coprod_{i+j=n} (L^iP \otimes_A L^jQ, L^i\alpha \otimes L^j\beta)]t^n.$$

Since $[L^iP \otimes_A L^jQ, L^i\alpha \otimes L^j\beta] = [L^iP \otimes L^jQ, 1_{L^iP} \otimes L^j\beta] + [L^iP \otimes L^jQ, L^i\alpha \otimes 1_{L^jQ}] = [L^iP] [L^jQ, L^j\beta] + [L^jQ] [L^iP, L^i\alpha]$, (using the $K_o(A)$-module structure of $K_1(A)$), we conclude from the formula above that

(3) $$L(P \oplus Q, \alpha \oplus \beta) = L[P]\, L(Q, \beta) + L[Q]\, L(P, \alpha).$$

This suggests that we introduce

$$L_1(P, \alpha) = L[P]^{-1}\, L(P, \alpha).$$

For then it follows from (2) that L_1 is still additive with respect to composition. Moreover, combining (3) and (1) we have

$$\begin{aligned} L_1(P \oplus Q, \alpha \oplus \beta) &= L[P \oplus Q]^{-1}\, L(P \oplus Q, \alpha \oplus \beta) \\ &= L[P]^{-1}\, L[Q]^{-1}\, (L[P]\, L(Q, \beta) \\ &\quad + L[Q]\, L(P, \alpha)) \\ &= L_1(P, \alpha) + L_1(Q, \beta). \end{aligned}$$

It follows therefore that L_1 induces an additive homomorphism

$$L_1\colon K_1(A) \longrightarrow K_1(A)[[t]].$$

Just as for L and K_o, this is a natural transformation.

(8.1) EXAMPLE. Let $L_A = \Lambda_A$, the exterior algebra. The conditions (i), (ii), and (iii) are well known and Λ^1 is the identity functor. $\Lambda^n(A) = 0$ for $n > 1$ so we have $\Lambda[A] = 1 + t$. Hence $\Lambda[A^n] = (1 + t)^n$.

If $u \in U(A)$ write $[u] = [A, u] \in K_1(A)$. Then $\Lambda(A, u) = [u]t$ so $L_1[u] = (1 + t)^{-1} [u]t = [u] (t - t^2 + t^3 - \dots)$.

(8.2) EXAMPLE. Let $L_A = S_A$, the symmetric algebra. Again conditions (i), (ii) and (iii) are well known, and S^1 is the identity functor. $S^n(A) \simeq A$ for all $n \geq 0$ so we have $S[A] = \Sigma_{n>0} t^n = (1 - t)^{-1}$. Thus

$$S[A](t) = \Lambda[A](-t)^{-1}.$$

We shall generalize this fact in (8.4) below.

If $u \in U(A)$ then $S_1[u] = (1 - t) (\Sigma_{n\geq 0} [u^n]t^n) = \Sigma_{n>0} ([u^n] - [u^{n-1}])t^n = \Sigma_{n>0} [u]t^n = [u] (\Sigma_{n>0} t^n)$.

Let P be any A-module and let $d: P \longrightarrow A$ be a linear functional. Then d extends to a derivation of $\Lambda(P)$ by

$$(4) \qquad d(x_1 \Lambda \dots \Lambda x_n) = \sum_{1 \leq i \leq n} (-1)^i d(x_i) (x_1 \Lambda \dots \hat{i} \dots \Lambda x_n).$$

This defines a positive complex, the "Koszul complex",

$$
(5) \qquad \cdots \longrightarrow \Lambda^{n}(P) \longrightarrow \Lambda^{n-1}(P) \longrightarrow \cdots \longrightarrow \begin{array}{c} \Lambda^{1}(P) \\ \| \\ P \end{array} \begin{array}{c} \longrightarrow \\ \\ \xrightarrow{\ d\ } \end{array} \begin{array}{c} \Lambda^{\circ}(P) \\ \| \\ A \end{array} \longrightarrow 0 \cdots,
$$

whose zero homology is Coker(d) = $A/\mathfrak{a}$, where $\mathfrak{a} = \mathrm{Im}(P \xrightarrow{d} A)$. We have the following well known criterion for the vanishing of the other homology (see, e.g., Serre [2], Chapter IV, Part A) which we quote without proof.

(8.3) PROPOSITION. *Let* $P = A^n$, *with basis* $(e_i)_{1\le i\le n}$, *and let* $d: P \longrightarrow A$ *by* $d(e_i) = a_i$. *Set* $\mathfrak{a}_i = \sum_{0\le j<i} Aa_j$, *and assume the image of* a_i *in* $A/\mathfrak{a}_i$ *is not a divisor of zero* $(1 \le i \le n)$. *Then* (5) *is a free resolution of* $A/\mathfrak{a}_{n+1}$.

This proposition applies notably when A is a polynomial ring, $A = B[a_1,\ldots,a_n] \approx S_B(B^n)$.

We shall apply (8.3) to the following situation. Let P be an A-module. Then $P = S^1(P) \subset S(P)$ so this inclusion induces an S(P)-linear map

$$d: S(P) \otimes_A P \longrightarrow S(P).$$

As above this extends to a derivation of the exterior algebra, $\Lambda_{S(P)}(S(P) \otimes_A P)$, whose zero homology is

$$A = S(P)/(\text{the ideal generated by } P = S^1(P)).$$

If P is free with basis $(e_i)_{1\le i\le n}$ then S(P) is the polynomial ring $A[e_1,\ldots,e_n]$. Moreover $S(P) \otimes_A P$ is a free S(P)-module with basis $(1 \otimes e_i)_{1\le i\le n}$, and $d(1 \otimes e_i) = e_i$. Therefore (8.3) implies $\Lambda_{S(P)}(S(P) \otimes_A P)$ is a projective resolution of A in this case. More generally, if $P \in \underline{\underline{P}}(A)$ then we can apply the last conclusion to all the localizations $P_{\mathfrak{m}} \in \underline{\underline{P}}(A_{\mathfrak{m}})$, which are free, and conclude again that $\Lambda_{S(P)}(S(P) \otimes_A P)$ is acyclic except in degree zero.

Note that $\Lambda_{S(P)}(S(P) \otimes_A P) = S(P) \otimes_A \Lambda_A(P)$, so that it is bigraded, by $S^n(P) \otimes_A \Lambda^m(P)$ $(n, m \ge 0)$. Moreover, since d is extended S(P)-linearly from the inclusion $P = S^1(P) \subset S(P)$ it follows from (4) that d induces homomorphisms

$$S^n(P) \otimes_A \Lambda^m(P) \longrightarrow S^{n+1}(P) \otimes_A \Lambda^{m-1}(P).$$

This shows that, as a complex of A-modules, $\Lambda_{S(P)}(S(P) \otimes_A P)$ is a direct sum of subcomplexes of the form

$$(6)_n \qquad 0 \longrightarrow S(P) \otimes_A \Lambda^n(P) \longrightarrow S^1(P) \otimes_A \Lambda^{n-1}(P) \longrightarrow \cdots \longrightarrow S^n(P) \otimes_A \Lambda^0(P) \longrightarrow 0,$$

one for each $n \geq 0$. We have seen above that if $P \in \underline{\underline{P}}(A)$ then $(6)_n$ is acyclic except for the one term complex $(6)_0$: $0 \longrightarrow S^0(P) \otimes_A \Lambda^0(P) \longrightarrow 0$. In this case, therefore, we have

$$\sum_{0 \leq i \leq n} (-1)^i [S^i(P))] [\Lambda^i(P)] = 0 \text{ in } K_0(A)$$

for all $n > 0$). This proves the following elegant formula:

(8.4) PROPOSITION. Let

$$\Lambda, S: K_0(A) \longrightarrow U_1(K_0(A)[[t]])$$

be the maps corresponding to the exterior algebra (see (8.1) and the symmetric algebra (see (8.2)), respectively. Then if $x \in K_0(A)$ we have

$$\Lambda(x)(t). S(x)(-t) = 1.$$

HISTORICAL REMARKS

The exposition in the early sections is derived largely from Bass [1] and Bass-Murthy [1]. Milnor's theorem is taken from some unpublished notes of Milnor. The G-sequence of a localization is due to Heller-Reiner [1], and the analogous K-sequence is stated, with an incomplete proof, in Bass-Murthy [1]. The calculation of $K_i\underline{\underline{FP}}$ is lifted from my Tata notes [4]. The formula relating Λ and S in §8 seems to be well known to the experts. I learned of it in a conversation with Bott.

Chapter X

FINITENESS THEOREMS FOR RINGS OF ARITHMETIC TYPE

Throughout this chapter we fix the following data and notation:

(0)

- R is a Dedekind ring.
- $X = \max(R)$.
- $L = S^{-1}R$ ($S = R - \{0\}$) is the field of fractions of R.
- Λ is a semi-simple finite L-algebra.
- A is an R-order in Λ which is an R-lattice (i.e. a finitely generated R-module, in this case).

Our intention is to prove that the abelian groups $K_i(A)$ and $G_i(A)$ ($i = 0, 1$) are finitely generated when $R = \underline{\underline{Z}}$ and (only sometimes for $i = 1$) when $R = \underline{\underline{F}}[t]$, a polynomial ring over a finite field $\underline{\underline{F}}$. The proofs consist of a reduction to classical theorems on the finiteness of class number and the finite generation of units. We also employ a technique of Swan [1], who first proved some of these results in the case when $A = R\pi$ is the group ring of a finite group π.

§1. SWAN'S TRIANGLE, AND THE CARTAN CONDITION

We keep the notation of (0). The inclusion $A \subset \Lambda =$

$S^{-1}A$ leads to a commutative diagram with exact rows:

$$\begin{array}{cccccccccc}
 & K_1(A) & \xrightarrow{k_1} & K_1(\Lambda) & \longrightarrow & K_0(A,\ S) & \longrightarrow & K_0(A) & \xrightarrow{k_0} & K_0(\Lambda) \\
 & & & (\simeq)\big\downarrow c_1(\Lambda) & & \big\downarrow c_0(A,\ S) & & \big\downarrow c_0(A) & & (\simeq)\big\downarrow c_0(\Lambda) \\
(1) & & & G_1(\Lambda) & \longrightarrow & G_0(A,\ S) & \longrightarrow & G_0(A) & \xrightarrow[g_0]{} & G_0(\Lambda) \\
 & & & & & \Vert & & & & \\
 & & & & & \coprod_{\underline{p} \in X} G_0(A/\underline{p}\ A) & & & &
\end{array}$$

The top row comes from (IX, 6.3), and the bottom row from (IX, 6.2). The verticals are the Cartan homomorphism (IX, §2). The coproduct on the bottom is identified with $G_0(A,\ S)$ by (IX, 6.9). Many results in this section will refer to the notation in (1).

Let $\underline{\underline{M}}_0(A) \subset \underline{\underline{M}}(A)$ be the full subcategory of (R-) torsion free modules $M \in \underline{\underline{M}}(A)$. If $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$ is an exact sequence in mod-A with $M \in \underline{\underline{M}}_0(A)$ then $P \in \underline{\underline{M}}_0(A) \Longleftrightarrow N \in \underline{\underline{M}}_0(A)$. Moreover, if $P \in \underline{\underline{P}}(A)$ then $N \in \underline{\underline{M}}_0(A)$ so it follows that $\underline{\underline{M}}_0(A)$ is an admissible subcategory of $\underline{\underline{M}}(A)$, and every $M \in \underline{\underline{M}}(A)$ fits into an exact sequence as above with $P \in \underline{\underline{P}}(A)$ and $N \in \underline{\underline{M}}_0(A)$. It follows therefore from (VIII, 4.6) that the inclusion $\underline{\underline{M}}_0(A) \subset \underline{\underline{M}}(A)$ induces isomorphisms

(2) $$K_i(\underline{\underline{M}}_0(A)) \longrightarrow K_i(\underline{\underline{M}}(A)) = G_i(A) \quad (i = 0,\ 1).$$

Let $\mathfrak{a}$ be a non zero ideal in R. Then the functor

$$\otimes_R R/\underline{\mathfrak{a}}\colon \underline{\underline{M}}(A) \longrightarrow \underline{\underline{M}}(A/A\underline{\mathfrak{a}}),$$

has an <u>exact</u> restriction to $\underline{\underline{M}}_0(A)$, which is the reason for introducing $\underline{\underline{M}}_0$. For if $M \in \underline{\underline{M}}_0(A)$ then $M \in \underline{\underline{P}}(R)$, so a short exact sequence of such modules <u>splits</u> as a sequence of R-modules. Thus we obtain induced homomorphisms $K_i(\underline{\underline{M}}_0(A))$

$\longrightarrow G_i(A/A\mathfrak{a})$ $(i = 0, 1)$. Combining these with (2) we obtain homomorphisms

$$\phi_{\mathfrak{a}}: G_i(A) \longrightarrow G_i(A/A\mathfrak{a}) \qquad (i = 0, 1).$$

(1.1) PROPOSITION ("Swan's Triangle"). *Let* $\mathfrak{a}$ *be a non zero ideal of* R. *Then there is a unique homomorphism* $\delta_{\mathfrak{a}}: G_0(\Lambda) \longrightarrow G_0(A/A\mathfrak{a})$ *such that*

$$\begin{array}{ccc} G_0(A) & \xrightarrow{\;g_0\;} & G_0(\Lambda) \\ & {\scriptstyle \phi_{\mathfrak{a}}} \searrow \quad \swarrow {\scriptstyle \delta_{\mathfrak{a}}} & \\ & G_0(A/A\mathfrak{a}) & \end{array}$$

commutes.

Proof. Since g_0 is surjective (see (1)) the proposition will follow once we show that $\phi_{\mathfrak{a}}(\mathrm{Ker}(g_0)) = 0$. Now $\mathrm{Ker}(g_0) = \mathrm{Im}(G_0(A, S) \longrightarrow G_0(A))$ and $G_0(A, S) = K_0(\underline{\underline{M}}_S(A))$ is generated by the classes of simple A-modules $M \in \underline{\underline{M}}(A/A\mathfrak{p})$ for all $\mathfrak{p} \in \max(R)$.

Let $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$ be exact with $P \in \underline{\underline{P}}(A)$. Then, by definition, $\phi_{\mathfrak{a}}[M] = [P/P\mathfrak{a}] - [N/N\mathfrak{a}]$. Suppose first that $\mathfrak{a} = aR$ is principal.

If $Ma \neq 0$ then, since M is simple, $M \xrightarrow{\cdot a} M$ is an automorphism. It follows that $P\mathfrak{a} \cap N = Na$ and so $0 \longrightarrow N/Na \longrightarrow P/P\mathfrak{a} \longrightarrow M/Ma = 0$ is an exact sequence, showing that $\phi_{\mathfrak{a}}[M] = 0$.

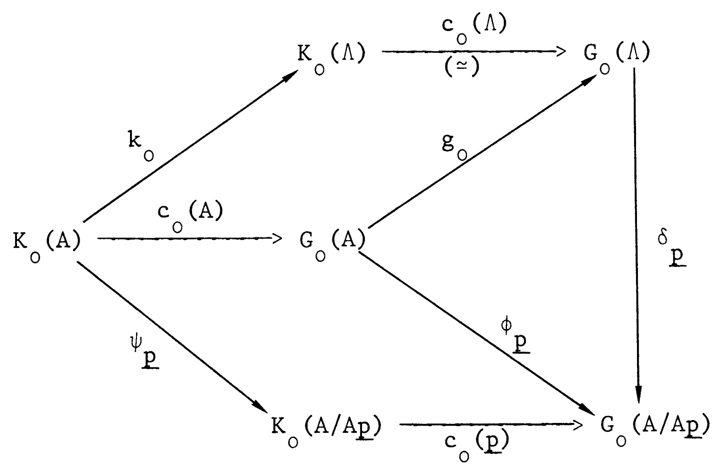
If, on the other hand, $Ma = 0$, then the exact sequence $0 \longrightarrow P\mathfrak{a}/N\mathfrak{a} \longrightarrow N/P\mathfrak{a} \longrightarrow P/P\mathfrak{a} \longrightarrow M \longrightarrow 0$ in $\underline{\underline{M}}(A/Aa)$ shows that $\phi_{\mathfrak{a}}[M] = [M] - [P\mathfrak{a}/Na] = [M] - [M] = 0$.

Finally, in case $\mathfrak{a}$ is not principal, set $S = 1 + \mathfrak{a}$ and localize to $S^{-1}R$. This does not alter any of the modules

$P/P\underline{a}$, $N/N\underline{a}$, etc. which are annihilated by $\underline{a}$. On the other hand $S^{-1}\underline{a} \subset \text{rad } S^{-1}R$ so $S^{-1}R$, being a Dedekind ring with non zero radical, is semi-local, and hence a principal ideal ring. Hence we can apply the arguments above over $S^{-1}R$ to conclude that $\phi_{\underline{a}}[M] = 0$. q.e.d.

(1.2) COROLLARY. *Suppose that* R *is local with maximal ideal* $\underline{p}$. *If the Cartan homomorphism* $c_0(\underline{p}) = c_0(A/A\underline{p})$: $K_0(A/A\underline{p}) \longrightarrow G_0(A/A\underline{p})$ *is a monomorphism then* k_0: $K_0(A) \longrightarrow K_0(\Lambda)$ *is likewise. Moreover, if* P, $Q \in \underline{\underline{P}}(A)$, *then* $P \otimes_R L \simeq Q \otimes_R L \Rightarrow P \simeq Q$. *If, further,* A *is right regular, then* k_0 *is an isomorphism.*

Proof. We have the commutative diagram



Since $A\underline{p} \subset \text{rad } A$ it follows from (IX, 1.3 (0)) that $\psi_{\underline{p}}$ is a monomorphism (in fact an isomorphism if R is $\underline{p}$-adically complete). Therefore $c_0(\underline{p})\psi_{\underline{p}} = \delta_{\underline{p}} c_0(\Lambda) k_0$ is a monomorphism, and hence k_0 is also. If A is right regular then $c_0(A)$ is an isomorphism so the surjectivity of k_0 follows from that of g_0.

Finally, if P, $Q \in \underline{\underline{P}}(A)$ and if $P \otimes_R L \simeq Q \otimes_R L$ then, since k_o is a monomorphism, $[P] = [Q]$ in $K_o(A)$, i.e. $P \oplus A^n \simeq Q \oplus A^n$ for some $n \geq 0$. Since A is semi-local it follows from (IV, 1.4) that $P \simeq Q$. q.e.d.

(1.3) DEFINITION. The _Cartan condition_ on A is as follows: For each $\underline{p} \in X$, the Cartan homomorphism

$$c_o(\underline{p}) = c_o(A/A\underline{p}) : K_o(A/A\underline{p}) \longrightarrow G_o(A/A\underline{p})$$

is a monomorphism. Equivalently, the Cartan matrix of $A/A\underline{p}$ has non zero determinant.

(1.4) COROLLARY. _Let_ A _satisfy the Cartan condition._ _Let_ $P \in \underline{\underline{P}}(A)$ _be such that_ $P \otimes_R L \simeq \Lambda^n$. _Then for all_ $\underline{p} \in X$, $P_{\underline{p}} \simeq A_{\underline{p}}^n$, _and_ P _has a direct summand isomorphic to_ A^{n-1}. _If_ R _is semi-local then_ $P \simeq A^n$.

Proof. The first assertion follows from (1.2) applied to $A_{\underline{p}}$ over $R_{\underline{p}}$. By virtue of the first assertion and the fact that $\dim X \leq 1$ the second assertion follows from Serre's Theorem (IV, 2.7). If R is semi-local the last assertion follows similarly because $\dim X = 0$ in this case. q.e.d.

(1.5) COROLLARY. _Let_ A _satisfy the Cartan condition and assume_ Λ _is a division algebra. Then every_ $P \in \underline{\underline{P}}(A)$ _is the direct sum of a free module and of a right ideal in_ A.

Proof. Clearly $P \otimes_R L \simeq \Lambda^n$ for some n so (1.4) implies $P \simeq Q \oplus A^{n-1}$, and necessarily Q is an A-lattice in $Q \otimes_R L \simeq \Lambda$. Hence Q is isomorphic to a right A-ideal.

(1.6) PROPOSITION. _Let_

$$C = \mathrm{Coker}(K_o(A, S) \xrightarrow{c_o(A, S)} G_o(A, S)).$$

Then there is a natural epimorphism

$$\coprod_{\underline{p} \in X} \mathrm{Coker}(c_o(\underline{p})) \longrightarrow C.$$

Hence, if A satisfies the Cartan condition, then C is a torsion group.

Proof. We have $G_o(A, S) = \coprod_{\underline{p} \in X} G_o(A/\underline{p}\, A)$ and $K_o(A, S) = K_o(\underline{\underline{H}}_S(A))$, where $\underline{\underline{H}}_S(A)$ is the category of torsion A-modules of finite homological dimension. Hence the proposition will follow once we show that $\underline{\underline{P}}(A/\underline{p}\, A) \subset \underline{\underline{H}}_S(A)$ for each $\underline{p} \in X$. Since any $P \in \underline{\underline{P}}(A/\underline{p}\, A)$ is a direct summand of some $(A/\underline{p}\, A)^n$ it suffices to show that $hd_A(A/\underline{p}\, A) < \infty$. But $\underline{p}A \simeq \underline{p} \otimes_R A \in \underline{\underline{P}}(A)$ because $\underline{p}$ is invertible. Therefore $0 \longrightarrow \underline{p}A \longrightarrow A \longrightarrow A/\underline{p}\, A \longrightarrow 0$ exhibits a finite $\underline{\underline{P}}(A)$-resolution. q.e.d.

(1.7) EXAMPLE. Suppose $B = \Pi B_i$ $(1 \leq i \leq n)$ where each B_i is a right Artin ring such that $B_i/\text{rad } B_i$ is simple. Then $c_o(B) = c_o(B_1) \oplus \ldots \oplus c_o(B_n)$, and each $c_o(B_i)$ is represented by a non zero one-by-one matrix. Hence c(B) is a monomorphism. Now any commutative Artin ring is a product of local rings, so the remark above implies:

If A is commutative then A satisfies the Cartan condition.

We further contend:

If A is a maximal R-order then A satisfies the Cartan condition.

Indeed, let $\underline{a}$ be any two sided ideal in A which is an R-lattice (e.g. $A\underline{p}$ for some $\underline{p} \in X$). Then we will show that $c(A/\underline{a})$ is injective.

According to (III, 8.6) we have a unique factorization $\underline{a} = \underline{p}_1^{n_1} \ldots \underline{p}_r^{n_r}$ with the $\underline{p}_i \in \max(A)$, the set of maximal two sided ideals. Set $\underline{q}_i = \Pi_{j \neq i}\, \underline{p}_j^{n_j}$ $(1 \leq i \leq n)$. By the unique factorization theorem $\underline{q}_i \not\subset \underline{p}_i$ so $\Sigma_j \underline{q}_j \not\subset \underline{p}_i$ for all i $(1 \leq i \leq n)$. It follows that $\Sigma_j \underline{q}_j = A$. Similarly we have $\underline{p}_i + \underline{q}_i =$

A for each i. Now, just as in the proof of the Chinese Remainder Theorem (III, 2.14), it follows that

$$A/\underline{a} \simeq \Pi\ A/\underline{p}_i^{\,n_i}.$$

Since each $\underline{p}_i \in \max(A)$, it follows that $B_i = A/\underline{p}_i^{\,n_i}$ satisfies the hypothesis made at the outset, so we conclude from the first remark above that $c(A/\underline{a})$ is injective. q.e.d.

(1.8) EXAMPLE. Let π be a finite group. Then $c_o(L\pi)$ is injective (cf. (XI, 4.5), and the ensuing remark). If char(L) does not divide $[\pi: 1]$ then $L\pi = \Lambda$ is semi-simple (see (XI, 1.2)), and the R-order $A = R\pi$ satisfies the Cartan condition. (Apply the first assertion to the fields $R/\underline{p}$.)

By virtue of the theory of maximal orders (cf. (III, §8) and (1.7) above) we can try to get information about A by comparing A with a maximal R-order B containing A. There is then a diagram analogous to (1) above for B, and we shall now write $k_o(A)$ in place of k_o, to distinguish it from its analogue $k_o(B)\colon K_o(B) \longrightarrow K_o(\Lambda)$. Similar conventions apply to k_i and g_i $(i = 0, 1)$.

For the rest of this section we assume B <u>is a maximal</u> R-<u>order containing</u> A and that B <u>is an</u> R-<u>lattice</u>. The last assumption guarantees that

$$\underline{c}_o = \{\alpha \in R \mid \alpha B \subset A\} = \mathrm{ann}_R(B/A)$$

is a non-zero R-ideal. Then $\underline{c}_o B$ is a two sided B-ideal contained in A, and it is an R-lattice in Λ. There is, in fact, a largest such ideal, $\underline{c}_{B/A}$, called the <u>conductor</u>:

$$\begin{aligned} \underline{c}_{B/A} &= \{b \in \Lambda \mid B\ b\ B \subset A\} \\ &= \{b \in B \mid B\ b\ B \subset A\} \\ &= \{b \in A \mid B\ b\ B \subset A\}. \end{aligned}$$

Let $T = R - (\bigcup_{\underline{p} \supset \underline{c}_o} \underline{p})$. This is a multiplicative set in R, and $T^{-1}R$ is a semi-local ring whose maximal ideals are the $T^{-1}\underline{p}$, where $\underline{p}$ ranges over primes containing $\underline{c}_o$.

(1.9) PROPOSITION. Keep the notation above.

(a) If $\underline{p} \in X$ and $\underline{p} \cap T \neq \phi$ then $A_{\underline{p}} = B_{\underline{p}}$. Hence T is regular for A (in the sense of (III, 6.7) so there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
K_o(A, T) & \longrightarrow & K_o(A) & \longrightarrow & K_o(T^{-1}A) & \longrightarrow & 0 \\
(\simeq)\ \downarrow c_o(A, T) & & \downarrow c_o(A) & & \downarrow c_o(T^{-1}A) & & \\
G_o(A, T) & \longrightarrow & G_o(A) & \longrightarrow & G_o(T^{-1}A) & \longrightarrow & 0
\end{array}$$

in which $c_o(A, T)$ is an isomorphism.

(b) $\mathrm{Coker}(c_o(A, S): K_o(A, S) \longrightarrow G_o(A, S))$ is a quotient of $\coprod_{\underline{p} \supset \underline{c}_o} \mathrm{Coker}(c_o(A/\underline{p}A))$, a finitely generated group.

(c) $K_o(A) \longrightarrow K_o(B)$ induces an epimorphism $\mathrm{Ker}(k_o(A)) \longrightarrow \mathrm{Ker}(k_o(B))$. If A satisfies the Cartan condition then the (non commutative) square

$$\begin{array}{ccc}
K_o(A) & \longrightarrow & K_o(B) \\
c_o(A)\ \downarrow & & (\simeq)\ \downarrow c_o(B) \\
G_o(A) & \longleftarrow & G_o(B)
\end{array}$$

induces a commutative square

$$\begin{array}{ccc}
\mathrm{Ker}(k_o(A)) & \longrightarrow & \mathrm{Ker}(k_o(B)) \\
\downarrow & & (\simeq)\ \downarrow \\
\mathrm{Ker}(g_o(A)) & \longleftarrow & \mathrm{Ker}(g_o(B)).
\end{array}$$

Hence the left and bottom arrows have the same image.

Proof. (a) If $\underline{p} \cap T \neq \phi$ then $\underline{p} \not\supset \underline{c}_0$. Hence $\underline{c}_0 R_{\underline{p}} = R_{\underline{p}}$ so $B_{\underline{p}} = \underline{c}_0 B_{\underline{p}} \subset A_{\underline{p}}$; this shows that $A_{\underline{p}} = B_{\underline{p}}$. According to (III, 8.7) B is hereditary, so, in particular, regular. Thus $A_{\underline{p}}$ is regular for all $\underline{p} \in X$ that meet T. By (III, 6.7) this implies that T is regular for A. The properties of the diagram now follow from (IX, 6.2) and (IX, 6.5).

(b) If $\underline{p} \cap T \neq \phi$ then $\underline{\underline{M}}(A/\underline{p}\, A) \subset \underline{\underline{H}}(A)$ because T is regular for A. Hence, under the homomorphism

$$c_0(A, S): K_0(A, S) \longrightarrow G_0(A, S) = \coprod_{\underline{p} \in X} G_0(A/\underline{p}\, A),$$

the image contains all terms for which $\underline{p} \cap T \neq \phi$. Moreover, as in the proof of (1.6), the image contains the image of $c_0(A/\underline{p}\, A): K_0(A/\underline{p}\, A) \longrightarrow G_0(A/\underline{p}\, A)$ for all $\underline{p} \in X$. Since each $G_0(A/\underline{p}\, A)$ is a free abelian group of finite rank part (b) follows from these observations.

(c) An element in $\mathrm{Ker}(k_0(B))$ can be written in the form $[P] - [F]$ where $F = B^n$ for some $n > 0$ and $P \otimes_R L \simeq \Lambda^n$. Since B satisfies the Cartan condition (see (1.7)) it follows from (1.4) that $T^{-1}P \simeq T^{-1}F$. Therefore we can choose a B-homomorphism $h: P \longrightarrow F$ such that $T^{-1}h$ is an isomorphism. It follows that h is a monomorphism, and $T^{-1}M = 0$, where $M = \mathrm{Coker}(h)$. Since T is regular for A it follows that $hd_A(M) < \infty$. In fact $M_{\underline{p}} = 0$ if $\underline{p} \cap T = \phi$, and $A_{\underline{p}}$ is hereditary otherwise, so $hd_A(M) \leq 1$ (see (III, 6.6)). Let $0 \longrightarrow P' \longrightarrow F' \longrightarrow M \longrightarrow 0$ be exact with $F' \in \underline{\underline{P}}(A)$; then $P' \in \underline{\underline{P}}(A)$ also.

We claim: (i) $M \otimes_A B \simeq M$; and (ii) $0 \longrightarrow P' \otimes_A B \longrightarrow F' \otimes_A B \longrightarrow M \otimes_A B \longrightarrow 0$ is exact. Once we know this it follows from Schanuel's Lemma (I, 6.3) that $F \oplus (P' \otimes_A B) \simeq (F' \otimes_A B) \oplus P$, and hence $[P] - [F] = [P' \otimes_A B] - [F' \otimes_A B] \in \mathrm{Im}(\mathrm{Ker}(k_0(A)) \longrightarrow \mathrm{Ker}(k_0(B))$, as required.

To prove (i) tensor $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$ with M over A. Since $(B/A)\underline{c}_o = 0$ and since M is annihilated by an element prime to $\underline{c}_o$ (because $T^{-1}M = 0$) it follows that $(B/A) \otimes_A M = 0$, and hence $M \longrightarrow M \otimes_A B$ is an epimorphism. Since these are R-modules of finite length it is an isomorphism.

To prove (ii) we note that $P' \otimes_A B \longrightarrow F' \otimes_A B$ is a homomorphism of torsion free R-modules (because P′ is projective) which becomes an isomorphism over L. Hence it is a monomorphism. The other exactness is standard. Thus, we have proved the first part of (c).

For the second part we start with a $[P]_{\underline{\underline{P}}(A)} - [F]_{\underline{\underline{P}}(A)} \in \mathrm{Ker}(k_o(A))$ with $F = A^n$ and $P \otimes_R L \simeq \Lambda^n$. The commutativity assertion means, explicitly, that $[P]_{\underline{\underline{M}}(A)} - [F]_{\underline{\underline{M}}(A)} = [P \otimes_A B]_{\underline{\underline{M}}(A)} - [F \otimes_A B]_{\underline{\underline{M}}(A)}$ in $G_o(A)$. Since A is now assumed to satisfy the Cartan condition we can apply the construction used above to obtain an exact sequence $0 \longrightarrow P \longrightarrow F \longrightarrow M \longrightarrow 0$ such that $T^{-1}M = 0$. Then assertions (i) and (ii) above (with P and F here replacing P′ and F′ there) apply unchanged, and we conclude that $[P]_{\underline{\underline{M}}(A)} - [F]_{\underline{\underline{M}}(A)} = [M]_{\underline{\underline{M}}(A)} = [P \otimes_A B]_{\underline{\underline{M}}(A)} - [F \otimes_A B]_{\underline{\underline{M}}(A)}$. q.e.d.

(1.10) PROPOSITION. Keep the notation of (1.9). Let $\underline{c}$ be a two sided B-ideal contained in A which is an R-lattice in Λ (e.g. $\underline{c} = \underline{c}_o B$ or $\underline{c} = \underline{c}_{B/A}$, the conductor). Then $A' = A/\underline{c}$ and $B' = B/\underline{c}$ are of finite length as R-modules, and we have an exact Mayer-Vietoris sequence,

$$K_1(A) \longrightarrow K_1(A') \oplus K_1(B) \longrightarrow K_1(B') \longrightarrow K_o(A)$$

$$\longrightarrow K_o(A') \oplus K_o(B) \longrightarrow K_o(B').$$

The groups $K_o(A')$ and $K_o(B')$ are free abelian of finite rank, and $U(B') \longrightarrow K_1(B')$ is surjective.

In case A is commutative then B is just the integral *closure of* A *in* Λ, *and we have an exact sequence*

$$0 \longrightarrow \frac{U(B')}{U(A')\cdot U(B)'} \longrightarrow \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(B) \longrightarrow 0,$$

where $U(B)' = \mathrm{Im}(U(B) \longrightarrow U(B'))$.

Proof. Since the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

is a fibre product (see (IX, 5.6)) it yields the Mayer-Vietoris sequence above, as well as

$$U(A) \longrightarrow U(A') \oplus U(B) \longrightarrow U(B') \longrightarrow \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(A') \oplus \mathrm{Pic}(B) \longrightarrow \mathrm{Pic}(B')$$

if A is commutative (see (IX, 5.3)). Both A′ and B′ are finitely generated torsion R-modules, and hence of finite length; therefore A′ and B′ are semi-local. It follows now from (IX, 1.4) that $K_0(A')$ and $K_0(B')$ are free abelian groups of finite rank, and that $U(B') \longrightarrow K_1(B')$ is surjective. Moreover, in case A is commutative it follows from (IX, 3.5) that $\mathrm{Pic}(A') = 0 = \mathrm{Pic}(B')$. The last exact sequence of the proposition follows from this and the Mayer-Vietoris sequence above for Pic. q.e.d.

§2. FINITENESS OF CLASS NUMBER

For the remainder of this chapter we specialize the data of (0) in §1 as follows:

$\underline{\underline{F}}$ is a finite field of characteristic p_o with $q = p_o^{n_o}$ elements.

(0)′ R = either $\underline{\underline{Z}}$ or $\underline{\underline{F}}[t]$, where t is an indeterminate.

$$R_+ = \begin{cases} \{\text{integers} \geq 0\} & \text{if } R = \underline{\underline{Z}} \\ R, \text{ if } R = \underline{\underline{F}}[t]. \end{cases}$$

It follows from (III, 8.5) that each R-order in Λ is an R-lattice, and that it can be embedded in a maximal R-order. We also introduce the homomorphism

$$| \; | : D(R) \longrightarrow U(\underline{\underline{R}}) \quad (\underline{\underline{R}} = \text{reals})$$

$$\left| \sum n_{\underline{p}} \underline{p} \right| = \Pi |\underline{p}|^{n_{\underline{p}}} \quad (\underline{p} \in X),$$

where $|\underline{p}| = \text{card}(R/\underline{p})$. If $\underline{a}$ is a fractional ideal of R we set

$$|\underline{a}| = |\; \text{div}(\underline{a}) \;|$$

and we abbreviate

$$|x| = |\; x R \;| \qquad (x \in U(L)).$$

Moreover, we shall agree that $|0| = 0$.

(2.1) PROPOSITION. (a) _If_ $\underline{a}$ _is a non zero ideal in_ R _then_

$$|\underline{a}| = \text{card}(R/\underline{a}).$$

Hence, if $a \in R$, _then_

$$|a| = \begin{cases} \underline{\text{ordinary absolute value}}, & \text{if } R = \underline{\underline{Z}} \\ q^{\deg(a)}, & \text{if } R = F[t]. \text{ (By convention, } \deg(0) = -\infty.) \end{cases}$$

(b) _If_ a, $b \in R$ _then_

$$|\; a + b \;| \leq |a| + |b|$$

and

$$|\; a + b \;| \leq \sup(|a|, |b|) \quad \underline{\text{if}}\ a, b \in R_+.$$

(c) <u>If</u> c <u>is a real number</u> ≥ 0 <u>then</u>

$$\text{card}\{a \in R \mid |a| \leq c\} < \infty$$

<u>and</u>

$$\text{card}\{a \in R_+ \mid |a| \leq c\} > c.$$

(d) <u>Let</u> $N(X_1,\ldots,X_m) \in R[X_1,\ldots,X_m]$ <u>be a homogeneous polynomial of degree</u> n. <u>Then there is a real number</u> $c > 0$ <u>such that, for all</u> $x = (x_1,\ldots,x_m) \in R^m$, <u>we have</u>

$$|N(x)| \leq c\,|x|^n,$$

<u>where</u> $|x| = \sup(|x_1|,\ldots,|x_m|)$.

<u>Proof</u>. (a) $|\underline{a}| = |\operatorname{div}(\underline{a})|$ and $\text{card}(R/\underline{a})$ are both multiplicative functions of $\underline{a}$, and they agree on the generators $\underline{a} = \underline{p} \in X$. If $a \neq 0$ in $\mathbb{Z}$, then clearly $\text{card}(\mathbb{Z}/a\mathbb{Z})$ is the ordinary absolute value of a. If $a \neq 0$ in $\mathbb{F}[t]$ and $\deg(a) = d$ then R/aR is a vector space over $\mathbb{F}$ with basis 1, $t,\ldots,t^{d-1}$. Hence $\text{card}(R/aR) = q^d$.

(b) Follows immediately from (a) and the definition of R_+, and the same applies to the first assertion of (c).

$$\{a \in R_+ \mid |a| \leq c\} = \begin{cases} \{0, 1,\ldots,[c]\}, & \text{if } R = \mathbb{Z} \\ \Sigma_{0 \leq i \leq [c]} \mathbb{F} t^i, & \text{if } R = \mathbb{F}[t]. \end{cases}$$

Hence the cardinality is $[c] + 1$, respectively, $q^{[c]+1}$. This proves the second assertion of (c).

(d) Write $N(x) = \Sigma\, a_\alpha x^\alpha$ where $\alpha = (i_1,\ldots,i_m)$ ranges over m-tuples such that $i_1 + \ldots + i_m = n$, and $x^\alpha = x_1^{i_1} \ldots x_m^{i_m}$. For each such α we have $|x^\alpha| = |x_1|^{i_1} \ldots |x_m|^{i_m} \leq |x|^{i_1} \ldots |x|^{i_m} = |x|^n$. Hence $|N(x)| \leq \Sigma\, |a_\alpha|\, |x^\alpha| \leq \Sigma\, |a_\alpha|\, |x|^n \leq c|x|^n$, where $c = \Sigma\, |a_\alpha|$. q.e.d.

Recall that if $M \in \underline{\underline{M}}(R)$ is a torsion module then we have

$$\chi(M) = \sum_{\underline{p} \in X} \ell_{\underline{p}}(M)\underline{p} \qquad \in D(R),$$

where $\ell_{\underline{p}}(M)$ is the length of the $R_{\underline{p}}$-module $M_{\underline{p}}$ (see (IX, 6.6)).

(2.2) PROPOSITION. (a) _If_ $M \in \underline{\underline{M}}(R)$ _is a torsion module then_ $\mathrm{card}(M)$ _is finite and equals_ $|\chi(M)|$.

(b) _Let_ $\alpha: R^n \longrightarrow R^n$ _have non zero determinant, and set_ $M = \mathrm{Coker}(\alpha)$. _Then_ $M \det(\alpha) = 0$ _and_

$$\mathrm{card}(M) = |\det(\alpha)|.$$

(c) _Let_ c _be a real number_ > 0. _Let_ W _denote the set of submodules_ $P \subset R^n$ _such that_ $\mathrm{card}(R^n/P) \leq c$. _Then there is a_ $d \neq 0$ _in_ R _such that_ $R^n d \subset P$ _for all_ $P \in W$. _Moreover_ $\mathrm{card}(W) < \infty$.

Proof. (a) M has a composition series with factors of the form $R/\underline{p}$ ($\underline{p} \in X$). Hence $\mathrm{card}(M)$ is finite. Moreover it is a multiplicative function of M. Since $|\chi(R/\underline{p})| = |\underline{p}| = \mathrm{card}(R/\underline{p})$ (by definition) it follows that $\mathrm{card}(M) = |\chi(M)|$ for all M as above.

Part (b) follows immediately from (IX, 6.6), with the aid of part (a).

(c) Suppose $P \in W$. Since R is a principal ideal ring P is free, so $P = \mathrm{Im}(\alpha)$ for some $\alpha \in \mathrm{End}_R(R^n)$. Therefore, by (b), $|\det(\alpha)| = \mathrm{card}(R^n/P) \leq c$. Let $\{d_1,\ldots,d_m\} = \{a \in R \mid 0 < |a| \leq c\}$ (using (2.1) (c)), and set $d = d_1 \ldots d_m$. Since $(R^n/P)\det(\alpha) = 0$ we have $(R^n/P)d = 0$. Hence $R^n \supset P \supset R^n d$. Since $R^n/R^n d = (R/Rd)^n$ is finite there are only finitely many $P \in W$. q.e.d.

Now we resume our discussion of R-orders, keeping the notation and conventions of (0) and of (0)´ above.

(2.3) PROPOSITION. _If Λ is a division algebra then there is only a finite number of isomorphism classes of right A-ideals._

Proof. If $\underline{a} \neq 0$ is a right A-ideal then $\underline{a}L = \Lambda$, because Λ has no proper right ideals. Thus $\underline{a}$ is an R-lattice in A so $c = \mathrm{card}(A/\underline{a})$ is finite.

Let $e_1, \ldots, e_n$ be an R-basis for A, and let

$$W = \{\Sigma_{1\leq i\leq n}\, e_i a_i \mid a_i \in R_+,\ |a_i| \leq c^{1/n} \quad (1 \leq i \leq n)\}$$

According to (2.1) (c) $\mathrm{card}(W) > (c^{1/n})^n = c$. Therefore there exist distinct u and v in W such that $u \equiv v \bmod \underline{a}$. Say $u = \Sigma e_i a_i$ and $v = \Sigma e_i b_i$. Then $w = u - v = \Sigma e_i c_i \neq 0$, where $|c_i| = |a_i - b_i| \leq \sup(|a_i|, |b_i|) \leq c^{1/n}$, thanks to (2.1) (b). Moreover, since $u \equiv v \bmod \underline{a}$, we have $w \in \underline{a}$.

Consider the norm, $N_{A/R}(x) = \det_R(A \xrightarrow{x\,\cdot} A)$. If $x = \Sigma e_i x_i \in A$ then $N_{A/R}(x)$ is a homogeneous polynomial of degree n in the variables $(x_1, \ldots, x_n)$ with coefficients in R. Therefore, if we write $|x| = \sup(|x_1|, \ldots, |x_n|)$ then (2.1) (d) implies there is a constant K (depending only on A) such that $|N_{A/R}(x)| \leq K|x|^n$ for all $x \in A$. Now it further follows from (2.2) (b) that $|N_{A/R}(x)| = \mathrm{card}(\mathrm{Coker}(A \xrightarrow{x\,\cdot} A)) = \mathrm{card}(A/xA)$, provided $N_{A/R}(x) \neq 0$.

Now we can apply this to the $w \neq 0$ constructed above. Since Λ is a division algebra we have $N_{A/R}(w) = N_{\Lambda/L}(w) \neq 0$, and hence $\mathrm{card}(A/wA) \leq K|w|^n \leq K(c^{1/n})^n = Kc$. Since $w \in \underline{a}$ we have an exact sequence

$$0 \longrightarrow \underline{a}/wA \longrightarrow A/wA \longrightarrow A/\underline{a} \longrightarrow 0$$

so $c(\mathrm{card}(\underline{a}/wA)) = \mathrm{card}(A/\underline{a})\ \mathrm{card}(\underline{a}/wA) = \mathrm{card}(A/wA) \leq Kc$. Therefore $\mathrm{card}(\underline{a}/wA) \leq K$. By (2.2) (c) there is a $d \neq 0$ in R, depending only on K, such that $(\underline{a}/wA)d = 0$. Thus $\underline{a}d \subset wA \subset \underline{a}$, i.e. $A \subset w^{-1}\underline{a} \subset d^{-1}A$. Now $\underline{a} \simeq w^{-1}\underline{a}$ as right ideals, and d depends only on K, therefore only on A. Since $d^{-1}A/A$ is finite, and since we have shown that any $\underline{a}$ is isomorphic to a right ideal sandwiched between A and $d^{-1}A$, the proposition is proved.

(2.4) THEOREM (Jordan-Zassenhaus). Let $V \in \underline{\underline{M}}(\Lambda)$. Then the cardinal number, $c_A(V)$, of isomorphism classes of A-lattices in V is finite.

Before the proof we record.

(2.5) COROLLARY. If $n > 0$ there is only a finite number of isomorphism classes of $M \in \underline{\underline{M}}(A)$ which are torsion free of rank $\leq n$ as R-modules.

Proof. Each such M is an A-lattice in $V = M \otimes_R L \in \underline{\underline{M}}(\Lambda)$, and $[V: L] \leq n$. Since Λ is semi-simple there are only finitely many such V (up to isomorphism), so the corollary follows from the theorem.

Proof of (2.4). Embed A in a maximal order B (see (III, 8.5)), and choose $\alpha \neq 0$ in R such that $B\alpha \subset A$. If M is an A-lattice in V, then MB is a B-lattice, and $MB\alpha \subset M \subset MB$. Suppose N is another A-lattice and $MB \simeq NB$ as B-lattices. Then, after applying an automorphism of V to N, we can assume $MB = NB$. In this case $MB\alpha = NB\alpha \subset N \subset NB = MB$. Thus every A-lattice N such that $MB \simeq NB$ has a representative (of its isomorphism class) sandwiched between $MB\alpha$ and MB. Since $MB/MB\alpha$ is finite, there are only a finite number of such lattices. Thus the finiteness of $c_A(V)$ follows from that of $c_B(V)$. In turn, the finiteness of $c_B(V)$ follows immediately from:

(2.6) PROPOSITION. If A is a maximal order then there *is only a finite number of isomorphism classes of indecomposable* $M \in \underline{\underline{M}}(A)$ *which are torsion free*.

Proof. According to (III, 8.7) the category of torsion free $M \in \underline{\underline{M}}(A)$ is just $\underline{\underline{P}}(A)$. This category is naturally equivalent to $\underline{\underline{P}}(B)$ for any R-algebra B such that mod-A and mod-B are equivalent. It follows from (III, 8.7 (b)) and (III, 8.9) that we can find such a B of the form $B = \Pi B_i$ where each B_i is a maximal R-order in a division algebra over L. Since $\underline{\underline{P}}(B) = \Pi\, \underline{\underline{P}}(B_i)$ we are thus reduced to proving the proposition when Λ is a division algebra. It follows from (III, 8.7) again that A is right hereditary and every $P \in \underline{\underline{P}}(A)$ is a direct sum of modules isomorphic to right ideals in A. Therefore every indecomposable $P \in \underline{\underline{P}}(A)$ is isomorphic to a right ideal. Now the conclusion follows from (2.3). q.e.d.

(2.7) THEOREM. (a) *The abelian groups* $K_o(A)$ *and* $G_o(A)$ *are finitely generated*.

(b) *Let* A *satisfy the Cartan condition. Then all homomorphisms in the square*

$$(1)\qquad \begin{array}{ccc} K_o(A) & \xrightarrow{\;k_o\;} & K_o(\Lambda) \\ {\scriptstyle c_o(A)}\downarrow & & \downarrow{\scriptstyle c_o(\Lambda)}\;(\simeq) \\ G_o(A) & \xrightarrow[\;g_o\;]{} & G_o(\Lambda) \end{array}$$

have finite kernels. Hence rank $K_o(A) \leq$ rank $G_o(A)$ = rank $G_o(\Lambda)$ = *the number of simple factors of* Λ

Proof. We shall carry out the proof in several steps.

(i) $\mathrm{Ker}(k_o)$ *is finite in case* (b), *and hence* $K_o(A)$ *is is finitely generated*.

An $x \in \mathrm{Ker}(k_o)$ can be written in the form $x = [P] - [A^n]$ with $P \otimes_R L \simeq \Lambda^n$. Now (1.4) implies $P \simeq Q \oplus A^{n-1}$ for some Q, and hence $x = [Q] - [A]$. Since Q is an A-lattice in $Q \otimes_R L \simeq \Lambda$ there are only finitely many such [Q]'s, by (2.4). Hence $\mathrm{Ker}(k_0)$ is finite. Since $K_0(\Lambda)$ is a free abelian group of finite rank it follows that $K_0(A)$ is finitely generated.

(ii) $G_0(A)$ is finitely generated.

We have seen that $G_0(A)$ is generated by all [M] where $M \in \underline{\underline{M}}(A)$ is torsion free. Thus M is an A-lattice in $V = M \otimes_R L$. By restricting to M a Λ-Jordan-Holder series for V, we obtain a finite filtration of M whose successive factors are A-lattices in simple Λ-modules. Therefore, if $V_1, \ldots, V_m$ represent the distinct simple Λ-modules, then $G_0(A)$ has $c_A(V_1) + \ldots + c_A(V_m)$ generators, in the notation of (2.4).

(iii) In case (b), g_0 and $c_0(A)$ have finite kernels.

From the diagram (1) of §1 we extract a commutative diagram

$$\begin{array}{ccccc}
K_0(A, S) & \longrightarrow & \mathrm{Ker}(k_0) & \longrightarrow & 0 \\
{\scriptstyle c_0(A,\,S)}\downarrow & & \downarrow{\scriptstyle h \ (\text{induced by } c_0(A))} & & \\
G\ (A, S) & \longrightarrow & \mathrm{Ker}(g_0) & \longrightarrow & 0
\end{array}$$

with exact rows. According to (1.6), $c_0(A, S)$, and hence also h, has a torsion cokernel. By part (i) $\mathrm{Ker}(k_0)$ is finite, and hence $\mathrm{Ker}(g_0)$ is torsion. But part (ii) implies $\mathrm{Ker}(g_0)$ is finitely generated; hence it is finite. Since $\mathrm{Ker}(c_0(A)) \subset \mathrm{Ker}(k_0)$ part (i) implies $\mathrm{Ker}(c_0(A))$ is finite. q.e.d.

(iv) $K_0(A)$ is finitely generated.

Embed A in a maximal order B and choose a B-ideal $\underline{c} \subset A$ as in (1.10). Then, in the notation of (1.9), we have an exact sequence

$$K_1(B') \longrightarrow K_o(A) \longrightarrow K_o(A') \oplus K_o(B) \longrightarrow K_o(B'),$$

where $A' = A/\underline{c}$ and $B' = B/\underline{c}$ are finite. Moreover $U(B') \longrightarrow K_1(B')$ is surjective, so $K_1(B')$ is finite, and $K_o(A')$ and $K_o(B')$ are free abelian groups of finite rank. Finally, B satisfies the Cartan condition so part (i) implies $K_o(B)$ is finitely generated. The exact sequence now shows that $K_o(A)$ is finitely generated. q.e.d.

This completes the proof of (2.7).

(2.8) COROLLARY. _Let_ B _be any finite_ R-_algebra. Then_ $K_o(B)$ _and_ $G_o(B)$ _are finitely generated abelian groups_.

Proof. According to (III, 8.10) there is a largest two sided nilpotent ideal $N \subset B$ such that $B/N \simeq T \times A$, where T is a finite semi-simple ring, and A is an R-order in a semi-simple algebra, as above. According to (IX, 1.3) we have isomorphisms $K_o(B) \xrightarrow{\simeq} K_o(B/N) \simeq K_o(T) \oplus K_o(A)$. Similarly we have isomorphisms $G_o(T) \oplus G_o(A) \simeq G_o(B/N) \xrightarrow{\simeq} G_o(B)$ from (IX, 2.3). Since T is semi-simple $K_o(T) = G_o(T)$ is a free abelian group of finite rank. Theorem (2.7) implies $K_o(A)$ and $G_o(A)$ are finitely generated, so the corollary now follows.

Finally, we treat the Picard group.

(2.9) THEOREM. _The (non abelian) group_ $Pic_R(A)$ (see (II, §5)) _is finite. Moreover there is an exact sequence_

$$1 \longrightarrow InAut(A) \xrightarrow{i} Aut_{R\text{-}alg}(A) \longrightarrow Pic_R(A),$$

so that Coker(i), _the group of "outer automorphisms" of the_ R-_algebra_ A, _is finite_.

Proof. The localization $A \longrightarrow \Lambda$ induces a group homomorphism $Pic_R(A) \longrightarrow Pic_L(\Lambda)$, and (III, 1.10) says $Pic_L(\Lambda)$ is a finite group. The kernel consists of elements [P] where P is an invertible A-A-bimodule (i.e. left

$A \otimes_R A^\circ$-module) such that $P \otimes_R L \simeq \Lambda$ as a bimodule.

Let C = center (Λ); then C is a finite product of field extensions of L. Since a tensor product of central simple algebras over a field is again central simple (see discussion about (III, 1.10)) it follows easily that $\Lambda \otimes_C \Lambda^\circ$ is a semi-simple L-algebra. Moreover the image, B, of $A \otimes_R A^\circ$ in $\Lambda \otimes_C \Lambda^\circ$ is an R-order. The bimodule P above can be viewed as a B-lattice in the $\Lambda \otimes_C \Lambda^\circ$-module Λ $(\simeq P \otimes_R L)$. If Q is another such bimodule then $P \simeq Q$ as A-bimodules $\Longleftrightarrow$ $P \simeq Q$ as left B-modules. It follows therefore from the the Jordan-Zassenhaus Theorem (2.4) that there are only finitely many such $[P] \in \mathrm{Ker}(\mathrm{Pic}_R(A) \longrightarrow \mathrm{Pic}_L(\Lambda))$. Hence $\mathrm{Pic}_R(A)$ is finite. The remaining assertions follow immediately from this together with (II, 5.3). q.e.d.

§3. FINITE GENERATION OF K_1 AND G_1.

We keep the notation and conventions of (0) and of (0)´ in §2. To these we add:

$$(0)'' \qquad \begin{aligned} C &= \text{center of } \Lambda. \\ R' &= \text{integral closure of } R \text{ in } C. \\ L_\infty &= \begin{cases} \underline{\underline{R}} & \text{if } R = \underline{\underline{Z}} \\ \underline{\underline{F}}((t^{-1})) & \text{if } R = \underline{\underline{F}}[t]. \end{cases} \end{aligned}$$

We shall further assume that C is separable over L. This implies that $\Lambda_\infty = \Lambda \otimes_L L_\infty$ is a semi-simple L_∞-algebra with center $C_\infty = C \otimes_L L_\infty$.

We start by quoting two classical finiteness theorems.

(3.1) THEOREM (Dirichlet). _The abelian group_ $U(R')$ _is finitely generated, and of rank_ $r_\infty - r_o$, _where_

r_o = _the number of simple factors of_ C (_or of_ A)

and

r_∞ = <u>the number of simple factors of</u> C_∞ <u>(or of</u> A_∞).

This can be found in almost any book on number theory. Since $U(R')$ is the direct product of its projections in the simple factors of C one reduces to the case when C is a field, i.e. $r_o = 1$. If $R = \mathbb{Z}$ then we have $C_\infty = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, with $r_1 + 2r_2 = [C: \mathbb{Q}]$, and $U(R')$ has rank $r_1 + r_2 - 1$.

(3.2) THEOREM (Siegel [1]; cf. also Borel-Harish-Chandra [1]).

Assume $R = \mathbb{Z}$. *Then* $SL_n(A)$, *and hence also* $GL_n(A)$, *are finitely generated groups for all* $n \geq 1$.

Here we have written $SL_n(A)$ for the kernel of

$$\det\ (= \mathrm{Nrd}_{A/C}) : GL_n(A) \longrightarrow U(C)$$

(see (III, §8) or (V, §9) for a discussion of the reduced norm). Since the elements of $M_n(A)$ are integral over R it follows that $\det(\alpha) \in R'$ for $\alpha \in M_n(A)$ (see (1) of (III, §8)). Therefore, for each two sided ideal $\underline{q}$ in A, we have an exact sequence of groups

$$1 \longrightarrow SL_n(A, \underline{q}) \longrightarrow GL_n(A, \underline{q}) \xrightarrow{\det} U(R'),$$

where we write $SL_n(A, \underline{q}) = GL_n(A, \underline{q}) \cap SL_n(A)$, as usual. In view of (3.1) we see why $GL_n(A)$ is finitely generated once $SL_n(A)$ is.

In the function field case the analogue of (3.2) is not true without exception. Indeed, $SL_2(\mathbb{F}[t])$ is *not* finitely generated. On the other hand O'Meara has shown that $SL_n(R')$ is finitely generated for all $n \geq 3$. Even more, it follows from (VI, 7.4) and (VI, 8.5) that:

(3.3) PROPOSITION. *For all* $n \geq 3$, $SL_n(R') = E_n(R')$, *and it is a finitely generated group*.

(3.4) CONJECTURE. For all $n \geq 3$, $SL_n(A)$ is a finitely generated group.

Of course Siegel's theorem affirms this in the number field case.

The main theorem of this section is:

(3.5) THEOREM. *Let* $\underline{q}$ *be a two sided ideal in* A, *and write* $SK_1(A, \underline{q}) = SL(A, \underline{q})/E(A, \underline{q})$, *so that we have an exact sequence*

$$(1) \qquad 0 \longrightarrow SK_1(A, \underline{q}) \longrightarrow K_1(A, \underline{q}) \xrightarrow{\det} U(R').$$

(a) $SK_1(A, \underline{q})$ *is a torsion group of bounded exponent. If* $GL_n(A)$ *is finitely generated for some* $n \geq 2$ *then* $SK_1(A, \underline{q})$ *is finite.*

(b) *Suppose* $A/\underline{q}$ *is finite. Then* det *in* (1) *has finite cokernel, so* $K_1(A, \underline{q})/(\text{torsion subgroup})$ *is free abelian rank* $r_\infty - r_0$, *in the notation of* (3.1). *If* $SK_1(A, \underline{q})$ *is finite then* $GL_n(A)$ *and* $SL_n(A)$ *are finitely generated groups for all* $n \geq 3$.

(c) *The Cartan homomorphism* $c_1(A): K_1(A) \longrightarrow G_1(A)$ *has finite cokernel.*

(3.6) COROLLARY. *In the number field case* ($R = \underline{\underline{Z}}$) *the sequence* $0 \longrightarrow SK_1(A, \underline{q}) \longrightarrow K_1(A, \underline{q}) \longrightarrow K_1(\Lambda)$ *is exact for all ideals* $\underline{q}$. *Moreover* $SK_1(A, \underline{q})$ *is finite, and hence* $K_1(A, \underline{q})$ *is finitely generated. Further* $G_1(A)$ *is finitely generated, and* $\text{Ker}(G_1(A) \longrightarrow G_1(\Lambda))$ *is finite.*

Proof. The first assertion follows from Wang's Theorem (V, 9.7). The finiteness of $SK_1(A, \underline{q})$ follows from (3.5) (a) and Siegel's Theorem (3.2). The homomorphism

$$\text{Ker}(K_1(A) \longrightarrow K_1(\Lambda)) = SK_1(A) \longrightarrow \text{Ker}(G_1(A) \longrightarrow G_1(\Lambda))$$

has finite cokernel, by (3.5) (c). The images of $K_1(A, \underline{q})$ and of $G_1(A)$ in $K_1(\Lambda)$ are identified, via det, with subgroups of the finitely generated group $U(R')$, so they also are finitely generated. q.e.d.

Proof of (3.5). We shall carry out the proof in several steps.

(i) *If* $\underline{q}$ *is a two sided ideal in* A, *there is another,* $\underline{q}'$, *such that* $\underline{q} \cap \underline{q}' = 0$ *and* $A/(\underline{q} + \underline{q}')$ *is finite. Moreover,* $K_1(A, \underline{q} + \underline{q}') = K_1(A, \underline{q}) \oplus K_1(A, \underline{q}')$, *and* $SK_1(A, \underline{q} + \underline{q}') = SK_1(A, \underline{q}) \oplus SK_1(A, \underline{q}')$.

For $\underline{q} \otimes_R L$ is a two sided ideal in the semi-simple algebra Λ. Let $\underline{q}' \subset A$ be a two sided A-lattice in the complementary two sided ideal. Then $\underline{q} \cap \underline{q}' = 0$ and $\underline{q} + \underline{q}'$ is an R-lattice in Λ. Hence $A/(\underline{q} + \underline{q}')$ is finite. The remaining assertions follow from (IX, 1.5) and the coordinatewise definition of the reduced norm.

Now assume $A/\underline{q}$ is finite. Let B be a maximal R-order containing A, and choose $a \neq 0$ in R so that $aB \subset \underline{q}$.

(ii) *The homomorphism*

$$K_1(A, \underline{q}) \longrightarrow K_1(B)$$

has finite cokernel, and there is a homomorphism

$$SK_1(B, a^2B) \longrightarrow SK_1(A, \underline{q})$$

with finite cokernel.

$GL_n(A, aB) = GL_n(B, aB)$ since both consist of all $\alpha \in GL_n(\Lambda)$ such that $I - \alpha$ and $I - \alpha^{-1}$ have coordinates in aB. Since B/aB is finite $GL_n(B, aB)$ has finite index in $GL_n(B)$. Hence, since $GL_n(A, aB) \subset GL_n(A, \underline{q})$, the latter has finite index in $GL_n(B)$. This fact for $n \geq 2$, implies the first assertion.

We next note that, for $n \geq 3$,

$$
\begin{aligned}
E_n(A, \underline{q}) &\supset E_n(A, \alpha B) \\
&\supset [GL_n(A, \alpha B), GL_n(A, \alpha B)] \qquad \text{(see (V, 4.3) and (V, 4.5))} \\
&= [GL_n(B, \alpha B), GL_n(B, \alpha B)] \\
&\supset E_n(B, \alpha^2 B) \qquad \text{(see (V, 1.5))}
\end{aligned}
$$

Evidently $SL_n(B, \alpha^2 B) = SL_n(A, \alpha^2 B) \subset SL_n(A, \underline{q})$, also, so we obtain a homomorphism $SK_1(B, \alpha^2 B) \longrightarrow SK_1(A, \underline{q})$, induced by the inclusions. We see as above that, since $SL_n(B, \alpha^2 B)$ has finite index in $SL_n(B)$, this homomorphism has finite cokernel.

(iii) det: $K_1(A, \underline{q}) \longrightarrow U(R')$ <u>has finite cokernel</u>.

With the aid of (ii) it suffices to show that det: $K_1(B) \longrightarrow U(R')$ has finite cokernel. B is a product of maximal orders in the simple factors of Λ, so we can assume Λ is simple; say $[\Lambda: C] = n^2$. Then if $\alpha \in U(R')$ we have $Nrd_{\Lambda/C}(\alpha) = \alpha^n$ (see (III, §8)). Since $R' \subset B$ we conclude that $\det(K_1(B))$ contains all n^{th} powers in $U(R')$. According to (3.1) $U(R')$ is finitely generated, so the desired conclusion now follows.

(iv) <u>For any two sided ideal</u> $\underline{q}$ <u>in</u> A, $SK_1(A, \underline{q})$ <u>is a torsion group of bounded exponent</u>.

With the aid of (i) above there is no loss in assuming that $A/\underline{q}$ is finite. Then we can replace $(A, \underline{q})$ by $(B, \alpha^2 B)$ as in (ii) above and further reduce to the case when A is a maximal order. Then A decomposes into a product of maximal orders in the factors of Λ, and $\underline{q}$ decomposes correspondingly, so we can assume Λ is simple. Let C_1 be a finite extension of C such that $\Lambda \otimes_C C_1 \simeq M_n(C_1)$. Let R_1

be the integral closure of R' in C_1 and put $A_1 = A \otimes_{R'} R_1$ and $\underline{q}_1 = \underline{q} \otimes_{R'} R_1$. Then there is a natural homomorphism

$$f: SK_1(A, \underline{q}) \longrightarrow SK_1(A_1, \underline{q}_1)$$

because the reduced norm is stable under base change. Moreover it follows from (IX, 4.7) that Ker(f) is annihilated by $[R_1: R]^2$. Hence the proof will be complete if we show that: $SK_1(A_1, \underline{q}_1)$ is finite. (The only point in this argument that prevents us from concluding that $SK_1(A, \underline{q})$ is finite is the lack of control over Ker(f) above. Thus the proof shows that $SK_1(A, \underline{q})$ is finite if, for example, the algebra Λ is split.)

Just as with $(A, \underline{q})$ above, we can embed A_1 in a maximal order B_1 and find an ideal $\underline{b}_1$ in B_1 such that $B_1/\underline{b}_1$ is finite and such that there is a homomorphism $SK_1(B_1, \underline{b}_1) \longrightarrow SK_1(A_1, \underline{q}_1)$ with finite cokernel. Since B is a maximal R_1-order in $M_n(C_1)$ it follows from (III, 8.8) that $B_1 = \mathrm{End}_{R_1}(P)$ for some $P \in \underline{\underline{P}}(R_1)$. Now it follows from the theory of (II, §§3-4) that $SK_1(B_1, \underline{b}_1) = SK_1(R_1, \underline{a})$ for some ideal $\underline{a} \neq 0$ in R_1. In the function field case it follows from (VI, 8.5) that $SK_1(R_1, \underline{a}) = 0$. In the number field case it follows from (VI, 7.3) that $SK_1(R_1, \underline{a})$ is a finite cyclic group. q.e.d.

(v) Proof of (a).

The first assertion follows from (iv). Since $SL_n(A, \underline{q}) \longrightarrow SK_1(A, \underline{q})$ is surjective for $n \geq 2$ the latter will be finitely generated if the former is. A finitely generated torsion group is finite, so this proves (a).

(vi) Proof of (b).

It follows from (iii) that det: $K_1(A, \underline{q}) \longrightarrow U(R')$

has finite cokernel, and from (iv) that it has a torsion kernel. Hence $K_1(A, \underline{q})/(\text{torsion})$ is a subgroup of finite index in $U(R')/(\text{torsion})$. According to (3.1) the latter is free abelian of rank $r_\infty - r_0$, and (b) follows from this.

(vii) Proof of (c).

Let B be a maximal order containing A and let $\underline{c}$ be the conductor from B to A. Let

$$N = \operatorname{Im}(G_1(A/\underline{c}) \longrightarrow G_1(A))$$

and consider the (non commutative) square

$$(*) \qquad \begin{array}{ccc} K_1(A) & \xrightarrow{\ j_*\ } & K_1(B) \\ {\scriptstyle c_1(A)}\big\downarrow & & \big\downarrow{\scriptstyle c_1(B)}\ (\simeq) \\ G_1(A) & \xleftarrow[\ j^*\ ]{} & G_1(B). \end{array}$$

According to (IX, 5.9) $G_1(A) = \operatorname{Im}(j\ ) + N$, since A is a fibre product of $A/\underline{c}$ and B. Moreover, since $A/\underline{c}$ is finite, it follows that $G_1(A/\underline{c})$ $(\simeq K_1((A/\underline{c})/\operatorname{rad}(A/\underline{c}))$ is finite, so N is a finite group. Part (iii) above shows that j has finite cokernel, and $c_1(B)$ is an isomorphism because B is regular. Hence the assertion, (c), that $c_1(A)$ has finite cokernel will follow if we show that (*) commutes modulo N.

Suppose $(P, \alpha) \in \Sigma\underline{\underline{P}}(A)$. Then if $h = j^* c_1(B) j_*$, we have $h[P, \alpha]_{\underline{\underline{P}}(A)} = [P \otimes_A B, \alpha \otimes_A B]_{\underline{\underline{M}}(A)}$. If we tensor the exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$, of two sides A-modules, with (P, α), we see that $[P \otimes_A B, \alpha \otimes_A B]_{\underline{\underline{M}}(A)} = [P, \alpha]_{\underline{\underline{M}}(A)} + [P \otimes_A M, \alpha \otimes_A M]_{\underline{\underline{M}}(A)})$ where $M = B/A$. But $M\underline{c} = 0$ so the second term in the right lies in N. q.e.d.

This concludes the proof of (3.5).

We close this section with a partial generalization of (3.5) in characteristic zero.

(3.7) THEOREM. *Let* B *be any finite* $\mathbb{Z}$-*algebra. Then* $GL_n(B)$ *is a finitely generated group for all* $n \geq 1$. *Moreover* $K_1(B)$ *and* $G_1(B)$ *are finitely generated abelian groups*.

Proof. As in (III, 8.10) there is a maximal nilpotent two sided ideal N in B, and $B/N = A \times T$ where T is a semi-simple ring, and where A is a $\mathbb{Z}$-order in a semi-simple $\mathbb{Q}$-algebra. Therefore we have an exact sequence.

$$1 \longrightarrow U(B, N) \longrightarrow U(B) \longrightarrow U(B/N) \longrightarrow 1$$

of groups, and a decomposition $U(B/N) = U(A) \times U(T)$. The group U(T) is finite and U(A) is finitely generated by Siegel's Theorem (3.2). According to (3.8) below, U(B, N) is also finitely generated. Therefore U(B) is finitely generated. Similarly $U(M_n(B)) = GL_n(B)$ is finitely generated for all $n \geq 1$. For $n \geq 2$ this implies $K_1(B)$ is finitely generated, by (V, 4.2).

It follows from (VIII, 2.3) that $G_1(B/N) \longrightarrow G_1(B)$ is an isomorphism. We have $G_1(B/N) = G_1(A) \oplus G_1(T)$, and $G_1(T)$ is finite. According to (3.6) $G_1(A)$ is finitely generated. Hence $G_1(B)$ is finitely generated. q.e.d.

(3.8) PROPOSITION. *Let* N *be a two sided ideal in a ring* B *and suppose* $N^{d+1} = 0$ *for some* $d \geq 0$. *Let* $G = 1 + N = U(B, N)$.

(a) *If* $d = 1$ *then* $N \longrightarrow G$, $a \longmapsto 1 + a$, *is a group isomorphism*.

(b) *If* N *is finitely generated as an additive group then* G *is a finitely generated group*.

(c) *If* $eN = 0$ *for some integer* $e > 0$ *then* G *has exponent* e^d; *i.e.* $x^{e^d} = 1$ *for all* $x \in G$.

Proof. (a) is obvious.

(b) and (c): We have a group extension

$$1 \longrightarrow 1 + N^d \longrightarrow 1 + N \longrightarrow 1 + (N/N^d) \longrightarrow 1,$$

whose kernel, by (a), is isomorphic to N^d. Hence the kernel is finitely generated, resp., of exponent e. By induction on d, the quotient is finitely generated, resp., of exponent e^{d-1}. Hence 1 + N is finitely generated, resp., of exponent e^d.

HISTORICAL REMARKS

The main results of this chapter derive essentially from classical number theory. The techniques for applying these classical finiteness theorems to K_o and G_o originated in the work of Swan [1]. The deployment of the Cartan condition, as we have done in §1, is taken largely from the thesis of Strooker [1] and Giorguitti [1]. The finite generation of $G_1(A)$ in Characteristic zero (cf. (3.6)) is due to Lam [1].

Chapter XI
INDUCTION TECHNIQUES FOR FINITE GROUPS

This chapter is devoted to the exposition of a basic technique developed by Swan in two fundamental papers (Swan [1] and [3]) and later axiomatized and extended by Lam [1].

Briefly, the idea is the following. Let R be a commutative ring, and write $R\pi$ for the group algebra of a group π. We write

$$G_R(\pi)$$

for the Grothendieck group of all right $R\pi$-modules which are finitely generated and projective as R-modules. Swan pointed out that this is the proper generalization of the "character ring" in classical representation theory. When π is finite and R is a field of characteristic zero then $G_R(\pi)$ can be identified with the character ring. The classical induction theorems (of Artin, Brauer, Witt, Berman,...) state, in this case, that $G_R(\pi)$ is generated by induced characters from certain restricted families of subgroups of π. Swan then showed how the functorial properties of $G_R(\pi)$ could be used to extend these induction theorems to a much more general setting. He further discovered that one could deduce similar results for $K_o(R\pi)$ with the aid of the fact that $K_o(R\pi)$ is a $G_R(\pi)$-module in a way which is compatible with the induction and restriction homomorphisms, and so that the scalar multiplication satisfied a "Frobenius reciprocity" identity, which is familar in representation theory. In Bass [3] I used this method to obtain information about $K_1(\underline{\underline{Z}}\pi)$ for π a finite group.

The axiomatization of Lam is based on the notion of a "Frobenius functor". This is just an abstraction of the properties of G_R above which are required for the basic induction arguments. If G is a Frobenius functor he introduces the category of "G-modules". The general induction agrument can then be formulated as saying that an induction theorem for G implies similar results for all G-modules. In the setting above both $K_o(R\pi)$ and $K_1(R\pi)$ are $G_R(\pi)$-modules, in this sense.

The first section contains a rapid review of induction and restriction for modules over group rings. Frobenius functors and their modules are introduced in §2. In §3 we assemble a number of results of Swan and Lam on the functorial behavior of "induction exponents". Then in §4 the classical induction theorems are quoted, without proof. These include a sharp quantitative refinement of the Artin Induction Theorem, which is due to Lam. Some standard applications of these theorems to representation theory are also indicated here.

Several of the principal results of Swan [3] on $K_o(R\pi)$ and $G_o(R\pi)$ are derived in §5, following Swan's arguments rather closely.

For precise calculations of $K_i(\underline{\underline{Z}}\pi)$, when π is finite abelian, it is important to determine the "conductor" from $\underline{\underline{Z}}\pi$ to its integral closure in $\underline{\underline{Q}}\pi$. This calculation is carried out quite explicitly in §6. It is used, in particular, in §7 where the methods of this chapter are applied to the groups $K_1(R\pi)$ and $G_1(R\pi)$, when π is finite and R is a ring of algebraic integers.

§1. GROUP RINGS, RESTRICTION, AND INDUCTION

Let R be a ring and let π be a monoid. Then the <u>monoid ring</u> (or <u>group ring</u> if π is a group) of π over R is the free R-module with basis π and with multiplication extended R-bilinearly from the multiplication in π. It is a functor of both R and of π. Explicitly, let f: R $\longrightarrow$ R$'$ be a ring homomorphism and let j: $\pi \longrightarrow \pi'$ be a homomorphism of monoids. Then we have f: $R\pi \longrightarrow R'\pi$ by $f(\Sigma_{x \in \pi} a_x x) = \Sigma f(a_x)x$, and j: $R\pi \longrightarrow R\pi'$ by $j(\Sigma_{x \in \pi} a_x x) = \Sigma a_x j(x)$. (cf. (IV, §5)).

In case $\pi' = \{1\}$ we obtain the augmentation, $R\pi \longrightarrow R$, $\Sigma a_x x \rightarrowtail \Sigma a_x$, whose kernel, I, is called the augmentation ideal. It is a two sided ideal generated as an R-module by all $1 - x$, $(x \in \pi)$. The augmentation defines an $R\pi$-module structure on R (π acting trivially). We shall call this the trivial $R\pi$-module, and denote it by R_π.

If π' is a subgroup of a group π we write $[\pi: \pi'] = \mathrm{card}(\pi/\pi')$ for the index of π' in π. The expression "π is a p-group" will always mean p is a prime number and every element of π has order a power of p. For finite groups this is equivalent to $[\pi: 1]$ being a power of p.

(1.1) PROPOSITION (Maschke). Let π be a group and let π' be a subgroup of finite index $n = [\pi: \pi']$. Let R be a ring such that $n \in U(R)$.

(a) An exact sequence $0 \longrightarrow M' \longrightarrow M \xrightarrow{p} M'' \longrightarrow 0$ of $R\pi$-modules splits if it splits as a sequence of $R\pi'$-modules.

(b) If $M \in$ mod-$R\pi$ then $\mathrm{hd}_{R\pi}(M) = \mathrm{hd}_{R\pi'}(M)$.

Proof. (a) Let h: $M'' \longrightarrow M$ be an $R\pi'$-homomorphism such that $ph = 1_{M''}$. Let $\pi = \cup \pi' x_i$ $(1 \leq i \leq n)$ be the coset decomposition and set $h'(m) = \Sigma\, h(mx_i^{-1})\, x_i$ $(1 \leq i \leq n)$. Since h is π'-linear $h(mx_i^{-1})\, x_i$ depends only on the coset $\pi' x_i$. If $x \in \pi$ then $h'(mx) = \Sigma\, h(mx\, x_i^{-1})\, x_i = (\Sigma\, h(mx\, x_i^{-1})\, x_i\, x^{-1})\, x = h'(m)\, x$, because the $x_i\, x^{-1}$ are also a set of coset representatives. Hence h' is π-linear. Moreover $ph'(m) = \Sigma_i ph(m\, x_i^{-1})\, x_i = \Sigma_i m x_i^{-1} x_i = m \cdot n$. Thus $n^{-1}h'$ is an $R\pi$-linear right inverse for p.

(b) If $\pi = \cup x_i \pi'$ $(1 \leq i \leq n)$ then clearly $R\pi = \amalg x_i R\pi'$, so $R\pi$ is a free $R\pi'$-module. Hence, if $M \in$ mod-$R\pi$, an $R\pi$-projective resolution of M is also an $R\pi'$-projective resolution, so $hd_{R\pi'}(M) \leq hd_{R\pi}(M)$. Conversely, suppose $hd_{R\pi'}(M) = n < \infty$ and choose an exact sequence $0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ in mod-$R\pi$ with P_i $R\pi$-projective $(0 \leq i < n)$. We must know P_n is $R\pi$-projective, knowing it is $R\pi'$-projective. But this follows by letting P_n play the role of M" in part (a). q.e.d.

(1.2) COROLLARY. Let π be a finite group of order $n = [\pi: 1]$. Let R be an integrally closed integral domain with field of fractions L of characteristic not dividing n. Then $L\pi$ is semi-simple, and every R-order in $L\pi$ containing $R\pi$ is contained in $n^{-1}R\pi$.

Proof. The first assertion follows immediately from (1.1) (a) (see (III, 1.5)). Let $\pi = \{x_1 = 1, x_2, \ldots, x_n\}$. The regular representation of $L\pi$ with respect to this basis represents each $x_i \neq 1$ by a permutation matrix with no diagonal entries. Hence $Tr_{L\pi/L}(x_i) = \delta_{1i} \cdot n$ (Kronecker delta). Therefore, if we set $x_i' = n^{-1}x_i^{-1}$ we have $Tr_{L\pi/L}(x_i x_j') = \delta_{ij}$ $(1 \leq i, j \leq n)$. Let B be an R-order in $L\pi$ containing $R\pi$, and let $b \in B$. Clearly $x_1', \ldots, x_n'$ is an L-basis of $L\pi$ so we can write $b = \Sigma\, x_j'(b_j \in L)$. Then we have $Tr_{L\pi/L}(x_i b) = \Sigma_j\, Tr_{L\pi/L}(x_i x_j')\, b_j = b_i$. Now $x_i b \in B$ is integral over R, so it follows from (III, 5.14) that the characteristic polynomial of the L-endomorphism $L\pi \xrightarrow{x_i b \cdot} L\pi$ has coefficients which are integral over R. Since R is integrally close they lie in R; in particular $b_i = Tr_{L\pi/L}(x_i b) \in R$. Thus $B \subset \Sigma\, x_j' R = n^{-1}R\pi$. q.e.d.

We next indicate what happens at the other extreme, when $char(L) = p > 0$ and π is a p-group.

(1.3) PROPOSITION. Let π be a finite p-group operating *on a finite set* S. *Then*:

(a) $\operatorname{card}(S^\pi) \equiv \operatorname{card}(S) \bmod p$,

where S^π = *the set of fixed points of* π; and

(b) π *is nilpotent*.

Proof. (a) If $s \in S$ write $\pi_s = \{x \in \pi \mid xs = s\}$, the stability subgroup. Then $x \longmapsto xs$ induces a bijection $\pi/\pi_s \longrightarrow \pi s$ (the orbit), so $\operatorname{card}(\pi s) = [\pi: \pi_s]$ is a positive power of p unless $s \in S^\pi$. Since S is the disjoint union of the orbits we have $\operatorname{card}(S) = \operatorname{card}(S^\pi) + N$, where N is a sum of positive powers of p.

(b) Let π operate on itself by conjugation. Then π^π = center (π) has cardinality $\equiv [\pi: 1] \bmod p$. Applying induction on $[\pi: 1]$ to π/center (π) this implies (b). q.e.d.

(1.4) COROLLARY. *Let* R *be a commutative local ring with residue class field* k *of characteristic* $p > 0$, *and let* π *be a finite* p *group. Then* $R\pi$ *is a local ring whose only simple module is* k_π.

Proof. Let $\underline{m}$ = rad R, so $k = R/\underline{m}$. Then $\underline{m}(R\pi) \subset$ rad $R\pi$, (see (III, 2.5)) so it suffices to show that $R\pi/\underline{m}(R\pi) = k\pi$ is local. We have $k_\pi = k\pi/I$ where I is the augmentation ideal. If we show that k_π is the only simple module it will follow that $I \subset$ rad $k\pi$. Hence we will have I = rad $k\pi$, a nilpotent ideal, and $k\pi$ is local. This latter condition is stable under base field extensions, so it suffices to prove it for the prime field $k_o \subset k$.

Let $M \neq 0$ be a simple $k_o\pi$-module. Then M is a finite set so (1.3) implies $\operatorname{card}(M^\pi) \equiv \operatorname{card}(M) \bmod p$. Since $\operatorname{card}(M)$ is a positive power of p it follows that $M^\pi \neq 0$. Therefore, since M^π is a $k_o\pi$-submodule and M is simple, we must have $M^\pi = M \simeq (k_o)_\pi$. q.e.d.

<u>Henceforth</u> R <u>will denote a commutative ring</u>. Let π be a group and let $j\colon \pi' \longrightarrow \pi$ be a group homomorphism. Then we have the <u>induction</u> and <u>restriction</u> functors

$$\text{mod-}R\pi' \underset{j^* = \text{res}}{\overset{j_* = (\cdot \otimes_{R\pi'} R\pi)}{\rightleftarrows}} \text{mod-}R\pi.$$

If $f\colon R' \longrightarrow R$ is a homomorphism of commutative rings we also have the functors

$$\text{mod-}R'\pi \underset{f^* = \text{res}}{\overset{f_* = (\cdot \otimes_{R'} R)}{\rightleftarrows}} \text{mod-}R\pi.$$

The compatibility of these functors is expressed by the following natural isomorphisms, arising from the commutative square

$$\begin{array}{ccc} R'\pi' & \longrightarrow & R'\pi \\ \downarrow & & \downarrow \\ R\pi' & \longrightarrow & R\pi. \end{array}$$

(i) $f_* j_* \simeq j_* f_*$: If $M \in \text{mod-}R'\pi'$ then $(M \otimes_{R'\pi'} R'\pi) \otimes_{R'} R \simeq M \otimes_{R'\pi'} R\pi \simeq (M \otimes_{R'} R) \otimes_{R\pi'} R\pi$, as $R\pi$-modules.

(ii) $j_* f^* \simeq f^* j_*$: If $M \in \text{mod-}R\pi'$ then $(f^*M) \otimes_{R'\pi'} R'\pi \simeq f^*(M \otimes_{R\pi'} R\pi)$ as $R'\pi$-modules.

(iii) $f_* j^* = j^* f_*$: If $M \in \text{mod-}R'\pi$ then $j^*M \otimes_{R'} R = j^*(M \otimes_{R'} R)$ as $R\pi'$-modules.

(iv) $f_* j_* = j_* f_*$: If $M \in \text{mod-}R\pi$ then $f_* j_* M = j_* f_* M$ as $R'\pi'$-modules.

In parts (iii) and (iv) the isomorphisms are equalities. In part (ii), $M \otimes_{R'\pi'} R'\pi \longrightarrow M \otimes_{R\pi'} R\pi$ is defined by $m \otimes x \longmapsto m \otimes x$ ($m \in M$, $x \in \pi$). The isomorphisms in part

(i) follow from the associativity of tensor products once we note the natural isomorphism $M \otimes_{R'} R \simeq M \otimes_{R'\pi} R\pi$ for $M \in$ mod-$R\pi$, and similarly for π'. Similarly, if $f'\colon R'' \longrightarrow R'$ is another ring homomorphism, and if $j'\colon \pi'' \longrightarrow \pi'$ is another group homomorphism then we have the transitivity formulas:

$$(jj')_* \simeq j_* j'_* , \qquad (jj')^* = j'^* j^*$$

$$(ff')_* \simeq f_* f'_* , \qquad (ff')^* = f'^* f^* .$$

For restriction these are equalities. For induction they correspond to the associativity of tensor products.

Next we introduce the additive bifunctor

$$\otimes_R\colon (\text{mod-}R\pi) \times (\text{mod-}R\pi) \longrightarrow \text{mod-}R\pi .$$

If $M, N \in$ mod-$R\pi$ then $M \otimes_R N$ is an R-module on which we let $x \in \pi$ operate by $(m \otimes n)\, x = mx \otimes nx$, and extend this action R-linearly to $R\pi$. Evidently the natural isomorphism $M \otimes_R N \simeq N \otimes_R M$ is an isomorphism of $R\pi$-modules. Note also that

$$M \otimes_R R_\pi \simeq M$$

is an isomorphism of $R\pi$-modules.

(1.5) PROPOSITION ("Frobenius Reciprocity"). Let R be a commutative ring and let $j\colon \pi' \longrightarrow \pi$ be a homomorphism of groups. For $M \in$ mod-$R\pi$ and $N \in$ mod-$R\pi'$ there is an isomorphism

$$\phi\colon j_*(j^* M \otimes_R N) \longrightarrow M \otimes_R j_* N$$

defined by $\phi((m \otimes n) \otimes x) = mx \otimes (n \otimes x)$ $(m \in M,\ n \in N,\ x \in \pi)$, and it defines an isomorphism of functors

$$(\text{mod-}R\pi) \times (\text{mod-}R\pi') \longrightarrow \text{mod-}R\pi .$$

Proof. Let $W = M \otimes_R j_* N = M \otimes_R (N \otimes_{R\pi'} R\pi)$. For $x \in \pi$ the expression $mx \otimes (n \otimes x) \in W$ is R-bilinear for $(m, n) \in$

$M \times N$ so it defines an R-linear map $h_x: M \otimes_R N \longrightarrow W$. Since $R\pi$ is R-free with basis π we can use the h_x's $(x \in \pi)$ to define an R-linear map $\phi': (M \otimes_R N) \otimes_R R\pi \longrightarrow W$ such that $\phi'((m \otimes n) \otimes x) = mx \otimes (n \otimes x)$. If $y \in \pi'$ then $\phi'((m \otimes n) \otimes j(y)x) = mj(y)x \otimes (n \otimes j(y)x) = mj(y)x \otimes (ny \otimes x)$, while $\phi'((m \otimes n)y \otimes x) = \phi'((mj(y) \otimes ny) \otimes x) = mj(y) \otimes (ny \otimes x)$. Thus ϕ' is $R\pi'$-bilinear so it induces a homomorphism $\phi: (M \otimes_R N) \otimes_{R\pi'} R\pi \longrightarrow W$. If $y \in \pi$ then $\phi((m \otimes n) \otimes x)y) = \phi((m \otimes n) \otimes xy) = mxy \otimes (n \otimes xy) = mxy \otimes (n \otimes x)y = (mx \otimes (n \otimes x))y = \phi((m \otimes n) \otimes x)y$, so ϕ is $R\pi$-linear.

To construct the inverse, suppose $m \in M$, $n \in N$, and $x \in \pi$. Writing $V = (M \otimes_R N) \otimes_{R\pi'} R\pi$, the expression $(mx^{-1} \otimes n) \otimes x \in V$ defines an R-linear map $N \longrightarrow V$. Fixing m and varying $x \in \pi$ we obtain an R-linear map $h_m': N \otimes_R R\pi \longrightarrow V$. If $y \in \pi'$ then $h_m'(n \otimes j(y)\, x) = (m\, (j(y)^{-1} \otimes n) \otimes j(y)\, x = (mx^{-1}\, j(y)^{-1} \otimes n)\, y \otimes x = (mx^{-1} \otimes ny) \otimes x = h_m'(my \otimes x)$. Hence h_m' induces an R-linear map $h_m: N \otimes_{R\pi'} R\pi \longrightarrow V$ such that $h_m(n \otimes x) = (mx^{-1} \otimes n) \otimes x$. Since this expression is R-linear in m we obtain $\psi: M \otimes_R (N \otimes_{R\pi'} R\pi) \longrightarrow V$ such that $\psi(m \otimes (n \otimes y)) = (mx^{-1} \otimes n) \otimes x$. Evidently ψ is an inverse for ϕ, so ϕ is an isomorphism.

Suppose $f: M \longrightarrow M_1$ in mod-$R\pi$ and $g: N \longrightarrow N_1$ in mod-$R\pi'$. Then $\phi \circ ((f \otimes_R g) \otimes_{R\pi'} R\pi)$ sends $(m \otimes n) \otimes x$ to $f(m)x \otimes (g(n) \otimes x)$, while $(f \otimes_R (g \otimes_{R\pi'} R\pi)) \circ \phi$ sends it to $f(mx) \otimes (g(n) \otimes x)$. These two images are equal because f is π-linear. Thus ϕ is natural. q.e.d.

(1.6) COROLLARY. _If_ $M \in$ mod-$R\pi$ _has underlying_ R-_module_ j^*M _then_

$$M \otimes_R R\pi \simeq j^* M \otimes_R R\pi$$

_as_ $R\pi$-_modules_.

Proof. Let j be the inclusion of the trivial subgroup and apply (1.5) with N = R.

We shall denote by

$$\underline{\underline{M}}_o(R\pi)$$

the full subcategory of all M ε mod-Rπ which are finitely generated and projective as R-modules.

(1.7) COROLLARY. _The tensor product induces functors_

$$\otimes_R: \underline{\underline{M}}_o(R\pi) \times \underline{\underline{M}}_o(R\pi) \longrightarrow \underline{\underline{M}}_o(R\pi)$$

and

$$\otimes_R: \underline{\underline{M}}_o(R\pi) \times \underline{\underline{P}}(R\pi) \longrightarrow \underline{\underline{P}}(R\pi)$$

which preserve short exact sequences in each variable.

Proof. The exactness is clear since short exact sequences in $\underline{\underline{P}}$ split, and, in $\underline{\underline{M}}_o$, they split as sequences of R-modules. Moreover it is clear that $M \otimes_R N \in \underline{\underline{M}}_o$ if $M, N \in \underline{\underline{M}}_o$. If, further, $P \in \underline{\underline{P}}$, it remains to be shown that $M \otimes_R P \in \underline{\underline{P}}$. By a direct sum argument it suffices to show this for $P = R\pi$, in which case it follows immediately from (1.6). q.e.d.

(1.8) DEFINITION. Let R be a commutative ring and let π be a group. We define

$$G_R(\pi) = K_o(\underline{\underline{M}}_o(R\pi)).$$

According to (1.7) we can use $\otimes_R$ to give $G_R(\pi)$ the structure of a _commutative ring_. Even more, if $P \in \underline{\underline{P}}(R)$ and $M \in \underline{\underline{M}}_o(R\pi)$ then $P \otimes_R M \in \underline{\underline{M}}_o(R\pi)$, clearly and we can use this to make $G_R(\pi)$ a $K_o(R)$-_algebra_. The identity element of $G_R(\pi)$ is $[R_\pi]$. We shall call the groups

$$K_i(\underline{\underline{M}}_o(R\pi)) \qquad (i = 0, 1)$$

and

$$K_i(\underline{\underline{P}}(R\pi)) = K_i(R\pi) \qquad (i = 0, 1)$$

the four basic G_R-modules. They are, indeed, $G_R(\pi)$-modules, with action defined by

$$[M]\ [N] = [M \otimes_R N] \quad (i = 0)$$

and

$$[M]\ [N, \alpha] = [M \otimes_R N, M \otimes_R \alpha] \quad (i = 1)$$

where $M \in \underline{\underline{M}}_o$, $N \in \underline{\underline{M}}_o$ or $\underline{\underline{P}}$, as the case may be, and $\alpha \in \mathrm{Aut}_{R\pi}(N)$. Our notation is meant to suggest that, for fixed R, we view G_R and the basic G_R-modules as functors of π. The sense in which they are such functors will be described in (1.10) below. First, however, we shall show how the category $\underline{\underline{M}}_o(R\pi)$ is related to the category $\underline{\underline{M}}(R\pi)$ in certain cases.

(1.9) PROPOSITION. Let R be a commutative regular ring and let π be a finite group. Then we have $\underline{\underline{P}}(R\pi) \subset \underline{\underline{M}}_o(R\pi) \subset \underline{\underline{M}}(R\pi)$, and the latter inclusion induces isomorphisms

$$K_i(\underline{\underline{M}}_o(R\pi)) \longrightarrow K_i(\underline{\underline{M}}(R\pi)) = G_i(R\pi), \ (i = 0,1).$$

In particular, $G_R(\pi) = G_o(R\pi)$.

Proof. If $P \in \underline{\underline{P}}(R\pi)$ then $P \in \underline{\underline{M}}_o(R\pi)$ because $R\pi$ is a free R-module of finite rank ($[\pi: 1]$). The second inclusion is obvious.

Suppose $M \in \underline{\underline{M}}(R\pi)$. Since π is finite M is also a finitely generated R-module, so $hd_R(M) = n < \infty$. Let $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_o \longrightarrow M \longrightarrow 0$ be an exact sequence of $R\pi$-modules with $P_i \in \underline{\underline{P}}(R\pi)$ $(0 \leq i < n)$. (Note that $R\pi$ is right noetherian so we can do this.) Then since $P_i \in \underline{\underline{P}}(R)$ also $(0 \leq i < n)$ it follows that $P_n \in \underline{\underline{P}}(R)$, i.e. $P_n \in \underline{\underline{M}}_o(R\pi)$. Now we can apply (VIII, 4.6) to conclude that

$K_i(\underline{\underline{M}}_o) \longrightarrow K_i(\underline{\underline{M}})$ (i = 0, 1) are isomorphisms. q.e.d.

(1.10) PROPOSITION. _Let_ R _be a commutative ring, let_ π _be a group, and let_ $j\colon \pi' \longrightarrow \pi$ _be the inclusion of a subgroup of finite index. Then restriction and induction induce exact functors_

$$\underline{\underline{M}}_o(R\pi') \underset{j^*}{\overset{j_*}{\rightleftarrows}} \underline{\underline{M}}_o(R\pi)$$

and

$$\underline{\underline{P}}(R\pi') \underset{j^*}{\overset{j_*}{\rightleftarrows}} \underline{\underline{P}}(R\pi).$$

Hence, if K_R _denotes any of the basic_ G_R_-modules, we have induced additive maps_

$$K_R(\pi') \underset{j^*}{\overset{j_*}{\rightleftarrows}} K_R(\pi).$$

These satisfy the following conditions:

(1) $j^*\colon G_R(\pi) \longrightarrow G_R(\pi')$ _is a ring homomorphism and_ $j^*\colon K_R(\pi) \longrightarrow K_R(\pi')$ _is_ j^*_-semi-linear. I.e._ $j^*(ab) = j^*(a)\, j^*(b)$ _for_ $a \in G_R(\pi)$, $b \in K_R(\pi)$.

(2) _If_ $a \in G_R(\pi)$, $a' \in G_R(\pi')$, $b \in K_R(\pi)$, _and_ $b' \in K_R(\pi')$, _then_

$$a \cdot j_* b' = j_*(j^* a \cdot b)$$

and

$$j_* a' \cdot b = j_*(a' \cdot j^* b).$$

Proof. It is obvious that restriction preserves $\underline{\underline{M}}_o$ and that induction preserves $\underline{\underline{P}}$. The other two assertions follow easily from the fact that $R\pi \in \underline{\underline{P}}(R\pi')$. (In fact $R\pi$ is a free $R\pi'$-module with the coset representatives as a basis.) This further implies that j_* is exact, and j^* obviously is exact. Therefore we obtain the indicated homomorphisms. Since j^* preserves $\otimes_R$ (clearly) part (1) follows immediately from the definition of the action of G_R on K_R. Similarly, part (2) follows immediately from Frobenius reciprocity (1.5) in case K_R is one of the K_o's. For the K_1's this applies equally well because of the naturality of the Frobenius reciprocity isomorphism. Explicitly, suppose, in the setting of (1.5), that $\alpha \in Aut_{R\pi}(M)$ and $\beta \in Aut_{R\pi'}(N)$. Then

$$\phi:(j_*(j^* M \otimes_R N),\ j_*(j^*\alpha \otimes_R \beta)) \longrightarrow (M \otimes_R j_*N,\ \alpha \otimes j_*\beta)$$

is an isomorphism in the category, Σ(mod-$R\pi$), of automorphisms of $R\pi$-modules. Setting α or β equal to the identity now yields the two formulas of (2) where K_R is a K_1. q.e.d.

(1.11) PROPOSITION. *In the setting of* (1.10) *let* $f: R' \longrightarrow R$ *be a homomorphism of commutative rings. Then* f_*: mod-$R'\pi \longrightarrow$ mod-$R\pi$ *preserves tensor products, in the sense that there is a natural isomorphism*

$$f_*(M \otimes_{R'} N) \simeq f_*M \otimes_R f_*N$$

of $R\pi$-*modules for* $M, N \in$ mod-$R'\pi$. *Moreover* f_* *induces exact functors* $\underline{\underline{M}}_o(R'\pi) \longrightarrow \underline{\underline{M}}_o(R\pi)$ *and* $\underline{\underline{P}}(R'\pi) \longrightarrow \underline{\underline{P}}(R\pi)$, *and hence also additive maps*

$$f_*: G_{R'}(\pi) \longrightarrow G_R(\pi) \text{ and } f_*: K_{R'}(\pi) \longrightarrow K_R(\pi).$$

The first of these is a ring homomorphism, and the second is semi-linear with respect to the first (i.e. $f_*(ab) = f_*(a)f_*(b)$ *for* $a \in G_{R'}(\pi)$, $b \in K_{R'}(\pi)$). *Moreover, with* $j: \pi' \longrightarrow \pi$ *as in* (1.10) *we have* $f_*j_* = j_*f_*$ *and* $f_*j^* = j^*f_*$.

In case R *is a finitely generated projective* R'-*module* (i.e. $R \in \underline{\underline{P}}(R')$) *then the restriction functor*

f^*: mod-$R\pi$ ⟶ mod-$R'\pi$ induces functors $\underline{\underline{M}}_o(R\pi)$ ⟶ $\underline{\underline{M}}_o(R'\pi)$ and $\underline{\underline{P}}(R\pi)$ ⟶ $\underline{\underline{P}}(R'\pi)$, and hence also an additive map

$$f^*: K_R(\pi) \longrightarrow K_{R'}(\pi).$$

The maps f^* also commute with j_* and j^*, and we have

$$f^*f_* = \text{multiplication by } [R]_{R'},$$

the class of R as an R′-module in $K_o(R')$ (or as a trivial) $R'\pi$-module in $G_{R'}(\pi)$).

Proof. We have $f_*M \otimes_R f_2N = (M \otimes_{R'} R) \otimes_R (N \otimes_{R'} R) \simeq M \otimes_{R'}(N \otimes_{R'} R) \simeq f_*(M \otimes_{R'} N)$, and these are easily seen to be $R\pi$-isomorphisms. It is clear from the definitions that f_* preserves $\underline{\underline{M}}_o$ and $\underline{\underline{P}}$. Its restrictions to these categories are exact because short exact sequences in $\underline{\underline{M}}_o(R'\pi)$ and $\underline{\underline{P}}(R'\pi)$ split over R′. The semi-linearity of f_* follows from the preservation of tensor products. The commutativity of f_* (and of f^*) with j_* and j^* was established in the discussion preceding (1.5).

In case $R \in \underline{\underline{P}}(R')$ then $P \in \underline{\underline{P}}(R')$ for all $P \in \underline{\underline{P}}(R)$. Therefore $f^*\underline{\underline{M}}_o(R\pi) \subset \underline{\underline{M}}_o(R'\pi)$. Similarly, $R\pi \in \underline{\underline{P}}(R'\pi)$ so $f^*\underline{\underline{P}}(R\pi) \subset P(R'\pi)$. If $M \in$ mod-$R'\pi$ then $f^*f_*M = f^*(M \otimes_{R'} R) \simeq M \otimes_{R'} R$, where we view R as an R′-module, or as an $R'\pi$-module with trivial π action. This concludes the proof.

§2. FROBENIUS FUNCTORS AND FROBENIUS MODULES

In order to axiomatize the treatment of the induction theorems in the following sections we introduce the notion of a Frobenius functor on a category $\underline{\underline{C}}$. It is simply a functor G: $\underline{\underline{C}}$ ⟶ $\underline{\underline{\text{Frob}}}$, so we must describe the category

$\underline{\underline{\text{Frob}}}$.

Its objects are commutative rings. A morphism A ⟶ B in

Frob is a pair (i_*, i^*) of additive maps

$$A \underset{i^*}{\overset{i_*}{\rightleftarrows}} B,$$

such that i^* is a ring homomorphism and such that

(1) $$b \cdot i_*a = i_*(i^*b \cdot a) \qquad (a \in A,\ b \in B).$$

If (j_*, j^*): $B \longrightarrow C$ is another morphism then

$$(j_*, j^*)\ (i_*, i^*) = (j_* \circ i_*,\ i^* \circ j^*).$$

To check that this is admissible suppose $a \in A$ and $c \in C$. Then $c \cdot j_*i_*a = j_*(j^*c \cdot i_*a) = j_*(i_*(i^*j_*c \cdot a))$.

If G and G′ are Frobenius functors on $\underline{\underline{C}}$ then we can speak of a morphism (= natural transformation) from G to G′.

(2.1) EXAMPLE. Let $\underline{\underline{G}}$ be the category whose objects are groups and whose morphisms are monomorphisms $j: \pi' \longrightarrow \pi$ of finite index, (i.e. $[\pi: j(\pi')]$ is finite). Let R be a commutative ring. Then it follows from (1.10) that

$$G_R: \underline{\underline{G}} \longrightarrow \underline{\underline{\text{Frob}}}, \qquad \pi \longmapsto G_R(\pi),$$

is a Frobenius functor, with respect to the (induction, restriction) homomorphisms. If $f: R' \longrightarrow R$ is a homomorphism of commutative rings then it follows from (1.11) that $f_*: G_{R'} \longrightarrow G_R$ is a morphism of Frobenius functors.

Let $G: \underline{\underline{C}} \longrightarrow \underline{\underline{\text{Frob}}}$ be a Frobenius functor. Then a Frobenius G-module K consists of the following:

(i) K assigns to each $\pi \in \underline{\underline{C}}$ a $G(\pi)$-module $K(\pi)$.

(ii) K assigns to each morphism $j: \pi' \longrightarrow \pi$ in $\underline{\underline{C}}$ a pair of additive maps $K(j) = (j_*, j^*)$,

$$K(\pi') \underset{j^*}{\overset{j_*}{\rightleftarrows}} K(\pi),$$

such that j^* is ($j^*: G(\pi) \longrightarrow G(\pi'))$-semi-linear (we write $G(j) = (j_*, j^*)$ also, by abuse of notation) and such that

$$(2)\quad \begin{aligned} j_*a' \cdot b &= j_*(a' \cdot j^*b) \quad (a' \in G(\pi'),\ b \in K(\pi)) \\ a \cdot j_*b' &= j_*(j^*a \cdot b') \quad (a \in G(\pi),\ b' \in K(\pi')). \end{aligned}$$

Moreover, we require that $j \longmapsto j_*$ and $j \longmapsto j^*$ should each make K into a functor, the latter contravariant. The Frobenius G-modules are themselves the objects of a category, which we denote by

G-mod.

If K, H $\in$ G-mod then a morphism f: K $\longrightarrow$ H is a collection of $G(\pi)$-homomorphisms $f(\pi): K(\pi) \longrightarrow H(\pi)$ ($\pi \in \underline{\underline{C}}$), which is a natural transformation simultaneously for the covariant and contravariant functors underlying K and H, respectively. We define $K \oplus H$ by $(K \oplus H)(\pi) = K(\pi) \oplus H(\pi)$, etc., and this makes G-mod an additive category. In fact it is easy to see that G-mod <u>is an abelian category</u>. For example Ker(f) exists and is the obvious thing: Ker(f) (π) = Ker(f(π)), with the morphisms those induced by j_* and j^*. Similarly for Coker(f), Im(f), etc.

The fact that G-mod is abelian is technically very useful, as we shall see in the later sections. In spirit, we can treat the Frobenius functors on a fixed category $\underline{\underline{C}}$ like commutative rings, and their Frobenius modules like modules over these rings. Note in particular that if $G' \longrightarrow G$ is a morphism of Frobenius functors then it permits us to view Frobenius G-modules as Frobenius G′-modules (by "restriction").

(2.2) EXAMPLE. Let $G_R: \underline{\underline{G}} \longrightarrow \underline{\underline{Frob}}$ be the Frobenius functor of example (2.1). Let K_R be one of the "basic G_R-modules" (see (1.8)). Then it follows from (1.10) that K_R defines a functor $\underline{\underline{G}} \longrightarrow \underline{\underline{Frob}}$ which is a Frobenius G_R-module. If f: R′ $\longrightarrow$ R is a homomorphism of commutative rings then it follows from (1.11) that $f_*: K_{R'} \longrightarrow K_R$ is

(f_*: $G_{R'} \longrightarrow G_R$)-semi-linear. Equivalently, if we view K_R as a $G_{R'}$-module, via f_*, then f_*: $K_{R'} \longrightarrow K_R$ is a $G_{R'}$-homomorphism. Therefore its kernel, cokernel, etc. are also $G_{R'}$-modules.

Let $\underline{\underline{FG}}$ denote the full subcategory of finite groups in $\underline{\underline{G}}$, and assume that R is a commutative regular ring. If $\pi \in \underline{\underline{FG}}$ then it follows from (1.9) that $K_i(\underline{\underline{M}}_o(R\pi)) = G_i(R\pi)$ ($i = 0, 1$). With this identification the inclusion $\underline{\underline{P}}(R\pi) \subset \underline{\underline{M}}_o(R\pi)$ induces the Cartan homomorphisms.

$$c_i(R\pi)\colon K_i(R\pi) \longrightarrow G_i(R\pi) \qquad (i = 0, 1).$$

These are evidently morphisms of Frobenius modules over G_R: $\underline{\underline{FG}} \longrightarrow \underline{\underline{Frob}}$. Therefore, as above, their kernels, cokernels, etc., are also Frobenius G_R-modules.

(2.3) DEFINITION. Let C be a class of objects in a category $\underline{\underline{C}}$, let G: $\underline{\underline{C}} \longrightarrow \underline{\underline{Frob}}$ be a Frobenius functor, and let $K \in G$-mod. Then we define, for each $\pi \in \underline{\underline{C}}$,

$$K_C(\pi) = \Sigma \operatorname{Im}(j_*),$$

and

$$K^C(\pi) = \bigcap \operatorname{Ker}(j^*),$$

where j ranges over all morphisms j: $\pi' \longrightarrow \pi$ with $\pi' \in C$ and where $K(j) = (j_*, j^*)$.

(2.4) PROPOSITION. _In the notation of_ (2.3) _we have_:

(a) $G(\pi)\, K_C(\pi) + G_C(\pi)\, K(\pi) \subset K_C(\pi)$, _and_ $G(\pi)\, K^C(\pi) + G^C(\pi)\, K(\pi) \subset K^C(\pi)$.

(b) $K_C(\pi)$ _and_ $K^C(\pi)$ _are_ $G(\pi)$-_submodules of_ $K(\pi)$.

(c) $G^C(\pi)\, K_C(\pi) = 0 = G_C(\pi)\, K^C(\pi)$.

(d) _If_ f: K $\longrightarrow$ H _is a morphism of Frobenius_ G-_modules then_ $f(\pi)\,(K_C(\pi)) \subset H_C(\pi)$ _and_ $f(\pi)\,(K^C(\pi)) \subset H^C(\pi)$.

(e) Suppose that for each morphism $j: \pi' \longrightarrow \pi$ in $\underline{\underline{C}}$, $j^*(K_C(\pi)) \subset K_C(\pi')$ (resp., $j_*(K^C(\pi')) \subset K^C(\pi)$). Then K_C (resp., K^C) is a Frobenius G-module.

Proof. (a) Suppose $j: \pi' \longrightarrow \pi$ is a morphism with $\pi' \in C$. If $a \in G(\pi)$ and $b' \in K(\pi')$ then $a \cdot j_*b = j_*(j^*a \cdot b)$ so $G(\pi)\ K_C(\pi) \subset K_C(\pi)$. Similarly, if $a' \in G(\pi')$ and $b \in K(\pi)$ then $j_*a \cdot b = j_*(a \cdot j^*b)$ so $G_C(\pi)\ K(\pi) \subset K_C(\pi)$, thus proving the first part of (a). The maps j^* are G_R-semi-linear in the sense that $j^*(a \cdot b) = j^*a \cdot j^*b$ for $a \in G(\pi)$ and $b \in K(\pi)$. The last part of (a) follows immediately from this.

(b) follows immediately from (a).

(c) With j as above, let $a \in G(\pi)$, $a' \in G(\pi')$, $b \in K(\pi)$, $b' \in K(\pi')$. Then $j_*a' \cdot b = j_*(a' \cdot j^*b)$ and $a \cdot j_*b' = j_*(j^*a \cdot b')$. Therefore if $b \in Ker(j^*)$ and $a \in Ker(j^*)$ we have $j_*a' \cdot b = 0 = a \cdot j_*b'$, thus proving (c).

(d) follows from the fact that f commutes with the j_*'s and the j^*'s.

(e) It follows immediately from the definitions that if $j: \pi' \longrightarrow \pi$ is a morphism in $\underline{\underline{C}}$ then j_* preserves K_C and j^* preserves K^C. Therefore if we assume further that j^* preserves K_C (resp., j_* preserves K^C) then $\pi \longmapsto K_C(\pi)$ (resp., $K^C(\pi)$) becomes a double functor whose value at π is a $G_R(\pi)$-module (by part (b)). The Frobenius reciprocity formulas required for K_C (resp., K^C) to be a G_R-module are satisfied because they are satisfied in K. q.e.d.

(2.5) DEFINITION. Let e be a positive integer. We shall say that a group π has exponent e if every element in π has order dividing e. (I.e. $x^e = 1$ for all $x \in \pi$ if π is multiplicative, or $ex = 0$ for all $x \in \pi$ if π is additive.) The set of exponents, if non empty, is the set of positive integers in some ideal in $\underline{\underline{Z}}$, and we denote the least positive exponent by

$$\exp(\pi)$$

if it exists. If A is an additive group and I is a subgroup we say I _has exponent_ e _in_ A if A/I has exponent e. In case A is a ring and I is a two sided ideal this is equivalent to the condition that the characteristic of A/I divides e, or that $e \cdot 1 \in I$.

(2.6) PROPOSITION. ("Induction and Restriction Principles") _Let_ $G: \underline{\underline{C}} \longrightarrow \underline{\underline{Frob}}$ _be a Frobenius functor. Let_ C _be a class of objects in_ $\underline{\underline{C}}$ _and let_ $\pi \in \underline{\underline{C}}$ _be such that_ $G_C(\pi)$ _has exponent_ e _in_ $G(\pi)$. _Then for all Frobenius_ G-_modules_ K, _the groups_ $K(\pi)/K_C(\pi)$ _and_ $K^C(\pi)$ _have exponent_ e.

Proof. Using (2.4) (a) we have $eK(\pi) = eG(\pi)\ K(\pi) \subset G_C(\pi)\ K(\pi) \subset K_C(\pi)$. Using (2.4) (c) we have $eK^C(\pi) = eG(\pi)\ K^C(\pi) \subset G_C(\pi)\ K^C(\pi) = 0$. q.e.d.

(2.7) COROLLARY. _Keep the notation and assumptions of_ (2.6). _Let_ C_π _denote the set of_ $\pi' \in C$ _for which there is a morphism_ $\pi' \longrightarrow \pi$. _Suppose that_ $K(\pi')$ _is torsion (resp., has exponent_ d) _for all_ $\pi' \in C_\pi$. _Then_ $K(\pi)$ _is torsion (resp., has exponent_ de).

Proof. The hypothesis implies $K_C(\pi)$ is torsion (resp., of exponent d), and (2.7) says $K(\pi)/K_C(\pi)$ has exponent e; the corollary follows immediately from this.

(2.8) COROLLARY. _Keep the notation and assumptions of_ (2.6). _Let_ $f: K \longrightarrow H$ _be a morphism of Frobenius_ G-_modules_. _Assume that_ $K(\pi)$ _is torsion free and that, for all_ $\pi' \in C_\pi$, $\mathrm{Ker}(f(\pi'))$ _is torsion (e.g. that_ $f(\pi')$ _is a monomorphism)_. _Then_ $f(\pi)$ _is a monomorphism_.

Proof. By (2.7), applied to Ker(f), Ker(f(π)) is torsion. Since K(π) is torsion free this implies Ker(f(π)) = 0. q.e.d.

§3. INDUCTION EXPONENTS

Recall from §2 that $\underline{\underline{G}}$ denotes the category whose objects are groups and whose morphisms are monomorphisms j: π´ ⟶ π of finite index (i.e. [π: j(π´)] is finite). Also $\underline{\underline{FG}}$ denotes the full subcategory of finite groups.

We shall fix a class C of objects of $\underline{\underline{G}}$. If R is a commutative ring, and if $\pi \in \underline{\underline{G}}$, we shall write

$$e_C(R, \pi)$$

for the exponent (see (2.5)) of $(G_R)_C(\pi)$ (see (2.3)) in $G_R(\pi)$. Since the latter is a ring and the former is an ideal, $e_C(R, \pi)$ is the least positive integer e (if one exists) such that $e \cdot [R_\pi] \in (G_R)_C(\pi)$. Except when we explicitly assert the existence of $e_C(R, \pi)$ its existence will be assumed. It is called the _induction exponent_ of (R, π) with respect to the class C. Its importance is explained by the following immediate corollary of the "induction and restriction principles" (2.6):

(3.1) PROPOSITION. _Let_ R _be a commutative ring and let_ G_R: $\underline{\underline{G}} \longrightarrow$ _Frob be the Frobenius functor of_ (2.1). _Then for any Frobenius_ G_R_-module_ K (e.g. $K = K_i(\underline{\underline{M}}_0(R\pi))$ _or_ $K = K_i(R\pi)$; i = 0, 1) _and for any_ $\pi \in \underline{\underline{G}}$, $K(\pi)/K_C(\pi)$ _and_ $K^C(\pi)$ _have exponent_ $e_C(R, \pi)$.

The results of this section describe the behavior of $e_C(R, \pi)$ as a function of R and of π.

(3.2) PROPOSITION. _Let_ f: R´ ⟶ R _be a homomorphism of commutative rings, and let_ $\pi \in \underline{\underline{G}}$.

(a) $e_C(R, \pi)$ _divides_ $e_C(R', \pi)$.

(b) *Suppose* $\max(R')$ *is a noetherian space of dimension* $\leq d$ *and that* R *is a projective* R'*-module of (constant) rank* n. *Then* $e_C(R', \pi)$ *divides* $e_C(R, \pi)\cdot n^{d+1}$. *Moreover, if* $n - [R]_{R'}$ *has finite (additive) order* m *in* $K_0(R')$ *then* $e_C(R', \pi)$ *divides* $e_C(R, \pi) \cdot n \cdot m$.

Proof. (a) $f_*: G_{R'} \longrightarrow G_R$ is a morphism of Frobenius functors. Hence $f_*(e_C(R', \pi) \cdot 1) = e_C(R', \pi) \cdot 1 \in (G_R)_C(\pi)$.

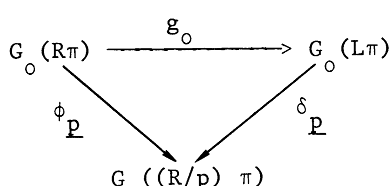
(b). Let $e = e_C(R, \pi)$. We have a restriction homomorphism $f^*: G_R \longrightarrow G_{R'}$, in this case, with commutes with restriction and induction homomorphisms induced from morphisms in $\underline{\underline{G}}$ (see (1.11)). Therefore $f^*(e \cdot 1) = e\, f^*(1) \in (G_{R'})_C\,(\pi)$. The element $1 \in G_R(\pi)$ here is $[R]_{R\pi}$, so $f^*(1) = [f^*R]_{R'\pi}$. Thus $(G_{R'})_C\,(\pi)$ contains $e[R]_{R'} \cdot G_{R'}(\pi)$, where $[R]_{R'}$ is the class of R in $K_0(R)$. It follows from (IX, 4.5) that $[R]_{R'} \cdot K_0(R')$ contains $n^{d+1} \cdot 1$. Moreover, if $m\ ([R]_{R'} - n) = 0$ then it contains $m[R]_{R'} = m\ (n + ([R]_{R'} - n)) = mn$. q.e.d.

(3.3) PROPOSITION. *Let* R *be a Dedekind ring with field of fractions* L, *and let* $\underline{p} \in \max(R)$. *Let* π *be a finite group of order not divisible by* $\operatorname{char}(L)$.

(a) $e_C(R/\underline{p}, \pi)$ *divides* $e_C(L, \pi)$.

(b) $e_C(R, \pi)$ *divides* $e_C(L, \pi)^2$.

Proof. By (1.2) $L\pi$ is semi-simple, so we have Swan's triangle (X, 1.1)



Using (1.9) we can identify this with a triangle

$$\begin{array}{ccc} G_R(\pi) & \longrightarrow & G_L(\pi) \\ & \searrow \quad \swarrow & \\ & G_{R/\underline{p}}(\pi) & \end{array}$$

The top and left side are induced by $R \longrightarrow L$ and $R \longrightarrow R/p$, respectively, so they are morphisms of Frobenius functors (on $\underline{\underline{FG}}$). The right side was deduced from surjectivity of the top, so it is also a morphism of Frobenius functors. Therefore $e(L, \pi) \cdot 1 \in (G_{R/\underline{p}})_C(\pi)$, thus proving (a).

From (IX, 6.9) we have an exact sequence

$$\coprod_{\underline{p} \in \max(R)} G_0((R/\underline{p})\pi) \xrightarrow{f} G_0(R\pi) \xrightarrow{g_0} G_0(L\pi) \longrightarrow 0,$$

where f is induced by the "restrictions" $\underline{\underline{M}}((R/\underline{p})\pi) \subset \underline{\underline{M}}(R\pi)$. In particular they commute with the induction and restriction homomorphisms arising from morphisms in $\underline{\underline{FG}}$. Let $e = e_C(L, \pi)$. Since $G_0(R\pi') \longrightarrow G_0(L\pi')$ is surjective for all $\pi' \subset \pi$ it follows that $(G_R)_C(\pi) \longrightarrow (G_L)_C(\pi)$ is surjective. Therefore we can choose $a \in (G_R)_C(\pi)$ such that $g_0(a) = e \cdot 1$. Thus $e \cdot 1 - a \in \mathrm{Ker}(g_0)$, so we can write $e \cdot 1 - a = f(b)$. According to part (a) we have $e \cdot b \in \coprod (G_{R/\underline{p}})_C(\pi)$ ($\underline{p} \in \max(R)$) and so $f(e \cdot b) = e\, f(b) = e^2 \cdot 1 - ea \in (G_R)_C(\pi)$. Since $ea \in (G_R)_C(\pi)$ we conclude that $e^2 \cdot 1 \in (G_R)_C(\pi)$, thus proving (b). q.e.d.

(3.4) COROLLARY. *For any field* L *and any finite group* π, $e_C(L, \pi)$ *divides* $e_C(\underline{\underline{Q}}, \pi)$. *For any commutative ring* R, $e_C(R, \pi)$ *divides* $e_C(\underline{\underline{Q}}, \pi)^2$.

Proof. If char(L) = 0 then L is a $\underline{\underline{Q}}$-algebra, and this follows from (3.2) (a). Similarly, in characteristic $p > 0$, (3.2) (a) makes it sufficient to prove this for the prime field, $\underline{\underline{Z}}/p\underline{\underline{Z}}$. In this case the assertion follows from (3.3) (a). Similarly, the last assertion follows from (3.3) (b) since R is a $\underline{\underline{Z}}$-algebra.

Next we shall fix R and vary π.

(3.5) PROPOSITION. _Assume that every subgroup of finite index in a group in_ C _is also in_ C. _Let_ $j: \pi' \longrightarrow \pi$ _be a morphism in_ $\underline{\underline{G}}$. _Then for any commutative ring_ R, $j^*((G_R)_C(\pi)) \subset (G_R)_C(\pi')$. _Hence_ $(G_R)_C$ _is a Frobenius_ G_R-_submodule of_ G_R. _Moreover_ $e_C(R, \pi')$ _divides_ $e_C(R, \pi)$.

Proof. Since j^* is a ring homomorphism it is clear that the last assertion follows from the first. The second does also, thanks to (2.4) (e).

Now let $i: \pi'' \longrightarrow \pi$ be a morphism in $\underline{\underline{G}}$ with $\pi'' \in C$. The proposition will follow if we show that $j^*(\mathrm{Im}(i_*)) \subset (G_R)_C(\pi')$. For convenience we shall identify π' and π'' with subgroups of π, so that j and i are inclusions. Given $M \in \underline{\underline{M}}_o(R\pi'')$ it suffices for us to show that $j^*i_*[M] \in (G_R)_C(\pi')$. Now $j^*i_* M = j^*(M \otimes_{R\pi''} R\pi)$, and it follows from the "Mackey Subgroup Theorem" (Curtis-Reiner [1], p. 324) that $j^*(M \otimes_{R\pi''} R\pi) \simeq \amalg_D M(D)$ where D ranges over the double cosets $D = \pi'' x \pi'$ and where M(D) is an induced module w.r.t. the morphisms $j(D): x^{-1}\pi''x \cap \pi' \longrightarrow \pi'$; say $M(D) = j(D)_*N_D$. (Curtis-Reiner assume R is a field, but this is nowhere required in their proof. There are minor differences also in formulation here, because we are dealing with right modules.) Since both π'' and π' have finite index in π it follows that $x^{-1}\pi''x \cap \pi'$ does also, and our hypothesis implies it is in C, being isomorphic to a subgroup of finite index in π''. As an R-module $j(D)_*N$ is a direct sum of copies of N. It is also a direct summand of M so we conclude that $N \in \underline{\underline{M}}_o(R[x^{-1}\pi''x \cap \pi'])$. Thus we have $j^*i_*[M] \in \Sigma_D \mathrm{Im}(j(D)_*) \subset (G_R)_C(\pi')$. q.e.d.

(3.6) PROPOSITION. Assume that if $\sigma \in C$ then every quotient, σ/σ', where σ' is a finite normal subgroup, is also in C. Let π be a group and let π' be a finite normal subgroup of order n (= $[\pi: 1]$). Let R be a commutative ring such that $n \in U(R)$. Then $e_C(R, \pi/\pi')$ divides $e_C(R, \pi)$.

Proof. We have $R\pi' = Re \oplus I$ where $e = n^{-1} \sum_{x \in \pi'} x$ is a central idempotent and I is the augmentation ideal. This is easy to see (use (1.1), for example, to split the exact sequence $0 \longrightarrow I \longrightarrow R\pi' \longrightarrow R_{\pi'} \longrightarrow 0$). This means that $R\pi'$ is a product of two rings, one factor corresponding to the trivial $R\pi'$-module $R_\pi \simeq Re$. Hence, for all $M \in$ mod-$R\pi'$, $M = Me \oplus MI$ is a canonical decomposition as $R\pi'$-module, and it is compatible with tensor products (e.g. $(M \otimes_R N)e = Me \otimes_R Ne$). In particular the functor $\otimes_{R\pi'} R_{\pi'}$ is an exact functor that preserves $\otimes_R$.

Let $J = R\pi \cdot I$. Since π' is normal in π it follows that J is a two sided ideal in $R\pi$, and clearly π' goes to 1 in $R\pi/J$. On the other hand it is clear that $I \subset \mathrm{Ker}(R\pi \longrightarrow R\pi'')$, where $\pi'' = \pi/\pi'$, so it follows that $R\pi'' = R\pi/J$. Now if $M \in$ mod-$R\pi$ we have $M \otimes_{R\pi} R\pi'' = M/MJ = M/M\cdot R\pi \cdot I = M/MI$ $= M \otimes_{R\pi'} R_{\pi'}$. Thus $\otimes_{R\pi} R\pi''$ is exact and preserves $\otimes_R$, thanks to the conclusions in the first paragraph. The latter also make it clear that if M is a finitely generated projective R-module then $M \otimes_{R\pi} R\pi''$ is also, since, as an R-module, it is isomorphic to the direct summand Me of M. Thus we have an exact functor $\otimes_{R\pi} R\pi''$: $\underline{\underline{M}}_0(R\pi) \longrightarrow \underline{\underline{M}}_0(R\pi'')$ which preserves $\otimes_R$, and hence a ring homomorphism $p: G_R(\pi) \longrightarrow G_R(\pi'')$.

Let $j: \sigma \longrightarrow \pi$ be a morphism in $\underline{\underline{G}}$ with $\sigma \in C$. If we can show that $p(\mathrm{Im}(j_*)) \subset (G_R)_C\ (\pi'')$ then it will follow that $p((G_R)_C\ (\pi)) \subset (G_R)_C\ (\pi'')$, and the proposition follows from this.

Let $\sigma' = j^{-1}(\pi')$ (a finite normal subgroup of σ), let $\sigma'' = \sigma/\sigma'$, and let $j'': \sigma'' \longrightarrow \pi''$ be the induced monomorphism (of finite index). Our hypothesis on C implies $\sigma'' \in C$.

Therefore it suffices for us to establish a natural isomorphism

$$(j_*M) \otimes_{R\pi} R\pi'' \simeq j_*''(M \otimes_{R\sigma} R\sigma'')$$

of $R\pi''$-modules, for $M \in$ mod-$R\sigma$. Explicitly, we want $(M \otimes_{R\sigma} R\pi) \otimes_{R\pi} R\pi'' \simeq (M \otimes_{R\sigma} R\sigma'') \otimes_{R\sigma''} R\pi''$. But both sides are isomorphic to $M \otimes_{R\sigma} R\pi''$. q.e.d.

§4. CLASSICAL INDUCTION THEOREMS AND THEIR APPLICATIONS

In this section we shall quote, without proof, some fundamental induction theorems for finite groups. Some applications to representation theory are deduced from them.

If e is an integer ≥ 1 let w_e be a primitive (say complex) e^{th} root of unity, and set $\mathbb{Z}_e = \mathbb{Z}[w_e]$, the ring of algebraic integers in the cyclotomic field $\mathbb{Q}_e = \mathbb{Q}[w_e]$. If R is a commutative ring, and if π is a finite group, we say that R <u>is large enough for</u> π if there is a ring homomorphism $\mathbb{Z}_e \longrightarrow R$, where $e = \exp(\pi)$ (see (2.5)). If R is a field this just means that R contains all the e^{th} roots of unity in its algebraic closure.

Let p be a prime. A finite group H is called p-<u>hyperelementary</u> if H is a semi-direct product, $H = N \times_{s-d} P$, where N is a cyclic normal subgroup of order prime to p, and where P is a p-group. H operates on N by conjugation and we shall write

$$A_H (= A_H(N)) \subset \text{Aut}(N)$$

for the image of H (or of P) under this action. In case $A_H = \{1\}$, i.e. if the semi-direct product is direct, we call H p-<u>elementary</u>.

We call a finite group H <u>hyperelementary</u> (resp., <u>elementary</u>) if it is p-hyperelementary (resp., p-elementary) for some prime p. Unless H is abelian p is then characterized as that prime for which the Sylow p-subgroup is not an abelian normal subgroup of H. Moreover, in the ambiguous case when H is abelian, $A_H = \{1\}$ for any choice of p, so A_H is essentially intrinsic.

Let L be a field of characteristic zero, and let $H = N \times_{s-d} P$ be a p-hyperelementary group, as above. Choose some isomorphism, j, of N with the group, jN, of $[N: 1]^{th}$ roots of unity in some algebraic closure of L. Then L(jN)/L is a galois extension whose galois group is determined by its action on jN. Using j to pull this back gives us an isomorphism of Gal(L(jN)/L) with a subgroup

$$A_L(N) \subset \mathrm{Aut}(N).$$

Changing j has the effect of conjugating $A_L(N)$ by an element of the abelian group Aut(N) ($\simeq U(\underline{\underline{Z}}/[N: 1]\underline{\underline{Z}})$). Therefore $A_L(N)$ depends, indeed, only on L and N. We shall call H a p-L-*elementary group* if

$$A_H(N) \subset A_L(N).$$

A group is called L-*elementary* if it is p-L-elementary for some prime p.

For example, *if* L *is large enough for* π *then every* L-*elementary subgroup of* π *is elementary*. For in this case, if $H \subset \pi$ then L contains the $[N: 1]^{th}$ roots of unity, so $A_L(N) = \{1\}$. At the other extreme we have: *Every hyperelementary group is* $\underline{\underline{Q}}$-*hyperelementary*. This follows because $A_{\underline{\underline{Q}}}(N) = \mathrm{Aut}(N)$ for any cyclic group N. The relevance here of these notions is explained by the following beautiful theorem.

(4.1) THEOREM (Witt, Berman). *Let* L *be a number field and let* (L-elem) *denote the class of finite* L-*elementary groups. Then for any finite group* π,

$$e_{(L\text{-elem})}(L, \pi) = 1.$$

This is proved in Curtis-Reiner [1], Theorem (42.3).

For a given finite group π the only L-elementary groups that intervene in determining $e_{(L\text{-elem})}(L, \pi)$ are the L-elementary subgroups of π. Using this we can prove:

(4.2) COROLLARY. _Let_ C _be a class of groups such that any subgroup of a group in_ C _also belongs to_ C. _Let_ R _be the ring of algebraic integers_ (= _integral closure of_ $\underline{\underline{Z}}$) _in a number field_ L. _Then for any commutative_ R-_algebra_ A, _and for any finite group_ π,

$$e_C(A, \pi) = \ell.c.m.\ \{e_C(A, H)\},$$

where H _ranges over the_ L-_elementary subgroups of_ π. _In particular_ $e_{(L\text{-elem})}(A, \pi) = 1$.

Proof. Let d denote the ℓ.c.m. above. It follows from (3.5) that d divides $e_C(A, \pi)$. To show the opposite divisibility we introduce the Frobenius G_A-module, $\pi' \longmapsto K(\pi') = G_A(\pi')/(G_A)_C(\pi')$. That this is, in fact, a G_A-module follows also from (3.5). By definition of d, K(H) has exponent d for all L-elementary subgroups H of π. It follows therefore from (2.7) that $K(\pi)$ has exponent $e_{(L\text{-elem})}(A, \pi) \cdot d$. Thus the corollary will be proved if we show that $e_{(L\text{-elem})}(A, \pi) = 1$. According to (3.2) (a), $e_{(L\text{-elem})}(A, \pi)$ divides $e_{(L\text{-elem})}(R, \pi)$. According to (3.3) (b), $e_{(L\text{-elem})}(R, \pi)$ divides $e_{(L\text{-elem})}(L, \pi)^2$. According to (4.1), $e_{(L\text{-elem})}(L, \pi) = 1$, so we conclude, as claimed, that $e_{(1\text{-elem})}(A, \pi) = 1$.

(4.3) COROLLARY. _Let_ C _be as in_ (4.2), _let_ π _be a finite group, and let_ A _be a commutative ring which is large enough for_ π. _Then_

$$e_C(A, \pi) = \ell.c.m.\ \{e_C(A, E)\},$$

where E _ranges over all elementary subgroups of_ π. _In_ *particular* $e_{(elem)}(A, \pi) = 1$.

Proof. We can take $R = \mathbb{Z}_e$ and $L = \mathbb{Q}_e$ (where e = exp (π)) in the corollary above. Then an L-elementary subgroup of π is elementary. q.e.d.

The next few results give applications of (4.1) to representation theory.

(4.4) THEOREM (Brauer). *Let* π *be a finite group of order* n. *Let* L *be a field whose characteristic does not divide* n, *and which is large enough for* π. *Then* $L\pi$ *is split (i.e. is a product of full matrix algebras over* L). *Equivalently, if* f: L ⟶ L´ *is any field extension, and if* M *is a simple* $L\pi$*-module, then* $M \otimes_L L'$ *is a simple* $L'\pi$*-module*.

Proof. The assertion is clearly equivalent to saying that $f_*(\pi): G_L(\pi) \longrightarrow G_{L'}(\pi)$ is an isomorphism. It is always a monomorphism (see (IX, 2.8)). Now f_* is a morphism of Frobenius G_L-modules, say with cokernel K. Since, by (4.3), $e_{(elem)}(L, \pi) = 1$ it suffices, by (2.7), to show that K(H) = 0 for all elementary subgroups $H = N \times P$ of π. Since H is nilpotent (cf. (1.3) (b)) it follows from a well known (and relatively easy) theorem that every simple L´H-module is deduced from a one dimensional L´H´-module for some subgroup H´ of H (cf. Curtis-Reiner [1], Theorem (52.1).) But since L contains all e^{th} roots of unity in L´ (e = exp (π)) we see that any such module is defined over L. Thus K(H) = 0. q.e.d.

(4.5) THEOREM (Brauer). *Let* L *and* π *be as in* (4.4). *Let* R *be a Dedekind ring with field of fractions* L, *and let* $k = R/\underline{m}$ *for some* $\underline{m} \in \max(R)$. *Consider the commutative diagram*

$$\begin{array}{ccccc}
K_o(R\pi) & \xrightarrow{c_o(R\pi)} & G_R(\pi) & \xrightarrow{g_o(\pi)} & G_L(\pi) \\
 & \searrow \psi_{\underline{m}}(\pi) & & \searrow \phi_{\underline{m}}(\pi) \quad \swarrow \delta_{\underline{m}}(\pi) & \\
 & K_o(k\pi) & \xrightarrow[c_o(k\pi)]{} & G_k(\pi) &
\end{array}$$

<u>where the</u> c_o<u>'s are the Cartan homomorphisms, and where the triangle is Swan's triangle</u> (X, 1.1).

(a) $\delta_{\underline{m}}(\pi)$ <u>is surjective</u>.

(b) $c_o(k\pi)$ <u>is injective and its cokernel has exponent</u> $[\pi_p : 1]$, <u>where</u> $p = \mathrm{char}(k)$ <u>and</u> π_p <u>is a Sylow p-subgroup of</u> π.

<u>Proof</u>. Since L is large enough for π so also is R. (A homomorphism $\underline{\underline{Z}}_e \longrightarrow L$ must land in the integrally closed ring R.) Since all arrows in the diagram are morphisms of Frobenius G_R-modules (cf. proof of (3.3) for the case of $\delta_{\underline{m}}$), and since $e_{(elem)}(R, \pi) = 1$(see (4.3)), it suffices to treat the case when π is elementary. In this case $\pi = \pi_p \times \pi'$ where π' has order prime to p. According to (4.4) $L\pi$ is split and $k\pi'$ is split. According to (1.4) $k\pi_p$ is a local ring, with residue class field k ($= k_{\pi_p}$). It follows easily from these observations that the decompositions $L\pi = L\pi_p \otimes_L L\pi'$ and $k\pi = k\pi_p \otimes_k k\pi'$ induce corresponding decompositions $G_L(\pi) = G_L(\pi_p) \otimes G_L(\pi')$, $G_k(\pi) = G_k(\pi_p) \otimes G_k(\pi')$ and $K_o(k\pi) = K_o(k\pi_p) \otimes K_o(k\pi')$. Therefore both $\delta_{\underline{m}}$ and c_o decompose into a tensor product, making it sufficient to prove the theorem separately for π_p and for π'.

<u>Case 1; $\pi = \pi_p$</u>. $k\pi$ is local so $c_o(k\pi)$ is represented by the one-by-one matrix, $(\mathrm{length}_{k\pi}(k\pi)) = ([\pi : 1])$, thus proving (b). Since $G_k(\pi) \simeq \underline{\underline{Z}}$ the ring homomorphism $\delta_{\underline{m}}$ must

be surjective.

Case 2; $\pi = \pi'$. Let R' and L' denote the $\underline{m}$-adic completions of R and L, respectively. ($R' = \varprojlim_n R/\underline{m}^n$, and L' is the field of fractions of the DVR R', with maximal ideal $\underline{m}' = \underline{m}R'$.) We still have $k = R'/\underline{m}'$. Moreover, since $L\pi$ is split, $G_L(\pi) \longrightarrow G_{L'}(\pi)$ is an isomorphism. Therefore, we can replace (R, L, k) by (R', L', k) without essentially changing the questions at hand, so we shall now assume that R is $\underline{m}$-adically complete. It follows that $R\pi$ is $\underline{m}R\pi$-adically complete, so (IX, 1.3 (0)) implies that $\psi_{\underline{m}}$ is an isomorphism.

Since $n = [\pi: 1]$ is prime to $p = \text{char}(k)$ it follows that $n \in U(R)$, and hence, by (1.2), $R\pi$ is a maximal order. According to (III, 8.7), therefore, $R\pi$ is regular. Moreover $n \in U(k)$ so $k\pi$ is semi-simple. It follows that $c_o(R\pi)$ and $c_o(k\pi)$ are isomorphisms, and hence all arrows in the left hand parallelogram above are isomorphisms. Consequently $\delta_{\underline{m}}$ must be surjective. (In fact, since g_o is surjective, both g_o and $\delta_{\underline{m}}$ are isomorphisms in this case.) q.e.d.

(4.5) COROLLARY. Let F be a field of characteristic $p > 0$. Let π be a finite group, and let π_p be a Sylow p-subgroup. Then $c_o(F\pi)$ is a monomorphism, and its cokernel has exponent $[\pi_p : 1]$.

Proof. Let F_o be the prime field of F, and let F' be an algebraic closure of F. Then $F\pi = F_o\pi \otimes_{F_o} F$, so $F\pi$ is "basically commutative" (see (IX, 2.8)), and therefore $K_o(F\pi) \longrightarrow K_o(F'\pi)$ and $G_o(F\pi) \longrightarrow G_o(F'\pi)$ are split monomorphisms. Thus we deduce inclusions $\text{Ker}(c_o(F\pi)) \subset \text{Ker}(c_o(F'\pi))$ and $\text{Coker}(c_o(F\pi)) \subset \text{Coker}(c_o(F'\pi))$, which make it sufficient to prove the corollary for F'. Let F_1 be the subfield of F' generated by all n^{th} roots of unity, for a large enough n so that $K_o(F_1\pi) \longrightarrow K_o(F'\pi)$ and $G_o(F_1\pi) \longrightarrow G_o(F'\pi)$ are isomorphisms. Then F_1 is a residue class field of $\underline{\underline{Z}}_n$, and $\underline{\underline{Q}}_n$ is large enough for π if we choose n divisible by, say, $[\pi: 1]$. We can apply (4.5) (b), therefore, to conclude that $c_o(F_1\pi)$ is a monomorphism with cokernel of exponent $[\pi_p: 1]$. q.e.d.

Remark. Let R be a Dedekind ring with field of fractions L, and let π be a finite group of order not divisible by char(L). Then $R\pi$ is an R-order in the semi-simple L-algebra $L\pi$. If $k = R/\underline{m}$, $\underline{m} \in \max(R)$ then $c_o(k\pi)$ is a monomorphism. For $k\pi$ is semi-simple if char(k) = 0, and otherwise it follows from (4.5). Thus $R\pi$ satisfies the "Cartan condition" of (X, 1.3), (cf. also (X, 1.8)).

Now we return to questions about induction exponents. In order to make the type of information we are seeking accessible to the methods of commutative algebra it is of interest to have induction theorems for, say, the class (abel) of abelian groups. In fact there is already a reasonably effective theorem of Artin for the class (cyclic) of cyclic groups. This theorem will now be stated in the very precise form recently proved by Lam [2]. Reference will be made in the theorem to the following groups, $(Q)_n$, $(D)_n$, and $(SD)_n$, defined by generators a, b, and relations:

$$(Q)_n:\ a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ bab^{-1} = a^{-1}$$

$$(D)_n:\ a^{2^n} = 1,\ b^2 = 1,\ bab^{-1} = a^{-1}$$

$$(SD)_n:\ a^{2^n} = 1,\ b^2 = 1,\ bab^{-1} = a^{-1+2^{n-1}}.$$

(4.6) THEOREM ("Artin-Lam cyclic exponent theorem"). *Let p be a prime. For a finite group π write*

$$e(\pi) = e_{(\text{cyclic})}(\underline{\underline{Q}}, \pi),$$

$e_p(\pi)$ = *the largest p^{th} power dividing $e(\pi)$, and*

π_p = *a Sylow p-subgroup of π.*

(a) $(G_{\underline{Q}})_{(\text{cyclic})}(\pi)$ _is generated additively by all_ $j_*[\underline{\underline{Q}}_{\pi'}]$ _where_ $j\colon \pi' \subset \pi$ _ranges over all cyclic subgroups of_ π. _Moreover_, $e(\pi)$ _divides_ $[\pi: 1]$ _and_ $e(\pi) = 1 \Longleftrightarrow \pi$ _is cyclic_.

(b) _Let_ $[\pi_p: 1] = p^{n+1}$. _Then_ $e(\pi_p)$ _divides_ $e_p(\pi)$ _and_,

$$e(\pi_p) = \begin{cases} 1 & \text{if } \pi_p \text{ is cyclic} \\ 2 & \text{if } p = 2 \text{ and } \pi_p \simeq (Q)_n, (D)_n \text{ or } (SD)_n \\ p^n & \text{otherwise.} \end{cases}$$

(c) $e_p(\pi) = \sup\{e_p(H)\}$ _where_ H _ranges over the p-hyperelementary subgroups of_ π. _If_ $\pi = N \underset{s-d}{\times} \pi_p$ _is p-hyperelementary then_

$$e_p(\pi) = \sup(e(\pi_p), [A_\pi(N): 1])$$

provided, if $p = 2$, π_p _is not one of the exceptional groups_ $(Q)_n$, $(D)_n$, _or_ $(SD)_n$.

Recall that $A_\pi(N)$ above denotes the image of π_p in Aut(N) under the representation by inner automorphisms. This theorem completely determines $e_p(\pi)$ except when $p = 2$ and π_p is one of the exceptional groups. To illustrate, suppose π_p is not exceptional. If there is a cyclic group N of order prime to p in π which is normalized by π_p, and on which π_p acts faithfully by inner automorphisms, then $e_p(\pi) = [\pi_p: 1]$. (Note that π_p must be abelian in this case, because Aut(N) is abelian.) In the contrary case we have $e_p(\pi) = e(\pi_p)$, and the latter is determined by part (b).

Artin's induction theorem corresponds to part (a), and is not difficult to prove. Lam's proof of parts (b) and (c) is rather long, and it invokes a variety of techniques.

(4.7) THEOREM. _Let_ C _be a class of finite nilpotent groups containing all cyclic groups. Assume that a subgroup_ *of a group in* C *is also in* C, *and that a product of two groups in* C *which have relatively prime orders is again in* C. *Let* π *be a finite group and let* L *be a field whose characteristic does not divide* $[\pi: 1]$.

(a) $e_C(L, \pi)$ *divides* $e_{(cyclic)}(L, \pi)$, *and hence it divides* $[\pi: 1]$.

(b) *If* L *is large enough for* π *then*

$$e_C(L, \pi) = \prod_p e_C(L, \pi_p),$$

where p *ranges over primes dividing* $[\pi: 1]$, *and* π_p *is a Sylow* p-*subgroup of* π.

Proof. (a) Clearly $e_C(L, \pi)$ divides $e_{(cyclic)}(L, \pi)$, and $e_{(cyclic)}(L, \pi)$ divides $e_{(cyclic)}(\mathbb{Q}, \pi)$, by (3.4). Now apply (4.6) (a).

(b) By (3.5), $e_C(L, \pi_p)$ divides $e_C(L, \pi)$, and part (a) implies $e_C(L, \pi_p)$ is a power of p. Hence $\prod_p e_C(L, \pi_p)$ divides $e_C(L, \pi)$. On the other hand (4.3) implies $e_C(L, \pi) = \ell.c.m. \{e_C(L, E)\}$ where E ranges over the elementary subgroups of π. Let E_p be a Sylow p-subgroup of E; we may assume $E_p \subset \pi_p$, and hence, again by (3.5), $e_C(L, E_p)$ divides $e_C(L, \pi_p)$. Therefore, if we prove the theorem for E then it will follow that $e_C(L, E)$ divides $\prod_p e_C(L, \pi_p)$, and the theorem will be proved. Since E is elementary it is the direct product, $E = \prod E_p$, of its Sylow subgroups. Moreover $e_C(L, E_p)$ is prime to $e_C(L, E_q)$ if $p \neq q$. Thus the theorem for E follows from the next proposition.

(4.8) PROPOSITION. *Let* C, L, *and* π *be as in* (4.7) (b), *and write* $K(\pi') = G_L(\pi')/(G_L)_C(\pi')$ *for any group* π'. *Suppose* $\pi = \pi_1 \times \pi_2$ *is the direct product of two groups of relatively prime orders. Then there is a natural isomorphism*

$$K(\pi_1 \times \pi_2) \simeq (K(\pi_1) \otimes G_L(\pi_2)) \oplus (G_L(\pi_1) \otimes K(\pi_2)).$$

Hence $e_C(L, \pi_1 \times \pi_2) = e_C(L, \pi_1) \cdot e_C(L, \pi_2)$.

Proof. For any subgroup $j: \pi' \subset \pi$ it follows from (1.2) and (4.4) that $L\pi'$ is a split semi-simple algebra. If $\pi' \in C$ then π' is nilpotent and is hence the direct product of its Sylow subgroups. Hence $\pi' = \pi_1' \times \pi_2'$ where $\pi_i' = \pi' \cap \pi_i$ $(i = 1, 2)$, because π_1 and π_2 have relatively prime orders. If $j_i: \pi_i' \subset \pi_i$ then, since all the group algebras are split, we can identify $j_*: G_L(\pi') \longrightarrow G_L(\pi)$ with $j_{1*} \otimes j_{2*}: G_L(\pi_1') \otimes G_L(\pi_2') \longrightarrow G_L(\pi_1) \otimes G_L(\pi_2)$. Now as π' ranges over all C-subgroups of π, π_1' and π_2' range independently over all C-subgroups of π_1 and of π_2, respectively, thanks to the hypotheses made on C. From this it follows that $(G_L)_C(\pi_1 \times \pi_2) = (G_L)_C(\pi_1) \otimes (G_L)_C(\pi_2)$. (We can identify the latter with its image in $G_L(\pi_1) \otimes G_L(\pi_2)$ since all these abelian groups are free.) By part (a) of (4.7) $K(\pi_1)$ and $K(\pi_2)$ have relatively prime order. Hence the first assertion of the proposition follows from: Let $M_o \subset M$ and $M_o' \subset M'$ be free abelian groups of finite rank such that M/M_o and M'/M_o' are finite and of relatively prime orders. Then $(M \otimes M')/(M_o \otimes M_o') \simeq ((M/M_o) \otimes N') \oplus (M \otimes (M'/M_o'))$. This can be seen, for example, by choosing bases for M and M_o so that M_o is the image of an endomorphism of M represented by a diagonal matrix, and similarly for $M_o' \subset M'$. We leave the details as an excercise.

Since $e_C(L, \pi) = \exp(K(\pi))$ the last assertion of the proposition follows from the first. q.e.d.

§5. APPLICATIONS TO $K_o(R\pi)$ AND $G_o(R\pi)$.

The applications we shall describe here are all due to Swan.

(5.1) PROPOSITION. Let R be a commutative semi-local ring and let $L = S^{-1}R$, where S is a multiplicative set of non divisors of zero. Let π be a finite group. Then $K_0(R\pi) \longrightarrow K_0(L\pi)$ is a monomorphism. Moreover, if $P, P' \in \underline{\underline{P}}(R\pi)$, then $P \otimes_R L \simeq P' \otimes_R L \Rightarrow P \simeq P'$.

Proof. The first assertion implies the last. For if $P \otimes_R L \simeq P' \otimes_R L$ then the first assertion implies $[P] = [P']$ in $K_0(R\pi)$. If $\underline{a}$ = rad R then $(R/\underline{a})\pi$ is an Artin ring. Since $[P/P\underline{a}] = [P'/P'\underline{a}]$ in $K_0((R/\underline{a})\pi)$, therefore, we conclude that $P/P\underline{a} \simeq P'/P'\underline{a}$. But $(R\pi)\underline{a} \subset \mathrm{rad}(R\pi)$ so (III, 2.12) implies $P \simeq P'$.

Write $K(\pi) = \mathrm{Ker}(K_0(R\pi) \longrightarrow K_0(L\pi))$. Then K is a Frobenius G_R-module. According to (IX, 1.4) $K_0(R\pi)$ is free abelian. Therefore we need only show that $K(\pi)$ is torsion. Since $e_{(\mathrm{cyclic})}(R, \pi)$ divides $e_{(\mathrm{cyclic})}(\underline{\underline{Q}}, \pi)^2$ (see (3.4)), it divides $[\pi: 1]^2$ (see (4.6) (a)). Therefore (see (2.7)) it suffices to show that $K(\pi)$ is torsion when π is cyclic. But in that case $R\pi$ is a commutative semi-local ring, so $K_0(R\pi) \longrightarrow H_0(R\pi)$ is an isomorphism. Since S consists of non divisors of zero for R, and hence also for $R\pi$, $H_0(R\pi) \longrightarrow H_0(L\pi)$ is a monomorphism (IX, 3.1). Now the commutative square

$$\begin{array}{ccc} K_0(R\pi) & \longrightarrow & K_0(L\pi) \\ \downarrow & & \downarrow \\ H_0(R\pi) & \longrightarrow & H_0(L\pi) \end{array}$$

shows that the top arrow is injective. q.e.d.

(5.2) THEOREM. Let R be an integral domain of characteristic zero. Let π be a finite group such that no prime divisor of $[\pi: 1]$ is invertible in R. Then if $P \in \underline{\underline{P}}(R\pi)$, the $R_{\underline{p}}\pi$-module $P_{\underline{p}}$ is free for all $\underline{p} \in \mathrm{spec}(R)$.

Proof. According to (5.1) it suffices to show that $P \otimes_R L$ is $L\pi$-free, where L is the field of fractions of R. Let $p_1,\dots,p_n$ be the prime divisors of $[\pi: 1]$. For each i let $\underline{p}_i$ be a minimal element among the prime ideals of R containing p_i. Let $S = R - \cup_i \underline{p}_i$ and let $R' = S^{-1}R$. If we prove the theorem for R' it will follow that $(S^{-1}P) \otimes_{R'} L = P \otimes_R L$ is $L\pi$-free. Hence it suffices to prove the theorem when R is semi-local and R/p_iR has a unique prime ideal for each i.

Let T be the trivial Frobenius G_R-module: $T(\pi) = \underline{\underline{Z}}$ for all π. If π' has finite index in π then

$$T(\pi') \underset{\text{id.}}{\overset{[\pi:\ \pi']}{\rightleftarrows}} T(\pi).$$

There is a canonical morphism $r: K_o \longrightarrow T$ of Frobenius G_R-modules on the category of subgroups of π defined by $P \longmapsto [P: R]$ for $P \in \underline{\underline{P}}(R\pi')$. We claim:

(i) $r(\pi')$ is a monomorphism for all $\pi' \subset \pi$.

(ii) If $P \in \underline{\underline{P}}(R\pi')$ then $[\pi': 1]$ divides $r[P] = [P: R]$.

If we know these two facts then it follows that $[P] - [(R\pi)^n] \in \operatorname{Ker}(r(\pi)) = 0$ for some n and hence, as in (5.1), $P \simeq (R\pi)^n$.

Proof of (ii). Let π_i be a Sylow p_i-subgroup of π'. It suffices to show that $[\pi_i: 1]$ divides $[P: R]$ for each i. By restriction of P to $\underline{\underline{P}}(R\pi_i)$ it suffices to establish (ii) when π' is a p_i-group. Since $[P: R] = [P/Pp_i: R/Rp_i]$ it suffices to show that P/Pp_i is a free $(R/Rp_i)\pi'$-module. But (1.4) implies the latter is a local ring so the contention follows from (III, 2.13).

Proof of (i). Since $K_o(R\pi)$ is torsion free (recall R is now semi-local) it suffices to show that $\operatorname{Ker}(r(\pi'))$ is torsion. As in (5.1) it suffices to show this when π' is cyclic, using the Artin induction theorem ((4.6) (a)). Since $R\pi'$ is then commutative and semi-local, $K_0(R\pi') \longrightarrow H_0(R\pi')$ is an isomorphism, so we need only show that $H_0(R\pi')$ has rank one, i.e. that there are no idempotents e in $R\pi$ except e = 0 or e = 1. But this is so, for otherwise $[e(R\pi'): R]$ could not be divisible by $[\pi': 1]$, contradicting (ii) above. q.e.d.

(5.3) THEOREM. _Let_ R _be a semi-local Dedekind ring with field of fractions_ L. _Then, for any finite group_ π,

$$g_0(\pi): G_R(\pi) \longrightarrow G_L(\pi)$$

is an isomorphism.

Proof. g_0 is a morphism of Frobenius G_R-modules and it is surjective because $R \longrightarrow L$ is a localization. Let $K = \text{Ker}(g_0)$. Since $e_{(\text{hypelem})}(R, \pi) = 1$ it suffices to show that $K(\pi) = 0$ when π is a hyperelementary group.

$K(\pi)$ is generated by classes, $[M] \in G_R(\pi) = G_0(R\pi)$ of R-torsion $R\pi$-modules. These have finite length so we can further restrict attention to simple modules M, by "devissage". In this case M is a simple $k\pi$-module where $k = R/\underline{m}$ for some $\underline{m} \in \max(R)$. We wish to show that $[M] = 0$ in $G_R(\pi)$. If π does not act faithfully on M then M is a $k\pi''$-module for some proper quotient π'' of π. By induction on order we can assume $[M]_{\pi''} = 0$ in $G_R(\pi'')$. But $[M]_\pi$ is the image of $[M]_{\pi''}$ under the restriction homomorphism $G_R(\pi'') \longrightarrow G_R(\pi'') \longrightarrow G_R(\pi)$.

We can therefore assume π acts faithfully on M. According to (5.1) $k_0(\pi): K_0(R\pi) \longrightarrow K_0(L\pi)$ is a monomorphism. Hence, if $\text{hd}_{R\pi}(M) < \infty$ we have $[M]_{\underline{\underline{H}}(R\pi)} = 0$ in $K_0(R\pi)$. But [M] is the image of $[M]_{\underline{\underline{H}}(R\pi)}$ under the Cartan homomorphism $K_0(R\pi) \longrightarrow G_0(R\pi)$. Therefore it suffices to show that $\text{hd}_{R\pi}(M) < \infty$. Since $\text{hd}_{R\pi}(k\pi) < \infty$ ($0 \longrightarrow \underline{m}R\pi \longrightarrow R\pi \longrightarrow k\pi \longrightarrow 0$ is a finite $R\pi$-projective resolution) it further suffices to show that $M \in \underline{\underline{P}}(k\pi)$. Therefore the theorem follows from:

(5.4) PROPOSITION. Let k be a field and let $\pi = N \times_{s-d} P$ be a p-hypelementary group. Let M be a simple $k\pi$-module on which π acts faithfully. Then $M \in \underline{\underline{P}}(k\pi)$.

Proof. Let $q = \mathrm{char}(k)$. If q does not divide $[\pi: 1]$ then $k\pi$ is semi-simple, by (1.2). Otherwise let π_q be a Sylow q-subgroup of π; then either $\pi_q \subset N$ or else $q = p$ and $\pi_q = P$.

Suppose $\pi_q \subset N$. Then π_q is an (abelian) normal subgroup of π. Let I be the augmentation ideal in $k\pi_q$. Then I is nilpotent (see (1.4)) and $J = I(R\pi) = (R\pi)I = \mathrm{Ker}(R\pi \longrightarrow R(\pi/\pi_q))$. It follows that J is nilpotent, so $J \subset \mathrm{rad}\, R\pi$, and hence $MJ = 0$ since M is simple. But then π_q acts trivially on M, contrary to assumption. Hence $\mathrm{char}(k) = p$, and $[N: 1]$ is prime to p. The argument above shows further that π has no normal p-subgroups, so the action of P on N by conjugation is faithful. (The kernel of that action is normal in P and centralized by N). Moreover kN is commutative and semi-simple.

Case 1. k contains the $[N: 1]^{th}$ roots of unity (i.e. k is large enough for N).

Then M is a direct sum of one dimensional kN-submodules. Let $M_o = ek$ be one of them. Then the induced $k\pi$-homomorphism $M_o \otimes_{kN} k\pi \longrightarrow M$ is surjective because M is simple. If we show that this is an isomorphism then, since $M_o \in \underline{\underline{P}}(kN)$, it will follow that $M \in \underline{\underline{P}}(R\pi)$.

If $N' \subset N$ acts trivially on M_o then N' is normal in N (because N is cyclic) and $M_o \otimes_{kN} k\pi = M_o \otimes_{k(N/N')} k(\pi/N')$, contrary to our assumption that the action of π is faithful.

If $x \in N$ then $ex = eh(x)$ for some character $h: N \longrightarrow U(k)$, which we have just seen to be a monomorphism. If $y \in P$ then $eyx = eyxy^{-1}y = eh(yxy^{-1})y = ey\, hy(x)$, where $hy(x) = h(yxy^{-1})$. Thus eyk is a kN-submodule of M with character hy. Moreover we have $h_{y_1y_2} = (h_{y_2})_{y_1}$ for $y_1, y_2 \in P$, clearly.

Therefore $h_{y_1} = h_{y_2} \Rightarrow (h_{y_1})_{y_2^{-1}} = h_1$, i.e. $h_{y_2^{-1}y_1} = h$, i.e. $y_2^{-1}y_1$ centralizes N. But P acts faithfully on N, as noted above, so $y_1 \neq y_2 \Rightarrow h_{y_1} \neq h_{y_2} \Rightarrow ey_1k \simeq ey_2k$ as kN-modules. Therefore the sum $M = \Sigma_{y \in P}$ eyk is direct, so $[M: k] = [P: 1] = [M_o \otimes_{kN} k\pi: k]$. Hence $M_o \otimes_{kN} k\pi \simeq M$. q.e.d.

General case. Let k′ be an extension of k containing all $[N: 1]^{th}$ roots of unity. If $M \otimes_k k' \in \underline{\underline{P}}(k'\pi)$ then $M \in \underline{\underline{P}}(k\pi)$ because $M \otimes_k k'$ is a direct sum of [k′: k] copies of M, qua $k\pi$-module. The same observation shows that the factors of a Jordan-Holder series of $M \otimes_k k'$ over $k\pi'$ must be π-faithful, since this is even true of $M \otimes_k k'$ as a $k\pi$-module. Therefore, case 1 implies the Jordan-Holder factors of $M \otimes_k k'$ are $k'\pi$-projective, and hence $M \otimes_k k'$ is the direct sum of its Jordan-Holder factors, and is $k'\pi$-projective. q.e.d.

For the remaining results we fix a Dedekind ring R with field of fractions $L = S^{-1}R$ ($S = R - \{0\}$) and a finite group π such that char(L) does not divide $[\pi: 1]$. Then, as in (X, §1, diagram (1)) we have a commutative diagram

(1)

$$\begin{array}{ccccccccccc}
K_1(R\pi) & \xrightarrow{k_1(R\pi)} & K_1(L\pi) & \longrightarrow & K_o(R\pi,\ S) & \longrightarrow & K_o(R\pi) & \xrightarrow{k_o(R\pi)} & K_o(L\pi) & & \\
 & & {\scriptstyle(\simeq)}\big\downarrow {\scriptstyle c_1(L\pi)} & & \big\downarrow {\scriptstyle c_o(R\pi,\ S)} & & \big\downarrow {\scriptstyle c_o(R\pi)} & & {\scriptstyle(\simeq)}\big\downarrow {\scriptstyle c_o(L\pi)} & & \\
 & & G_1(L\pi) & \longrightarrow & G_o(R\pi,\ S) & \longrightarrow & G_o(R\pi) & \xrightarrow[g_o(R\pi)]{} & G_o(L\pi) & \longrightarrow & 0 \\
 & & & & \parallel & & & & & & \\
 & & & & \left(\prod_{\underline{p} \,\varepsilon\, \max(R)} G_o((R/\underline{p})\pi)\right) & & & & & &
\end{array}$$

(5.5) THEOREM. In the diagram (1) above $c_o(R\pi, S)$ is an epimorphism.

Proof. It follows from (1.2) that $R\pi$ is contained in a maximal R-order B in $L\pi$ and that $nB \subset R\pi$, where $n = [\pi: 1]$. If char(L) = p > 0 then $n \in U(R)$ so $R\pi = B$ and is regular. Hence $c_o(R\pi, S)$ is an isomorphism.

Otherwise let $T = R - (\cup \underline{p})$ where the union is over all primes containing nR. It follows from (X, 1.9) that T is regular for $R\pi$, and hence that $K_o(R\pi, T) \longrightarrow G_o(R\pi, T)$ is an isomorphism.

For each $\underline{p} \in \max(R)$ let $\underline{\underline{M}}_{\underline{p}}(R\pi)$ be the category of $M \in \underline{\underline{M}}(R\pi)$ which are annihilated by a power of $\underline{p}$. Let $\underline{\underline{H}}_{\underline{p}}(R\pi)$ be the full subcategory of $M \in \underline{\underline{M}}_{\underline{p}}(R\pi)$ which have finite homological dimension. Then since any torsion module over $R\pi$ is canonically the direct sum of its components in each $\underline{\underline{M}}_{\underline{p}}(R\pi)$ we see that $c_o(R\pi, S)$ is the direct sum of the homomorphisms $K_o(\underline{\underline{H}}_{\underline{p}}(R\pi)) \longrightarrow K_o(\underline{\underline{M}}_{\underline{p}}(R\pi))$. If $\underline{p} \cap T = \phi$ then $\underline{\underline{M}}_{\underline{p}}(R\pi) = \underline{\underline{M}}_{T_{\underline{p}}^{-1}}(T^{-1}R\pi)$, and similarly for $\underline{\underline{H}}_{\underline{p}}$. Hence, by separating the $\underline{p}$'s which meet and do not meet T, respectively, we obtain a split short exact sequence of homomorphisms,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_o(R\pi, T) & \longrightarrow & K_o(R\pi, S) & \longrightarrow & K_o(T^{-1}R\pi, S) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c_o(R\pi, T)} & & \downarrow{\scriptstyle c_o(R\pi, S)} & & \downarrow{\scriptstyle c_o(T^{-1}R\pi, S)} & & \\
0 & \longrightarrow & G_o(R\pi, T) & \longrightarrow & G_o(R\pi, S) & \longrightarrow & G_o(T^{-1}R\pi, S) & \longrightarrow & 0
\end{array}
$$

We have seen that $c_0(R\pi, T)$ is an isomorphism, so it suffices to show that $c_0(T^{-1}R\pi, S)$ is surjective. Consider the commutative diagram

$$\begin{array}{ccccc}
K_1(L\pi) & \longrightarrow & K_0(T^{-1}R\pi, S) & \longrightarrow & 0 \\
(\simeq)\ \big\downarrow\ c_0(L\pi) & & \big\downarrow\ c_0(T^{-1}R\pi, S) & & \\
G_1(L\pi) & \longrightarrow & G_0(T^{-1}R\pi, S) & \longrightarrow & 0
\end{array}$$

It follows from the exact G-sequence of $T^{-1}R\pi \longrightarrow L\pi$ and from (5.3), which applies because $T^{-1}R$ is semi-local, that the bottom row is exact. Therefore $c_0(T^{-1}R\pi, S)$ is surjective, and this concludes the proof.

(5.6) COROLLARY. *In the setting of* (5.5) *let* B *be a maximal* R-*order in* $L\pi$ *containing* $R\pi$. *Then the natural diagram*

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Ker}(c_o(R\pi)) & \longrightarrow & \mathrm{Ker}(k_o(R\pi)) & \xrightarrow{c_o(R\pi)} & \mathrm{Ker}(g_o(R\pi)) & \longrightarrow & 0 \\
 & & & & \downarrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \mathrm{Ker}(k_o(B)) & \xrightarrow[c_o(B)]{} & \mathrm{Ker}(g_o(B)) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

is commutative with exact rows and columns.

Proof. The top row is extracted from the diagram (1) above, with the zero on the right inserted with the aid of (5.5). The commutative square and the bottom zero come from (X, 1.9 (c)), thanks to the fact that $R\pi$ satisfies the Cartan condition; i.e. $c_0((R/\underline{p})\pi)$ is injective for all $\underline{p} \in \max(R)$. The top zero is then forced by the surjectivity of $c_0(R\pi)$ above. Finally, since B is regular, $c_0(B)$ is an isomorphism. q.e.d.

Corollary (5.6) shows that all of the groups appearing there are either quotients or subgroups of

$$\mathrm{Ker}(k_0(R\pi) : K_0(R\pi) \longrightarrow K_0(L\pi)).$$

We can therefore estimate the exponents of all of them by estimating that of $\mathrm{Ker}(k_0(R\pi))$. With the aid of Artin induction we obtain such an estimate as soon as we have one when π is a cyclic group.

Suppose more generally that π is abelian. Then $\mathrm{Ker}(k_0(R\pi)) = Rk_0(R\pi)$. This is easy to see directly, but it follows also from (5.1). Moreover, since R is Dedekind and π is finite we have $\dim(R\pi) \leq 1$, so it follows from (IX, 3.8) that $Rk_0(R\pi) \simeq \mathrm{Pic}(R\pi)$. In this case also the maximal order B above is just the integral closure of $A = R\pi$ in $L\pi$. If $\underline{c}$ is the "conductor" (see §6 below) from B to A, i.e. the largest B-ideal contained in A, then the square

$$\begin{array}{ccc} A & \subset & B \\ \downarrow & & \downarrow \\ A/\underline{c} & \subset & B/\underline{c} \end{array}$$

is a fibre product (see (IX, 5.6)) from which we deduce an exact Mayer-Vietoris sequence (IX, 5.3):

$$U(A/\underline{c}) \oplus U(B) \longrightarrow U(B/\underline{c}) \longrightarrow \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(B) \longrightarrow 0.$$

We have suppressed the terms $\mathrm{Pic}(A/\underline{c})$ and $\mathrm{Pic}(B/\underline{c})$ which should appear here, because they are zero, due to the fact that $A/\underline{c}$ and $B/\underline{c}$ are Artin rings. Thus we obtain information about $\mathrm{Pic}(R\pi)$ ($= \mathrm{Pic}(A)$) once we know $\mathrm{Pic}(B)$, and once we know which units of $B/\underline{c}$ lift to units of B. Needless to say this last problem's analysis requires an explicit description of B and of the conductor, $\underline{c}$. This matter will be taken up in the next section.

In a certain case it is convenient to approximate $\mathrm{Pic}(R\pi)$ with the aid of a somewhat different fibre product. We shall close this section with a discussion of this example, which is due to Milnor.

Fix a prime p, and let $\underline{\underline{F}} = \underline{\underline{Z}}/p\underline{\underline{Z}}$. For each $n \geq 0$ let π_n denote a cyclic group of order p^n, with group ring $A_n = \underline{\underline{Z}}\,\pi_n$, and let $R_n = \underline{\underline{Z}}[w_n]$, where w_n is a primitive p^nth root of unity. Let T be an indeterminate. Then $A_{n+1} = \underline{\underline{Z}}[T]/(T^{p^{n+1}}-1)$, and $R_{n+1} = \underline{\underline{Z}}[T]/(\phi_{p^{n+1}}(T))$, where

$$T^{p^{n+1}} - 1 = (T^{p^n} - 1)\,\phi_{p^{n+1}}(T),$$

$$\phi_{p^{n+1}}(T) = \phi_p(T^{p^n}), \text{ and}$$

$$\phi_p(T) = 1 + T + \ldots + T^{p-1}.$$

Let t be the image of T in A_{n+1}. Then $\phi_{p^{n+1}}(t)$ and $t^{p^n} - 1$ generate ideals in A_{n+1} with zero intersection, and with corresponding factor rings R_{n+1} and A_n, respectively. If we factor out the sum of these two ideals we obtain the quotient of A_n by the ideal generated by $\phi_{p^{n+1}}(s)$, where s is the image of t in A_n, a generator of π_n. Thus $\phi_{p^{n+1}}(s) = \phi_p(s^{p^n}) = \phi_p(1) = p$, so $A_{n+1}/((\phi_{p^{n+1}}(t)) + (t^{p^n} - 1)) = A_n/pA_n = \underline{\underline{F}}\pi_n$. Therefore (see (IX, 5.5)) we have a cartesian square

$$\begin{array}{ccc} A_{n+1} & \xrightarrow{\quad f \quad} & A_n \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\ R_{n+1} & \xrightarrow[\quad f' \quad]{} & \mathbb{F}\pi_n \end{array} \tag{2}$$

Since $\mathbb{F}\pi_n$ is finite $\text{Pic}(\mathbb{F}\pi_n) = 0$, so the Mayer-Vietoris sequence of (2) yields an exact sequence

$$\begin{aligned} U(R_{n+1}) \oplus U(A_n) &\xrightarrow{\ h\ } U(\mathbb{F}\pi_n) \longrightarrow \text{Pic}(A_{n+1}) \\ &\longrightarrow \text{Pic}(R_{n+1}) \oplus \text{Pic}(A_n) \longrightarrow 0. \end{aligned} \tag{3}$$

This sequence provides a basis for computing $\text{Pic}(A_n)$ by induction on n. To carry this out it is necessary to determine $\text{Coker}(h)$, and this has not yet been done except for $n = 0$ (see (5.8) below). The following description of $U(\mathbb{F}\pi_n)$ is useful for this purpose.

(5.7) PROPOSITION. *As above, let* $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$, *and let* π_n *be a cyclic group of order* p^n *with generator* s. *Put* $d = s - 1$, *so that* $I = d\mathbb{F}\pi_n$ *is the augmentation ideal. We have*

$$U(\mathbb{F}\pi_n) = U(\mathbb{F}) \times (1 + I).$$

Let $J = \{i \in \mathbb{Z} \mid 0 < i < p^n;\ p \nmid i\}$; *if* $i \in J$ *define* $e(i)$ *by* $p^{n-e(i)} \le i < p^{n-e(i)+1}$. *Then* $1 + I$ *is the direct product of the cyclic groups with generators* $1 + d^i$ $(i \in J)$, *and* $1 + d^i$ *has order* $p^{e(i)}$.

Proof. We have $d^{p^n} = (s^{p^n} - 1) = 0$, so $\mathbb{F}\pi_n = \mathbb{F}[d]$ and $d^m = 0$, where $m = p^n$. This relation implies $[\mathbb{F}\pi_n : \mathbb{F}] \le m$, so it is already a defining relation for $\mathbb{F}\pi_n$. Suppose now that $m > 0$ is arbitrary and that $B = \mathbb{F}[d]$ with defining relation $d^m = 0$. Then clearly $U(B) = U(\mathbb{F}) \times (1 + I)$ where I is the nilpotent ideal dB. The order of the group 1 + I is card(I) $= p^{[I:\,\underline{\underline{F}}]} = p^{m-1}$, so 1 + I is a p-group. If $0 < i < m$, therefore, the order of $1 + d^i$ is the least $r \geq 0$ such that $(1 + d^i)^{p^r} = 1$, i.e. such that $d^{ip^r} = 0$, i.e. such that $ip^r \geq m$. Therefore $1 + d^i$ has order p^r if $ip^r \geq m$ and $ip^{r-1} < m$, i.e. if $mp^{-r} \leq i < mp^{-r+1}$. Thus, if e(i) is defined by $mp^{-e(i)} \leq i < mp^{-e(i)+1}$, $1 + d^i$ has order $p^{e(i)}$. Let $J = \{i \in \underline{\underline{Z}} \mid 0 < i < m;\ p \nmid i\}$. We claim that 1 + I is the direct product of the cyclic groups generated by the $1 + d^i$ $(i \in J)$. Once this is shown the proposition will be proved.

For m = 1 these elements generate (because I = 0), so by induction on m, they generate modulo $d^{m-1}B$ if $m > 1$. But $d^{m-1}B = d^{m-1}\underline{\underline{F}}$, clearly, and $1 + d^{m-1}\underline{\underline{F}} = \mathrm{Ker}(U(B) \longrightarrow U(B/d^{m-1}B))$ is the cyclic group generated by $1 + d^{m-1}$. If $m - 1 = p^r i$ with i prime to p then $1 + d^{m-1} = (1 + d^i)^{p^r}$. This proves that $1 + d^i$ $(i \in J)$ generate 1 + I.

To see that the $1 + d^i$ $(i \in J)$ are independent generators it suffices to show that the elements $1 + d^{ip^{e(i)-1}}$ $(i \in J)$, which have order p, are independent. The conclusion of the last paragraph shows that these elements generate the group G of elements of order p in 1 + I, and there are card (J) such generators. Hence it suffices to show that $[G: 1] = p^{\mathrm{card}(J)}$. If $x \in I$ then $(1 + x)^p = 1 \Longleftrightarrow x^p = 0 \Longleftrightarrow x \in d^rB$, where $r = [m/p]$. Therefore $G = 1 + d^rB$ and it has cardinality pm-r. But $m - r = m - [m/p]$ is the number of integers between o and m minus the number which are multiples of p, i.e. $m - r = \mathrm{card}(J)$. q.e.d.

In order to see which units of $\underline{\underline{F}}\pi_n$ lift to units of $R_{n+1} = \underline{\underline{Z}}[w_{n+1}]$, we look for some natural source of units in R_{n+1}. One such source is the following (see (6.3) below): If u and v are roots of unity of the same order then α =

$(1 - u)/(1 - v)$ is a unit in $\underline{\underline{Z}}[v]$. For $u = v^i$ for some $i > 0$, and hence $a = 1 + v + \ldots + v^{i-1} \in \underline{\underline{Z}}[v]$, and similarly $a^{-1} = 1 + u + \ldots + u^{j-1} \in \underline{\underline{Z}}[v]$ for some $j > 0$. The homomorphism $f': R_{n+1} \longrightarrow \underline{\underline{F}}\pi_n$ is defined by $f'(w_{n+1}) = s$, and hence $f'(w_1) = f'(w_{n+1}{}^{p^n}) = s^{p^n} = 1$. We have $a = 1 + w_1 + \ldots + w_1{}^{i-1} \in U(R_{n+1})$ $(0 < i < p)$, as we saw above, and $f'(a) = i$. Thus

(4) $\quad \text{Im}(f': U(R_{n+1}) \longrightarrow U(\underline{\underline{F}}\pi_n)) \supset U(\underline{\underline{F}})$.

Moreover this image contains $f'(w_{n+1}) = s = 1 + d$, which generates a subgroup of order p^n of $1 + I$.

(5.8) COROLLARY. In the sequence (3) above, Coker(h) is a finite group of exponent p^n whose order divides $p^{p^n-(n+1)}$.

Proof. $U(\underline{\underline{F}}\pi_n) = U(\underline{\underline{F}}) \times (1 + I)$ and the remarks above show that Coker(h) is a quotient of $1 + I$ modulo a subgroup of order p^n. It follows from (5.7) that $1 + I$ has exponent p^n and order p^{p^n-1}. q.e.d.

(5.9) COROLLARY. As above, let π_n be a cyclic group of order p^n, and let $R_n = \underline{\underline{Z}}[w_n]$, where w_n is a primitive p^nth root of unity. Then

(5) $\quad \text{Pic}(\underline{\underline{Z}}\pi_{n+1}) \longrightarrow \text{Pic}(R_{n+1}) \oplus \text{Pic}(\underline{\underline{Z}}\pi_n)$,

from (3) above, is an epimorphism whose kernel is of exponent p^n and of order dividing $p^{p^n-(n+1)}$. Hence

$$\text{Pic}(\underline{\underline{Z}}\pi_n) \simeq \begin{cases} 0 & \text{, if } n = 0 \\ \text{Pic}(R_1) & \text{, if } n = 1 \\ 0 & \text{, if } p \leq 3 \text{ and } n \leq 2. \end{cases}$$

Proof. The first assertions follow immediately from (5.8) and the exact sequence (3). They imply that (5) is an isomorphism when n = 0, in which case $\mathrm{Pic}(\underline{\underline{Z}}\pi_n) = \mathrm{Pic}(\underline{\underline{Z}}) = 0$. In case p = 2 the first assertions again imply (5) is an isomorphism for $n \leq 1$. Moreover $R_0 = R_1 = \underline{\underline{Z}}$, in this case, and R_2 is the ring of Gaussian integers, a euclidean ring. Hence $\mathrm{Pic}(R_n) = 0$ for $n \leq 2$ if p = 2. The same is true if p = 3. In this case $R_1 = \underline{\underline{Z}}\left[\frac{1+\sqrt{-3}}{2}\right]$ has class number one, i.e. $\mathrm{Pic}(R_1) = 0$ (see Borevich-Shafarevich [1], Table 4). The only further point to be checked when p = 3 is that $f': U(R_2) \longrightarrow U(\underline{\underline{F}}\pi_1)$ is surjective. The image contains $U(\underline{\underline{F}})$ and 1 + d, as we have seen, and $U(\underline{\underline{F}}\pi_1)$ is generated by these elements plus $1 + d^2$. But $(1 + d^2)^{-1} = 1 - d^2 = 1 - (s^2 - 2s + 1) = 2s - s^2 = -s(1 + s) = f'(-w_2)\, f'(1 + w_2)$, and $1 + w_2 \in U(R_2)$. q.e.d.

§6. THE CONDUCTOR OF AN ABELIAN GROUP RING

If R is the ring of integers in a number field then it follows from the theorems of (X, §2) that all the groups appearing in Corollary (5.6) above are finite groups. With the aid of the induction theorems we can obtain information about their exponents provided we know how they behave when π is an abelian (or even cyclic) group. When π is abelian the integral closure of $R\pi$ is a product of rings of algebraic integers, so its arithmetic is relatively well understood. In order to make a careful comparison of $R\pi$ with its integral closure we must first compute the conductor. That is the purpose of this section.

To start with, let A be a commutative ring in which the zero ideal is an irredundant intersection of minimal prime ideals:

$$(0) = \underline{p}_1 \cap \ldots \cap \underline{p}_n .$$

Then $A_i = A/\underline{p}_i$ is an integral domain $(1 \leq i \leq n)$ and $A \subset B = \Pi A_i$. The projections $\rho_i: A \longrightarrow A_i$ are then induced by the coordinate projections in B. The conductor from B to A is

$$\underline{c} = \underline{c}_{B/A} = \{a \in A \mid aB \subset A\}.$$

It is the largest B-ideal contained in A. Being a B-ideal it is the direct sum of its components:

$$(1) \quad \begin{aligned} \underline{c} &= \amalg \underline{c}_i \\ \underline{c}_i &= \{a \in \underline{c} \mid \rho_j(a) = 0 \text{ for all } j \neq i\} \end{aligned}$$

Thus $\underline{c}_i \subset \bigcap_{j \neq i} \underline{p}_j = \{a \in A \mid \rho_j(a) = 0 \text{ for all } j \neq i\}$. Since $\underline{p}_i \cdot (\bigcap_{j\neq i} \underline{p}_j) = 0$ we have $\underline{c}_i \subset \text{ann}_A(\underline{p}_i)$. But $\text{ann}_A(\underline{p}_i)$ is a module over $A/\underline{p}_i$, hence it is a B-ideal, and hence it is contained in $\underline{c}$. Thus we conclude that:

$$(2) \quad \underline{c}_i = \bigcap_{j \neq i} \underline{p}_j = \text{ann}_A(\underline{p}_i) \neq 0.$$

Let $I = \{1,\ldots,n\}$. If $J \subset I$ set $B_J = \Pi_{j \in J} A_j$. If $f = f_J\colon A \longrightarrow B_J$ is the map induced by the projection $B \longrightarrow B_J$ we shall write $A_J = \text{Im}(f)$ and $\underline{a} = \text{Ker}(f)$. Assume that $\text{ann}_A(\underline{a}) = N \cdot A$, the principal ideal generated by some $N \in A$. Since $\text{ann}_A(\underline{a})$ contains all $\underline{c}_j$ $(j \in J)$ it follows that $f(N)$ is not a zero divisor in A_J, (or even in B_J; we use the fact that $\underline{c}_j \neq 0$ for all j). Now we claim:

(3) <u>For each</u> $j \in J$, $\text{ann}_A(\underline{p}_j) = N \cdot f^{-1}(\text{ann}_{A_J}(f(\underline{p}_j)))$, <u>and hence</u> $f(\text{ann}_A(\underline{p}_j)) = f(N) \cdot \text{ann}_{A_J}(f(\underline{p}_j))$.

For since $\underline{a} \subset \underline{p}_j$ we have $\text{ann}_A(\underline{a}) = N \cdot A \supset \text{ann}_A(\underline{p}_j)$. If $a \in \text{ann}_A(\underline{p}_j)$ write $a = Nb$, and apply f to the equation $Nb\underline{p}_j = 0$. As remarked above $f(N)$ is not a divisor of zero so $f(b) \in \text{ann}_{A_J}(f(\underline{p}_j))$. Thus $\text{ann}_A(\underline{p}_j) \subset N \cdot f^{-1}(\text{ann}_{A_J}(f(\underline{p}_j)))$. Suppose, conversely, that $b \in A$ and $f(b) \in \text{ann}_{A_J}(f(\underline{p}_j))$.

Then $b\underline{p}_j \subset \mathrm{Ker}(f) = \underline{a}$. Therefore, since $N \cdot A = \mathrm{ann}_A(\underline{a})$, we have $Nb\underline{p}_j = 0$, i.e. $Nb \in \mathrm{ann}_A(\underline{p}_j)$. The second equation follows by applying f to the first.

Now assume further that $\mathrm{ann}_{A_J}(f(\underline{p}_j)) = f(d)A_J$, a principal ideal. Then we claim, for $j \in J$:

(4) $\underline{c}_j = NdA$, (if $\mathrm{ann}_A(\ker(f_J)) = NA$ and if

$$\mathrm{ann}_{A_J}(f_J(\underline{p}_j)) = f_J(d)A_J).$$

For if $a \in A$ then $f(da\underline{p}_j) = f(a)\ f(d)\ f(\underline{p}_j) = 0$ so $da\underline{p}_j \subset \mathrm{Ker}(f) = \underline{a}$, and hence $Nda\ \underline{p}_j = 0$ since $N\underline{a} = 0$. Thus $NdA \subset \mathrm{ann}_A(\underline{p}_j) = \underline{c}_j$. Conversely, suppose $a\underline{p}_j = 0$. Then (3) implies $a = Nb$ where $f(b)\ f(\underline{p}_j) = 0$. Hence $f(b) = f(d)\ f(c)$ for some c, i.e. $b - dc \in \mathrm{Ker}(f) = \underline{a}$. Therefore, since $N\underline{a} = 0$, $a = Nb = Ndc$. q.e.d.

Now we shall apply these remarks to the following data:

(5)
- R = an integrally closed integral domain.
- L = field of fractions of R, of characteristic zero.
- π = finite abelian group of order $n = [\pi: 1]$.
- $L\pi = \prod_{i \in I} L_i$, where L_i are (finite cyclotonic) field extensions of L.
- $A = R\pi$
- $B = \prod_{i \in I} A_i$, where A_i is the projection of A in L_i.

Let $\underline{p}_i = \mathrm{Ker}(\rho_i)$ where $\rho_i: A \longrightarrow A_i \subset L_i$ is the projection. Then the $\underline{p}_i$ are prime ideals and

$$\bigcap_{i \in I} \underline{p}_i = (0)$$

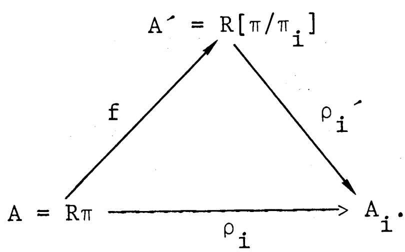
because $A \subset L\pi = \prod L_i$. It is easy to see that this intersection is irredundant, i.e. that $\bigcap_{j \in J} \underline{p}_j \neq 0$ for all proper subsets $J \subset I$. This follows, for example, from the fact that A is an R-lattice in $L\pi$. Alternatively one can use primary decomposition theory (which has not been discussed in these notes).

If $m \geq 1$ write μ_m for the group of m^{th} roots of unity (in the algebraic closure of any field we choose to consider here). Then $L_i = L[\mu_{m_i}]$ where $\mu_{m_i} = \rho_i(\pi)$. We define the kernel π_i by the exact sequence

(6) $$1 \longrightarrow \pi_i \longrightarrow \pi \xrightarrow{\rho_i} \mu_{m_i} \longrightarrow 1.$$

Since $A = R\pi$ we have $A_i = \rho_i(A) = R[\mu_{m_i}]$.

We propose to compute $\underline{c} = \underline{c}_{B/A} = \underline{c}_i$ $(i \in I)$ by the method described above. Fix an $i \in I$ and consider the commutative triangle



Then, in the notation introduced above, $A' = A_J$ where $J = \{j \mid \pi_i \subset \pi_j\} = \{j \mid \rho_j(\pi_i) = \{1\}\}$. The ideal $\underline{a} = \mathrm{Ker}(f)$ is generated by all $1 - x$ $(x \in \pi_i)$. We claim:

(7) $$\mathrm{ann}_A(\underline{a}) = N_i A, \qquad \text{where } \underline{a} = \mathrm{Ker}(R\pi \longrightarrow R[\pi/\pi_i])$$
$$\text{and } N_i = \Sigma_{x \in \pi_i} x.$$

Since A is a free $R\pi_i$-module and since $\underline{a}$ is generated by elements of $R\pi_i$ it suffices to prove this when $\pi = \pi_i$. It is clear that $xN_i = N_i$ for all $x \in \pi_i$, so $N_i\underline{a} = 0$. Conversely, if $a = \Sigma a_x x$ $(x \in \pi_i)$ and $a \cdot \underline{a} = 0$ then $ya = a$ for all $y \in \pi_i$ so $a_x = a_{yx}$ for all $x \in \pi_i$. Thus $a = a_1 \cdot N_i$, as claimed.

Let $\sigma_i = \pi/\pi_i$, a cyclic group of order m_i. Choose a generator $s = f(t)$ of σ_i $(t \in \pi)$. Then $R\sigma_i = R[s] = R[S]/(P(S))$, where

$$P(S) = S^{m_i} - 1 = \prod_{j \in J} P_j(S)$$

is the factorization of P into monic irreducible polynomials in L[S]. Note that this is the same set J introduced above, and we have $L\sigma_i = L[s] = L[S]/(P(S)) = \prod_{j \in J} L[S]/(P_j(S))$. Since R is integrally closed the coefficients of each P_j are in R, and $A_j = R[S]/(P_j(S)) = A'/f(\underline{p}_j)$, where $f(\underline{p}_j) = P_j(s)A'$. We can describe $P_j(S)$ as the minimal polynomial of $\rho'_j(s) = \rho_j(t)$ over L. In particular, when $j = i$, $w = \rho_j(t)$ is a primitive m_ith root of unity, and $A_i = R[w]$. Set

$$Q(S) = \prod_{j \in J,\ j \neq i} P_j(S) = (S^{m_i} - 1)/P_i(S).$$

Then it is clear that the annihilator in A' of $f(\underline{p}_i) = P_i(s)R[s]$ is $Q(s)R[s]$. Note that $Q(s) = Q(f(t)) = f(Q(t))$.

Now we can apply (4) above. The N_i here plays the role of N there, thanks to (7), and Q(t) here plays the role of d in (4). We shall recapitulate the setting, and formulate the conclusion, as a proposition.

(6.1) PROPOSITION. Keep the notation of (5) and (6) above. Let $\underline{p}_i = \mathrm{Ker}(\rho_i : A \longrightarrow A_i)$ $(i \in I)$, and let $\underline{c} = \underline{c}_{B/A}$ be the conductor. Then

$$\underline{c} = \prod \underline{c}_i \quad (i \in I)$$

where $\underline{c}_i = \mathrm{ann}_A(\underline{p}_i)$, and $\rho_i(\underline{c}) = \rho_i(\underline{c}_i)$. Choose $t \in \pi$ such *that* $w = \rho_i(t)$ *generates* $\mu_{m_i} = \rho_i(\pi)$, *and set* $Q(T) = (T^{m_i} - 1)/P(T)$, *where* $P(T)$ *is the minimal polynomial of* w *over* L. *Let* $N_i = \Sigma_{x \in \pi_i} x$ *where* $\pi_i = \mathrm{Ker}(\rho_i : \pi \longrightarrow \mu_{m_i})$. *Then we have*

(8) $\underline{c}_i = N_i\, Q(t)A,$

and hence

(9) $\rho_i(\underline{c}_i) = (m/m_i)\, Q(w)\, R[w].$

In order to complete this calculation we would like a more explicit description of the ideal $Q(w)\, R[w]$ in (9) above. We shall undertake this now in case R is a ring of cyclotomic integers.

For each integer $m \geq 1$ write

$$R_m = \underline{\underline{Z}}[\mu_m], \text{ and}$$

$$L_m = \underline{\underline{Q}}[\mu_m].$$

(These were denoted $\underline{\underline{Z}}_m$ and $\underline{\underline{Q}}_m$, respectively, in §4.) It is known that R_m is the full ring of algebraic integers (= integral closure of $\underline{\underline{Z}}$) in L_m. Set

$$\Phi_m = \{\text{primitive } m^{th} \text{ roots of unity}\}$$
$$= \{\text{generators of } \mu_m\}.$$

Then μ_m is the disjoint union of the Φ_d $(d|m)$, and $[L_m : \underline{\underline{Q}}] = \mathrm{card}(\Phi_m) = \phi(m)$ (the Eucler ϕ-function).

$$T^m - 1 = \Pi\ \phi_d(T) \quad (d|m)$$

where

$$\phi_d(T) = \Pi\ (T - w) \quad (w \in \Phi_d)$$

is the d^{th} cyclotomic polynomial.

(6.2) PROPOSITION. *Suppose* $w \in \Phi_m$ $(m > 1)$. *Then*

$$N_{L_m/\mathbb{Q}}(1 - w) = \phi_m(1) = \begin{cases} p & \text{if } m \text{ is a power of the prime } p \\ 1 & \text{if } m \text{ is composite.} \end{cases}$$

Proof. Since Φ_m is the set of conjugates of w over $\mathbb{Q}$ we have $N_{L_m/\mathbb{Q}}(1 - w) = \Pi_{u \in \Phi_m}(1 - u) = \phi_m(1)$. If p is prime then $\phi_p(T) = 1 + T + \ldots + T^{p-1}$ and $\phi_{p^n}(T) = \phi_p(T^{p^{n-1}})$ for $n \geq 1$. Hence $\phi_{p^n}(1) = p$, thus proving the proposition for prime powers. Suppose m is composite and that we know the result, by induction, for all proper divisors of m. We have

$$1 + T + \ldots + T^{m-1} = \Pi\ \phi_d(T) \quad (d|m,\ d > 1),$$

and so $m = \Pi\ \phi_d(1)$ $(d|m,\ d > 1)$. The right side is $\phi_m(1)$ times the product of all $\phi_d(1)$ $(d|m,\ 1 < d < m)$. The composite d's among these contribute $\phi_d(1) = 1$, by induction. If $m = p^n m'$ where p is a prime not dividing m' then we get n factors $\phi_{p^i}(1) = p$ $(1 \leq i \leq n)$, and these contribute, altogether, a factor p^n. Letting p vary now we see that $\Pi\ \phi_d(1)$ $(d|m,\ 1 < d < m)$ equals m. The equation above therefore implies $m = \phi_m(1) \cdot m$ so $\phi_m(1) = 1$. q.e.d.

(6.3) COROLLARY. *Suppose* $u, v \in \Phi_m$ $(m > 1)$. *Then*

$$(1 - u)/(1 - v)$$

is a ("cyclotomic") unit in R_m. *If* m *is composite then* $1 - u$ *itself is a unit.*

Proof. We can write $u = v^i$ so that $(1 - u)/(1 - v) =$

$1 + v + \ldots + v^{i-1} \in R_m$. By symmetry its inverse is also in R_m. Alternatively, (6.2) implies it has norm one (over $\underline{Q}$), and, since it is an algebraic integer, it must be a unit. This argument applies equally well to 1 - u when m is composite. q.e.d.

This corollary shows that 1 - u and 1 - v generate the same ideal in R_m, and hence also in any ring containing R_m. Moreover, it is the unit ideal if m is composite. If $m = p^n$, p a prime, then this ideal depends only on p^n. Since $p = \prod_{v \in \Phi_{p^n}} (1 - v)$, and since $\text{card}(\Phi_{p^n}) = \phi(p^n) = (p - 1)p^{n-1}$ we conclude that:

(10) *If* p *is a prime and if* $u \in \Phi_{p^n}$ *then*

$$(1 - u)^{\phi(p^n)} = (p), \quad \text{\underline{or}} \quad (1 - u) = (p)^{1/\phi(p^n)}.$$

These are to be interpreted as equations between principal ideals in R_{p^n}, or, more generally, in any integral domain containing R_{p^n}. The second equation signifies that (1 - u) is the unique principal ideal whose $\phi(p^n)^{th}$ power is (p). We shall also record the equation above:

(11) $(p) = \Pi\ (1 - u) \quad (u \in \Phi_{p^n})$ *for each* $n \geq 1$.

(6.4) PROPOSITION. *Let* m *and* n *be positive integers with prime factorizations* $m = \Pi\ p^{m_p}$ *and* $n = \Pi\ p^{n_p}$. *Let* $h_p = \max(m_p, n_p)$ *and* $r_p = \min(m_p, n_p)$, *so that* $h = \Pi\ p^{h_p} = \ell.c.m.(m, n)$ *and* $r = \Pi\ p^{r_p} = g.c.d.\ (m, n)$.

Let $w \in \Phi_m$ *have minimal polynomial* P(T) *over* L_n, *and set* $Q(T) = (T^m - 1)/P(T)$. *Then*

(12) $Q(w)\ R_n[w] = \Pi\ (p)^{s_p} \quad (p|m)$,

where

$$s_p = \begin{cases} 1/(p-1) & \text{if } p \nmid n \text{ (i.e. if } n_p = 0) \\ r_p = \min(m_p, n_p) & \text{if } p | n \text{ (i.e. if } n_p > 0). \end{cases}$$

If $p|m$ and if $\underline{p}$ is a prime ideal of $R_n[w]$ (= R_h) which divides p then

$$(13) \qquad v_{\underline{p}}(Q(w)\ R_n[w]) = \begin{cases} p^{h_p - 1} & \text{if } p \nmid n \\ r_p(p-1)p^{h_p - 1} & \text{if } p|n. \end{cases}$$

Remark. It is clear that $R_n[w] = R_h$. Moreover it follows from the fact that p ramifies completely in p^{th} power cyclotomic fields, and not at all in R_m when m is prime to p, that $v_{\underline{p}}(p) = \phi(p^{h_p}) = (p-1)p^{h_p - 1}$, for $\underline{p}$ as above. Therefore $v_{\underline{p}}((p)^{s_p}) = s_p\ \phi(p^{h_p})$, so (13) follows from (12).

Note that, in the extreme cases, the proposition gives:

$$(14) \qquad Q(w)\ R_n[w] = \begin{cases} \Pi\ (p)^{1/(p-1)} \quad (p|m) & \text{if g.c.d. } (m, n) = 1 \\ (m) & \text{if } m|n. \end{cases}$$

Proof of (6.4). Thanks to the remark above we need only prove (12). Let C denote the set of conjugates over L_n of w. Then $C \subset \Phi_m$ and $Q(T) = \Pi(T - u)$ $(u \in \mu_m - C)$. The ideal generated by $w - u = w(1 - uw^{-1})$ depends only on the order of uw^{-1}, and it is the unit ideal if uw^{-1} has composite order (see (6.3)). Moreover, according to (10), if uw^{-1} has order a power of the prime p then the ideal $(1 - uw^{-1})$ is some fractional power of the ideal (p). Hence we have some formula of the type

$$Q(w)\, R_n[w] = \Pi\, (p)^{s_p} \qquad (p|m),$$

and we must determine the rational numbers s_p.

Fix a prime divisor p of m and put $q = p^{m_p}$. We must find out which $u \in \mu_m - C$ are such that uw^{-1} has p^{th} power order. Write $w = w_0 w_1$ where $w_0 \in \mu_q$ and w_1 has order prime to p. (Since $w \in \Phi_m$ it then follows that $w_0 \in \Phi_q$.) Similarly factor $u = u_0 u_1$. Then $uw^{-1} \in \mu_q \iff u_1 = w_1$. In this case u is not L_n-conjugate to $w \iff u_0$ is not L_n-conjugate to w_0. Thus, if C_p is the set of L_n conjugates of w_0, we have $(p)^{s_p} = \Pi\, (1 - u_0 w_0^{-1})\ (u_0 \in \mu_q - C_p)$.

Now $\mathrm{Aut}(\mu_q) \simeq U(\underline{\underline{Z}}/q\underline{\underline{Z}}) = U(\underline{\underline{Z}}/p^{m_p}\underline{\underline{Z}})$, and the automorphisms induced by $Gal(L_n(w_0)/L_n)$ correspond to the "congruence group" of level p^{r_p} = g.c.d. (q, n), i.e. the automorphisms that fix $\mu_{(p^{r_p})} = \mu_q \cap L_n$. (It is here that we make essential use of the fact that L_n is cyclotomic, and not any number field.) It follows from this that

$$C_p = \begin{cases} \Phi_q = \mu_{(p^{m_p})} - \mu_{(p^{m_p - 1})} & \text{if } p \nmid n \\ w_0 \mu_{(p^{t_p})}, \quad t_p = \max(0,\, m_p - n_p) & \text{if } p|n. \end{cases}$$

Therefore

$$w_0^{-1}(\mu_{(p^{m_p})} - C_p) = \begin{cases} w_0^{-1}\, \mu_{(p^{m_p - 1})} & \text{if } p \nmid n \\ w_0^{-1}\, \mu_{(p^{m_p})} - w_0\, \mu_{(p^{t_p})} & \\ = \mu_{(p^{m_p})} - \mu_{(p^{t_p})} & \text{if } p|n. \end{cases}$$

Now $w_o^{-1}\mu_{(p^{m_p-1})} \subset \Phi_{(p^{m_p})}$, and $\mu_{(p^{m_p})} - \mu_{(p^{t_p})} = \bigcup_{t_p < j \leq m_p} \Phi_{(p^j)}$. Therefore we can apply (10) and (11) to obtain

$$(p)^{s_p} = \prod_{u \in (\mu_q - C_p)} (1 - uw_o^{-1}) = \begin{cases} (p)^{p^{m_p-1}/\phi(p^{m_p})} = (p)^{1/(p-1)} & \text{if } p \nmid n \\ (p)^{m_p - t_p} & \text{if } p \mid n. \end{cases}$$

Since $m_p - t_p = m_p - \max(0, m_p - n_p) = \min(m_p, n_p)$ the proposition is now proved.

(6.5) COROLLARY. In the setting of (6.1) assume that $R = R_n$. Then in the notation of (6.1), $B = \Pi A_i$ is the integral closure of $A = R_n\pi$ in $L_n\pi$. Moreover

$$\rho_i(\underline{c}_i) = (m/m_i) \prod (p)^{s_p} \qquad (p \mid m_i)$$

where

$$s_p = \begin{cases} 1/(p-1) & \text{if } p \nmid n \\ \min(v_p(m_i), v_p(n)) & \text{if } p \mid n. \end{cases}$$

Hence $mB \subset \underline{c}$ and ${}^B\!\sqrt{\underline{c}} = {}^B\!\sqrt{mB}$. (Here $v_p(m)$ denotes the power to which p divides m.) The conductor $\underline{c}$ is its own radical in B if and only if $m = [\pi : 1]$ is square free (so π is cyclic) and either g.c.d. (m, n) = 1 or g.c.d. (m, n) = 2 and $4 \nmid n$.

Proof. B is integral over A and integrally closed, as remarked above; hence the first assertion. According to (6.1) $\rho_i(\underline{c}_i) = (m/m_i)\, Q(w)\, R_n[w]$, and (6.4) tells us that

$Q(w)\, R_n[w] = \Pi\, (p)^{s_p}\ (p|m_i)$, as above. Since each $s_p > 0$ it follows that every prime in A_i dividing p divides $\rho_i(\underline{c}_i)$ if $p|m_i$, and the same is clearly true if $p|(m/m_i)$. Therefore $\rho_i(\underline{c}_i)$ has the same radical in A_i as mA_i. It follows from (1.2) that $mB \subset \underline{c}$. The last assertion follows from a simple, but tedious, case analysis, using (13). We leave the details to the reader. The case when m and n are relatively prime can be deduced readily from the following, more precise, statement:

(6.6) COROLLARY. *Suppose, in* (6.5), *that* g.c.d. (m, n) = 1 (*e.g. that* n = 1, *in which case* $A = \underline{\underline{Z}}\pi$). *Then*

$$\rho_i(\underline{c}_i) = (m/m_i)\ \Pi\ (p)^{1/(p - 1)} \qquad (p|m_i).$$

Let p *be a prime, let* $\underline{p}$ *be a prime ideal of* A_i *dividing* p, *and let* $t = v_p(m_i)$. *Then*

$$v_{\underline{p}}(\rho_i(\underline{c}_i)) = \begin{cases} p^{t-1}\,(v_p(m/m_i)\,(p-1) + 1), & \text{if } t > 0 \\ v_p(m/m_i) & \text{, if } t = 0 \end{cases}$$

Proof. The first assertion follows from (6.5). The second follows from the first by virtue of the fact that $v_{\underline{p}}(p) = \phi(p^t)$, and $\phi(p^t) = (p - 1)p^{t-1}$ if $t > 0$. q.e.d.

For later applications we shall also want the following formulas:

(6.7) PROPOSITION. *Let* $A = \underline{\underline{Z}}\pi$ *where* π *is an abelian group of order* $m=[\pi: 1]$. *Let* B *be the integral closure of* A *in* $\underline{Q}\pi$, *and let* $\underline{c}$ *be the conductor from* B *to* A. *If* p *is a prime dividing* m *write* $\pi = \pi_p \times \pi_p{}'$, *where* π_p *is the Sylow* p-*subgroup. Then we have*:

$$h_o(A) = 1$$

$$h_o(B) = h_o(\underline{\underline{Q}}\pi) = \Pi h_o(\underline{\underline{Q}}\pi_p) \qquad (p|m).$$

$$h_o(A/\underline{c}) = \Sigma\, h_o(\underline{\underline{F}}_p\, \pi_p{}') \qquad (p|m)$$

$$h_o(B/\underline{c}) = \Sigma\, h_o(\underline{\underline{Q}}\pi_p)\, h_o(\underline{\underline{F}}_p\, \pi_p{}') \qquad (p|m).$$

<u>The abelian group</u>

$$M = \mathrm{Coker}(H_o(B) \oplus H_o(A/\underline{c}) \longrightarrow H_o(B/\underline{c}))$$

<u>is free of rank</u> $h_o(A) - (h_o(B) + h_o(A/\underline{c})) + h_o(B/\underline{c})$. <u>It vanishes if and only if</u> m <u>is a prime power</u>.

<u>Proof</u>. It follows from (5.2) that $h_o(A) = 1$, and it is clear that $h_o(B) = h_o(\underline{\underline{Q}}\pi)$.

If m and n are relatively prime then the fields L_m and L_n are linearily disjoint over $\underline{\underline{Q}}$, so $L_m \otimes_{\underline{\underline{Q}}} L_n \simeq L_{mn}$, and $R_m \otimes_{\underline{\underline{Z}}} R_n \simeq R_{mn}$. Suppose π' has order n and $A' = \underline{\underline{Z}}\pi'$ has integral closure B' in $\underline{\underline{Q}}\pi'$. Then it follows from the remarks just made, since $\underline{\underline{Q}}[\pi \times \pi'] = \underline{\underline{Q}}\pi \otimes_{\underline{\underline{Q}}} Q\pi'$ and similarly for $\underline{\underline{Z}}[\pi \times \pi']$, that $h_o(Q[\pi \times \pi']) = h_o(Q\pi)\, h_o(Q\pi')$ and that $B \otimes_{\underline{\underline{Z}}} B'$ is the integral closure of $\underline{\underline{Z}}[\pi \times \pi']$. The first of these conclusions implies that $h_o(Q\pi) = \Pi\, h_o(\underline{\underline{Q}}\pi_p)$ $(p|m)$.

Since $\underline{c}$ and mB have the same radical in B (see (6.5)) and hence also in A, we can use mB in place of $\underline{c}$ to compute h_o's. Moreover $m^2B \subset mA \subset mB$ so we have

$$h_o(B/\underline{c}) = h_o(B/mB) \text{ and } h_o(A/\underline{c}) = h_o(A/mA).$$

Evidently $h_o(A/mA) = \Sigma\, h_o(A/pA) = \Sigma\, h_o(\underline{\underline{F}}_p\pi) = \Sigma\, h_o(\underline{\underline{F}}_p\pi_p{}')$. The summation is over primes p dividing m, and the last equality results from the fact that π_p acts trivially on the simple $\underline{\underline{F}}_p\pi$-modules.

Similarly we have $h_o(B/mB) = \Sigma\, h_o(B/pB)$. Given p write $B = B_p \otimes_{\mathbb{Z}} B_p'$ corresponding to the decomposition $\pi = \pi_p \times \pi_p'$, as in the paragraph above. In each factor of B_p, p ramifies completely, so B_p/pB_p is a product of $h_0(\mathbb{Q}\pi_p)$ Artin local rings with residue class fields $\mathbb{F}_p$. Moreover, since π_p' has order prime to p, B_p' and $Z\pi_p'$ have the same $\mathbb{Z}$-localization at p, so $B_p'/pB_p' = (Z\pi_p')/p(\mathbb{Z}\pi_p') = \mathbb{F}_p\pi_p'$. Therefore, modulo a nilpotent ideal, $B/pB = (B_p/pB_p) \otimes_{\mathbb{Z}} (B_p'/pB_p')$ becomes a product of $h_0(\mathbb{Q}\pi_p)$ copies of $\mathbb{F}_p\pi_p'$. Thus $h_0(B/\underline{c}) = \Sigma\, h_0(\mathbb{Q}\pi_p)\, h_0(\mathbb{F}_p\pi_p')$.

The cartesian square

$$\begin{array}{ccc} A & \subset & B \\ \downarrow & & \downarrow \\ A/\underline{c} & \subset & B/\underline{c} \end{array}$$

yields an exact sequence (see (IX, 5.11))

$$0 \longrightarrow H_0(A) \longrightarrow H_0(B) \oplus H_0(A/\underline{c}) \xrightarrow{\ h\ } H_0(B/\underline{c})$$

in which $M = \mathrm{Coker}(h)$ is a torsion free abelian group. Being finite generated, clearly, we conclude that M is free of the indicated rank. If m is a power of the prime p then the formulas above show that $h_0(A/\underline{c}) = 1$ and $h_0(B/\underline{c}) = h_0(\mathbb{Q}\pi)$, and hence $M = 0$.

Finally, suppose $m = m_q m'$ where m_q is a power of a prime q and $m' > 1$ is prime to q; we claim now that $M \neq 0$. For the rank of M is

$$\begin{aligned} r(\pi) &= 1 - h_0(\mathbb{Q}\pi) + \Sigma_p\, h_0(\mathbb{Q}\pi_p)\, (h_0(\mathbb{F}_p\pi_p') - 1) \\ &= 1 - h_0(\mathbb{Q}\pi_q)\, h_0(\mathbb{Q}\pi_q') \\ &\quad + \Sigma_{p \neq q}\, h_0(\mathbb{Q}\pi_p)\, (h_0(\mathbb{F}_p\pi_p') - 1) \\ &\quad + h_0(\mathbb{Q}\pi_q)\, (h_0(\mathbb{F}_q\pi_q') - 1) \end{aligned}$$

$$= 1 - h_o(\underline{\underline{Q}}\pi_q{}')$$
$$+ \Sigma_{p \neq q} h_o(\underline{\underline{Q}}\pi_p)\,(h_o(\underline{\underline{F}}_p\pi_p{}') - 1)$$
$$+ h_o(\underline{\underline{Q}}\pi_q)\,(h_o(\underline{\underline{F}}_q\pi_q{}') - h_o(\underline{\underline{Q}}\pi_q{}'))$$
$$> r(\pi_q{}') + h_o(\underline{\underline{Q}}\pi_q)\,(h_o(\underline{\underline{F}}_q\pi_q{}') - h_o(\underline{\underline{Q}}\pi_q{}'))$$
$$\geq 0.$$

The last inequality holds since $r(\pi_q{}') \geq 0$ and $h_o(\underline{\underline{F}}_q\pi_q{}') \geq h_o(\underline{\underline{Q}}\pi_q{}')$, the latter since $\pi_q{}'$ has order prime to q. The strict inequality occurs when we replace the terms $h_o(\underline{\underline{F}}_p\pi_p{}')$ above (with $p \neq q$) by the strictly smaller terms $h_o(\underline{\underline{F}}_p\pi_p{}'')$, where $\pi_p{}' = \pi_q \times \pi_p{}''$. This concludes the proof.

§7. APPLICATIONS TO $K_1(R\pi)$ AND $G_1(R\pi)$.

For the first part of this section we shall fix the following data:

R is the ring of algebraic integers in a number field L.

π is an abelian group of order $m = [\pi : 1]$.

(0) $A = R\pi$

B is the integral closure of A in $L\pi$, and

$\underline{c}$ is the conductor from B to A.

As in (6.1) we have $B = \Pi A_i$, and the projections $\rho_i : A \longrightarrow A_i$ are surjective. Moreover $\underline{c} = \amalg \underline{c}_i$, where $\underline{c}_i$ projects isomorphically onto its image in A_i.

(7.1) THEOREM. (a) (Higman) Every unit of finite order in $U(R\pi)$ is of the form ux, where u is a root of unity in R, and $x \in \pi$.

(b) $U(R\pi)$ is a finitely generated abelian group of rank $h_o(\underline{\underline{R}} \otimes_{\underline{\underline{Q}}} L\pi) - h_o(L\pi)$.

(c) (Milnor) Let $U'(R\pi)$ be the subgroup of $U(R\pi)$ generated by all $U(R\pi')$, where π' ranges over the cyclic subgroups of π. Then $U(R\pi)/U(R\pi')$ is a finite group of exponent $e_{(cyclic)}(L, \pi)^2$.

Recall from (3.4) that $e_{(cyclic)}(L, \pi)$ divides $e_{(cyclic)}(\underline{\underline{Q}}, \pi)$, and the latter divides m (see (4.6)).

Proof. (a) The proof uses the orthogonality relations for characters, as indicated below.

We can clearly assume that L is large enough for π. Let $e_1, \ldots, e_m$ be the primitive idempotents in $L\pi \simeq L^m$. If $x \in \pi$ we have $x = \Sigma\, \rho_i(x) e_i$. Hence, if $\alpha = \Sigma\, \alpha_x x$ $(x \in \pi)$ then $\rho_i(\alpha) = \Sigma\, \alpha_x\, \rho_i(x)$. The consequence of the "orthogonality relations" that we require is the formula (see Curtis-Reiner [1], p. 263):

$$\alpha_x = m^{-1}\, \Sigma\, \rho_i(\alpha)\, \rho_i(x) \qquad (1 \le i \le m),$$

where we view each term in the sum as belonging to L. Now suppose $\alpha \in U(R\pi)$ and α has finite order. For any embedding of L into $\underline{\underline{C}}$ we have

$$|\alpha_x| \le |m|^{-1}\, \Sigma |\rho_i(\alpha)\, \rho_i(x)| \qquad (1 \le i \le m)$$
$$\le 1$$

because $\rho_i(\alpha)$ and $\rho_i(x)$ are roots of unity for each i. Hence, $N_{L/\underline{\underline{Q}}}(\alpha_x)$, being a product of (complex) conjugates of α_x, has absolute value ≤ 1. But $\alpha_x \in R$ is an algebraic integer, so we must have $|N_{L/\underline{\underline{Q}}}(\alpha_x)| = 0$ or 1. If $\alpha_x \ne 0$ therefore, we must have $\rho_i(\alpha)^{-}\rho_i(x) = \alpha_x$ for all i, for otherwise the inequality above would be strict. Thus $\rho_i(\alpha) = \alpha_x\, \rho_i(x) = \rho_i(\alpha_x x)$ for all i, so $\alpha = \alpha_x x$, where x is chosen so that $\alpha_x \ne 0$.

(b) follows from the Dirichlet Unit Theorem (X, 3.1) since U(A) has finite index in U(B). (This is because U(A) contains $U(B, \underline{c}) = \mathrm{Ker}(U(B) \longrightarrow U(B/\underline{c}))$.

(c) We can identify $U(R\pi)$ with the image of $K_1(R\pi) \longrightarrow K_1(L\pi)$, and, as such, it is a Frobenius module over the Frobenius functor G_R. As such $U'(R\pi)$ is simply $U_{(cyclic)}(R\pi)$, so it has exponent $e_{(cyclic)}(R, \pi)$ in $U(R\pi)$, by (3.1). By (3.3) $e_{(cyclic)}(R, \pi)$ divides $e_{(cyclic)}(L, \pi)^2$. q.e.d.

Theorem (7.1), part (c), can be used to obtain a set of generators and relations for a subgroup of finite index in $U(\underline{\underline{Z}}\pi)$. (See Bass [3]).

(7.2) THEOREM. *Assume, above, that* $R = \underline{\underline{Z}}$.

(a) *The natural homomorphism*

$$SK_1(A, \underline{c}) \longrightarrow SK_1(A)$$

is surjective, and $SK_1(A, \underline{c}) = SK_1(B, \underline{c}) = \prod SK_1(A_i, \rho_i(\underline{c}_i))$.

(b) *Let* $\pi_i = \mathrm{Ker}(\rho_i(\pi)$ *and write* $m_i = [\pi : \pi_i]$ *and* $n_i = [\pi_i : 1]$. *Then*

$$SK\ (A_i, \rho_i(\underline{c}_i)) \simeq \mu_{r_i},$$

the r_i^{th} *roots of unity in* A_i, *where*

$$r_i = \begin{cases} 1, & \text{if } m_i \leq 2 \\ 2\ \text{g.c.d.}\ (m_i, n_i), & \text{if } m_i > 2 \text{ is odd and } 4 \mid n_i \\ \text{g.c.d.}\ (m_i, n_i), & \text{otherwise}. \end{cases}$$

Hence $SK_1(A, \underline{c})$, *and therefore also* $SK_1(A)$, *has exponent* e, *where* $e = \exp(\pi)$.

Proof. (a) Since $A/\underline{c}$ is semi-local the surjectivity of $SK_1(A, \underline{c}) \longrightarrow SK_1(A)$ follows from (V, 9.3). The final assertion of (a) follows from (IX, 5.8).

(b) We have $A_i = \underline{\underline{Z}}[\mu_{m_i}]$, where $\mu_{m_i} = \rho_i(\pi)$, so the the number of roots of unity in A_i is

$$m_i' = \begin{cases} m_i & \text{if } 2 \mid m_i \\ 2m_i & \text{if } 2 \nmid m_i. \end{cases}$$

It follows from (VI, 7.3) that $SK_1(A_i, \rho_i(\underline{c}_i)) \simeq \mu_{r_i}$, where $r_i = 1$ if $m_i \leq 2$ (i.e. if $A_i = \underline{\underline{Z}}$), and otherwise r_i is defined as follows:

$$r_i = \Pi\, p^{j_p} \qquad (p \mid m_i'),$$

where j_p is the nearest integer in the interval $[0, v_p(m_i')]$ to

$$\min_{\underline{p} \mid p \text{ in } A_i} \left[\frac{v_{\underline{p}}(\rho_i(\underline{c}_i))}{v_{\underline{p}}(p)} - \frac{1}{p - 1} \right].$$

According to (6.6)

$$\rho_i(\underline{c}_i) = (m/m_i)\, \Pi\, (p)^{1/(p - 1)} \quad (p \mid m_i).$$

Thus, if $p \mid m_i$ and $\underline{p} \mid p$, then, since $n_i = m/m_i$,

$$v_{\underline{p}}(\rho_i(\underline{c}_i)) = v_{\underline{p}}(n_i) + (1/p - 1)\, v_{\underline{p}}(p),$$

and

$$v_{\underline{p}}(n_i)/v_{\underline{p}}(p) = v_p(n_i), \text{ so}$$

$$j_p = \min(v_p(m_i), v_p(n_i)).$$

This shows that r_i and g.c.d. (m_i, n_i) agree in all factors corresponding to primes that divide m_i. Since r_i is divisible only by primes dividing m_i' this leaves only the case p = 2 and m_i is > 2 and odd. If $\underline{p}|2$ then $v_{\underline{p}}(\rho_i(\underline{c})) = v_{\underline{p}}(m/m_i) = v_{\underline{p}}(n_i)$, and 1/(2 - 1) = 1, so $j_2 = 2$ the nearest integer in the interval $[0, v_2(m_i')(= 1)]$ to $v_2(n_i) - 1$. Thus $j_2 = 0$ if $v_2(n_i) \leq 1$ (i.e. $4 \nmid n_i$) and $j_2 = 2$ if $4|n_i$. This establishes the formula for r_i. The latter shows that $r_i|m_i$ or $r_i|2m_i$ if m_i is odd and m is even. Hence in any case r_i divides e = exp(π). It now follows from part (a) that $SK_1(A, \underline{c})$ and $SK_1(A)$ have exponent e. q.e.d.

(7.3) PROPOSITION. _In the setting of_ (7.2), _assume, for some prime_ p, _that the Sylow_ p-_subgroup_, π_p, _of_ π _is cyclic. Then_ $SK_1(\underline{\underline{Z}}\pi)$ _has no_ p-_torsion_. Hence $SK_1(\underline{\underline{Z}}\pi) = 0$ _if_ π _itself is cyclic_.

Proof. Let $p^n = [\pi_p : 1]$. We argue by induction on n, the case n = 0 following from (7.2) (b). Assume n > 0 and let $\pi' = \pi/\sigma$, where σ is the subgroup of order p. Let $\underline{a} = \text{Ker}(\underline{\underline{Z}}\pi \longrightarrow \underline{\underline{Z}}\pi')$. Then we have an exact sequence

$$SK_1(\underline{\underline{Z}}\pi, \underline{a}) \longrightarrow SK_1(\underline{\underline{Z}}\pi) \longrightarrow SK_1(\underline{\underline{Z}}\pi'),$$

and the right hand term has no p-torsion, by the induction hypothesis. It is clear that $\underline{a}$ contains all components $\underline{c}_i$ of $\underline{c}$ such that $\rho_i(\sigma) \neq \{1\}$. Moreover, if $\underline{b}$ is the sum of these $\underline{c}_i$ then $\underline{b}$ has finite index in $\underline{a}$. (They are $\underline{\underline{Z}}$-lattices in the same two sided ideal of $\underline{\underline{Q}}\pi$.) According to (IX, 3.11) therefore $SK_1(\underline{\underline{Z}}\pi, \underline{b}) \longrightarrow SK_1(\underline{\underline{Z}}\pi, \underline{a})$ is surjective. Moreover (IX, 5.8) implies that $SK_1(\underline{\underline{Z}}\pi, \underline{b}) = SK_1(B, \underline{b})$, because $\underline{b}$ is a B-ideal. Now $SK_1(B, \underline{b})$ is the direct sum of those $SK_1(A_i, \rho_i(\underline{c}_i))$ for which $\rho_i(\sigma) \neq \{1\}$, so the proof will be complete if we show that none of these have p-torsion. The condition $\rho_i(\sigma) \neq \{1\}$implies ρ_i is faithful on π_p, because the latter is cyclic. Therefore, in the notation (7.2), $p^n|m_i$ and $p \nmid n_i$. Thus, in this case, $SK_1(A_i, \rho_i(\underline{c}_i)) \simeq \mu_{r_i}$ where

$v_p(r_i) \leq \min(v_p(n_i), v_p(m_i) = 0$. q.e.d.

Remark. There are no examples known of abelian π as above for which $SK_1(\underline{\underline{Z}}\pi) \neq 0$. Lam [1] and Kervaire (unpublished) have shown that $SK_1(\underline{\underline{Z}}\pi) = 0$ if π is an abelian p-group with two generators, one of order p. Lam has also shown that $SK_1(\underline{\underline{Z}}\pi)$ has no p-torsion if π is any finite abelian group such that $[\pi_p : 1] = p^2$. If there is an example for which $SK_1(\underline{\underline{Z}}\pi) = 0$ the elementary p-groups seem a likely place to look for one. The group of type (p, p, p) with p = 3 is the first unsettled case.

(7.4) THEOREM. Assume, in (0) above, that L is a cyclotomic field. Then

$$f: G_1(B) \longrightarrow G_1(A)$$

is an isomorphism, and $G_1(B) \simeq \prod_i U(A_i)$.

Proof. We have a commutative triangle

$$\begin{array}{ccc} G_1(B) & \xrightarrow{\quad f \quad} & G_1(A) \\ & {\scriptstyle g_1}\searrow \quad \swarrow & \\ & G_1(L\pi) & \end{array} .$$

Since B is regular we have $G_1(B) = K_1(B) = \Pi K_1(A_i) = \Pi U(A_i)$ (by (VI, 7.4)), and the last equality implies g_1, and hence also f, is a monomorphism. It follows from (IX, 5.9) that

$$G_1(B) \oplus G_1(A/\underline{c}) \longrightarrow G_1(A)$$

is surjective. Hence it suffices to show that the image of $G_1(A/\underline{c}) \longrightarrow G_1(A)$ lies in Im(f). If $M \in \underline{\underline{M}}(A, \underline{c})$ then M has a characteristic finite filtration, $0 = M_0 \subset M_1 \subset \ldots \subset M_n$ such that each M_j/M_{j-1} is annihilated by some $\underline{m} \in \operatorname{spec}(A)$. Since $A/\underline{c}$ is Artinian we must have $\underline{m} \in \max(A)$. Let $\underline{p}$ be a minimal prime of A contained in $\underline{m}$. Then $A/\underline{p} = A_i$ for some i, so M_j/M_{j-1} is a B-module. (This uses the fact that L is a cyclotomic field, which implies that the projections ρ_i: $A \longrightarrow A_i$ are surjective.) Now if $\alpha \in \mathrm{Aut}_A(M)$ then α leaves each M_j invariant and induces, say α_j, on M_j/M_{j-1}. Then in $G_1(A)$, $[M, \alpha] = \Sigma[M_j/M_{j-1}, \alpha_j] \in \mathrm{Im}(f)$. q.e.d.

The results above for abelian groups, together with Artin induction, now imply the following result for arbitrary finite groups.

(7.5) THEOREM. *Let* π *be any finite group, and consider the commutative diagram*

$$(1) \qquad \begin{array}{ccccccc} 0 & \longrightarrow & SK_1(\underline{\underline{Z}}\pi) & \longrightarrow & K_1(\underline{\underline{Z}}\pi) & \xrightarrow{k_1} & K_1(\underline{\underline{Q}}\pi) \\ & & & & \downarrow{\scriptstyle c_1(\underline{\underline{Z}}\pi)} & & \downarrow{\scriptstyle c_1(\underline{\underline{Q}}\pi)} \\ & & & & G_1(\underline{\underline{Z}}\pi) & \xrightarrow[g_1]{} & G_1(\underline{\underline{Q}}\pi) \end{array}$$

with exact top row (see (X, 3.6)).

(a) $SK_1(\underline{\underline{Z}}\pi)$ *is a finite group of exponent* $e_{(\text{cyclic})}(\underline{\underline{Q}}, \pi)^2$.

(b) $\mathrm{Im}(k_1)$ *is a finitely generated group of rank* = (rank $K_0(\underline{\underline{R}}\pi)$ - rank $K_0(\underline{\underline{Q}}\pi)$), *and whose torsion subgroup has exponent* $e_{(\text{cyclic})}(\underline{\underline{Q}}, \pi)^2 \cdot e$, *where* $e = \exp(\pi)$, *or* $2\exp(\pi)$ *if* $[\pi: 1]$ *is odd.*

(c) *The cokernel of* $c_1(\underline{\underline{Z}}\pi)$ *is finite, and* $\mathrm{Ker}(g_1)$ *is a finite group of exponent* $e_{(\text{cyclic})}(\underline{\underline{Q}}, \pi)^2$.

Proof. The diagram (1) consists of morphisms of Frobenius modules over the Frobenius functor $G_{\underline{\underline{Z}}}$. Hence all all of the estimates or exponents follow from (3.1) and

(3.3), once we verify the appropriate estimates when π is cyclic.

If π is cyclic then $SK_1(\underline{\underline{Z}}\pi) = 0$ (see (7.3)) and g_1 is injective (this follows immediately from (7.4)). Moreover (7.1) (a) implies $\mathrm{Im}(k_1)$ has torsion subgroup $\pm\pi$ when π is cyclic. The formula for rank $K_1(\underline{\underline{Z}}\pi)$ follows from (X, 3.5 (b)) in the general case. This establishes everything except the finiteness of $\mathrm{Coker}(c_1(\underline{\underline{Z}}\pi))$, and that follows from (X, 3.5 (c)). q.e.d.

HISTORICAL REMARKS

As stated in the introduction, the major portion of the material in this chapter is taken from Swan [1] and [3] and from Lam [1]. The material on the classical induction theorems can all be found in Curtis-Reiner [1], though the exposition here was greatly influenced by Serre [4].

The example discussed at the end of §5 was communicated to me by Milnor. The calculation in §6 is adapted from Bass-Murthy [1]. The applications to K_1 in §7 are based, in part, on results of Higman [1], Milnor (see Bass [3]), Bass-Milnor-Serre [1], Lam [1], and Kervaire (unpublished).

Chapter XII

POLYNOMIAL AND RELATED EXTENSIONS: THE FUNDAMENTAL THEOREM

Let T be an infinite cyclic group with generator t. This chapter studies the groups $K_i(A[t])$ and $K_i(A[T])$ (i = 0, 1), as well as $G_0(A[t])$. The results are most effective when A is regular, so we begin (in §2) by proving Hilbert's Syzygy Theorem which asserts that A[t] and A[T] are regular whenever A is. This is, of course, important for induction arguments when extending the theorems to several variables.

The first main result is a theorem of Grothendieck which asserts that $G_0(A) \longrightarrow G_0(A[t]) \longrightarrow G_0(A[T])$ are isomorphisms for noetherian A. The analogue for K_0 follows from this when A is regular.

In §5 we compute $\text{Ker}(K_1(A[t]) \longrightarrow K_1(A))$, via the the augmentation $t \longmapsto 1$, and show that its elements are represented by unipotents of the form $I + \nu(t - 1)$ where ν is a nilpotent matrix over A. It then follows, when A is regular, that $K_1(A[t]) \longrightarrow K_1(A)$ is an isomorphism.

In general the kernel above is not zero, and we show that it is isomorphic to the Grothendieck group, Nil(A), of the category of pairs (P, ν) ($P \in \underline{\underline{P}}(A)$, $\nu \in \text{End}_A(P)$, ν nilpotent) modulo those pairs of the form (P, 0).

In contrast with its analogue for K_0, the map

$K_1(A[t]) \longrightarrow K_1(A[T])$ is not an isomorphism, even when A is regular. Indeed, for commutative A, this is obvious because $t \in U(A[T])$ but t is not invertible in A[t]. Thus $K_1(A[T])$ contains at least a copy of T in addition to $K_1(A)$. It turns out that it even contains a copy of $K_o(A) \otimes T$, and that this gives a natural embedding of $K_o(A)$ as a direct summand of $K_1(A[T])$. The precise formulation of this result (Theorem (7.4)) shows that there is a canonical decomposition

$$K_1(A[T]) = K_1(A) \oplus Nil_+(A) \oplus Nil_-(A) \oplus K_o(A),$$

where $Nil_{\pm}(A)$ are two copies of Nil(A).

This theorem has a number of important applications. One principle to which it gives rise is that general theorems about K_1 imply (via the Fundamental Theorem) general theorems about K_o. The first application of this is that the Fundamental Theorem itself has an analogue for K_o. In the latter there appears a functor which bears the same relation to K_o that K_o bears to K_1. Accordingly, it is called K_{-1}. Now this procedure can be iterated yielding K_{-2}, K_{-3},... Finally, in §8, these functors are fitted into a "long Mayer-Vietoris sequence". Further, similar considerations apply to the functor Nil, and to various others to which the construction gives rise.

In order to organize notation efficiently we introduce, in §7, the notation of a "contracted functor". The definition is contrived so that the Fundamental Theorem says, essentially, that K_1 is a contracted functor. It further says that $K_o = LK_1$ and $Nil = NK_1$, where, for any functor F from rings to abelian groups, LF and NF are certain functors derived from F. The formalism then consists in showing that, if F is a contracted functor then NF and LF are also, and NLF = LNF. The Fundamental Theorem can be abbreviated by writing $K_1(A[T]) = (1 + 2N + L)\, K_1(A)$, with $NK_1 = Nil$ and $LK_1 = K_o$. This first formula applies to any contracted functor, in particular to LK_1 and NK_1. Therefore an induction on n shows, for example that

$$K_1(A[T^n]) = (1 + 2N + L)^n K_1(A),$$

where $A[T^n]$ is the group ring of a free abelian group of rank n.

The long Mayer-Vietoris sequence is derived by showing that if (F_1, F_0) are contracted functors which fit into a (six term) Mayer-Vietoris sequence, then the same holds for (NK_1, NF_0) and (LF_1, LF_0). Since $(K_1, K_0) = (K_1, LK_1)$ is such a pair, so also is $(L^nK_1, L^{n+1}K_1)$ for all $n \geq 0$, and we can therefore splice together a long Mayer-Vietoris sequence.

In §9 the results here are used to compute K_0 of a category which, for a commutative ring A, is equivalent to the category of algebraic vector bundles on the projective line over A. The computation is not quite definitive, however.

Some of the main calculations of Bass-Murthy [1] are deduced in §10. These are greatly clarified by the introduction of the operations L and N above.

In §11 we prove a theorem of Stallings on K_1 of a free product. By the general principle stated above, this implies the corresponding result for K_0. The latter is a theorem of Gersten which was proved in precisely this way.

§1. THE CHARACTERISTIC SEQUENCE OF AN ENDOMORPHISM.

Throughout this section t will denote an indeterminate. If A is a ring and if $M \in$ mod-A we shall identify $M \otimes_A A[t]$ with

$$M[t] = \{\Sigma_{i \geq 0} m_i t^i \mid m_i \in M \text{ and } m_i = 0 \text{ for almost all } i\}.$$

If $f: M \longrightarrow N$ is a homomorphism of A-modules then we have

$$f[t]: M[t] \longrightarrow N[t]; \; f[t]\,(\Sigma\, m_i t^i) = \Sigma\, f(m_i) t^i,$$

which is the A[t]-homomorphism corresponding to $f \otimes_A A[t]$. Since A[t] is A-free the functor $M \longmapsto M[t]$ is exact.

If $M \in$ mod-A[t] then it is determined completely as an A[t]-module by (i) the underlying A-module M, and (ii) the A-endomorphism $f: M \longrightarrow M$, $f(m) = mt$. Moreover these data may be prescribed arbitrarily. Thus, if $M \in$ mod-A and if $f \in \mathrm{End}_A(M)$ then we define an A[t]-module,

$$M_f,$$

to be M as additive group, and with A[t]-operation $m(\Sigma\, a_i t^i) = \Sigma\, f^i(m) a_i$. In this way we obtain an isomorphism of mod-A[t] with the category of endomorphisms of objects of mod-A.

Given $M \in$ mod-A and $f \in \mathrm{End}_A(M)$ as above, there is a canonical A[t]-epimorphism

$$\phi_f: M[t] \longrightarrow M_f, \; \phi_f(\Sigma\, m_i t^i) = \Sigma\, f^i(m_i).$$

The _characteristic sequence_ of f is the exact sequence (1) below.

(1.1) PROPOSITION. _Let_ $M \in$ mod-A _and_ $f \in \mathrm{End}_A(M)$. _Then_

$$(1) \qquad 0 \longrightarrow M[t] \xrightarrow{t \cdot 1_{M[t]} - f[t]} M[t] \xrightarrow{\phi_f} M_f \longrightarrow 0$$

is an exact sequence in mod-A[t]. _Moreover_ $(M, f) \longmapsto (1)$ _defines an exact functor from the category of endomorphisms of A-modules (which we can identify with_ mod-A[t]) _to the category of short exact sequences in_ mod-A[t].

Proof. It is clear that (1) is functorial in (M, f). It is exact because $(M, f) \longmapsto M_f$ and $(M, f) \longmapsto M[t]$ are (evidently) exact.

It remains to show that (1) is an exact sequence. We have seen that ϕ_f is surjective, and we have

$$\phi_f(t \cdot 1_{M[t]} - f[t]) (\Sigma\, m_i t^i)$$
$$= \phi_f(\Sigma(m_i t^{i+1} - f(m_i)t^i)$$
$$= \Sigma(f^{i+1}(m_i) - f^i(f(m_i))) = 0.$$

Since $t \cdot 1_{M[t]} - f[t]$ raises degree by one, and preserves "leading coefficient", it is a monomorphism. Finally suppose $x = \Sigma\, m_i t^i \in \mathrm{Ker}(\phi_f)$, i.e. $\Sigma\, f^i(m_i) = 0$. Then $x = x - \Sigma\, f^i(m_i) = \Sigma_{i \geq 0} (m_i t^i - f^i(m_i)) = \Sigma_{i > 0}(t^i \cdot 1_{M[t]} - f^i)(m_i)$ $= (t \cdot 1_{M[t]} - f[t])\, \Sigma_{i > 0}\, h_i(m_i) \in \mathrm{Im}(t \cdot 1_{M[t]} - f[t])$, where $h_i = \Sigma_{0 \leq j < i}\, t^j 1_{M[t]}\, f^{i-j}[t]$. q.e.d.

Remark. Suppose above that A is commutative and that M is a free A-module with basis $e_1, \ldots, e_n$. Then $M[t]$ is a free $A[t]$-module with the same basis, and $f[t]$ is represented by the same matrix as f. Thus

$$P_f(t) = \det(t \cdot 1_{M[t]} - f[t])$$

is the characteristic polynomial of f. But (see (IX, 6.6 (a)) over a commutative ring B, if $g: B^n \longrightarrow B^n$ is a B-homomorphism, $\mathrm{Coker}(g) \cdot \det(g) = 0$. Applying this to the characteristic sequence of f above we see that $P_f(t)$ annihilates M_f. However the endomorphism of M_f defined by $P_f(t)$ is just $P_f(f)$. Thus we have the Cayley-Hamilton Theorem: $P_f(f) = 0$.

§2. THE HILBERT SYZYGY THEOREM.

It asserts that rt. gl. dim. A[t] = 1 + rt. gl. dim.A (see (2.2)).

(2.1) PROPOSITION (Kaplansky). _Let_ B _be a ring and let_ $A = B/tB$ _where_ t _is a non divisor of zero which lies in the center of_ B. _If_ $M \neq 0$, $M \in$ mod-A, _and if_ $hd_A(M) = n < \infty$, _then_ $hd_B(M) = n + 1$.

Proof. The exact sequence $0 \longrightarrow B \xrightarrow{t} B \longrightarrow A \longrightarrow 0$ shows that a projective A-module P has $hd_B(P) \leq 1$. If $P \neq 0$ equality must hold because, since $Pt = 0$, P cannot be B-projective. By induction now, assume $n > 0$, and let $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$ with P A-projective. Then $N \neq 0$ and $hd_A(N) = n - 1$ so, by induction, $hd_B(N) = n$, and $hd_B(P) = 1$. It follows now from (I, 6.8) that $hd_B(M) \leq n + 1$, with equality if $n > 1$.

If $n = 1$ write $M = Q/H$ with Q B-projective. Then we have exact sequences of A-modules,

$$0 \longrightarrow H/Qt \longrightarrow Q/Qt \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow Qt/Ht \longrightarrow H/Ht \longrightarrow H/Qt \longrightarrow 0.$$

Since Q/Qt is A-projective so also is H/Qt (because $n = 1$) and hence the second sequence splits. Thus $M \simeq Qt/Ht$ is a direct summand of H/Ht, showing that H/Ht cannot be A-projective. Therefore H cannot be B-projective, so $hd_B(M) > 1$. q.e.d.

(2.2) THEOREM (Hilbert). _Let_ A _be a ring and let_ t _be an indeterminate. Then_

$$\text{rt. gl. dim } A[t] = \text{rt. gl. dim. } A[t, t^{-1}]$$
$$= 1 + \text{rt. gl. dim. } A.$$

Proof. We shall carry out the proof in three steps, which establish more precise results. Put $B_o = A[t]$ and $B_1 = A[t, t^{-1}]$.

(i) *If* $N \in$ mod-B_i *then* $hd_A(N) \leq hd_{B_i}(N)$ $(i = 0, 1)$.

For since B_i is A-free, a B_i-projective resolution of N is also, by restriction, an A-resolution.

(ii) *If* $M \in$ mod-A *then* $hd_{B_i}(M \otimes_A B_i) = hd_A(M)$ $(i = 0, 1)$.

As A-module, $M \otimes_A B_i$ is a direct sum of copies of M, so part (i) implies $\geq$. If $P \longrightarrow M$ is an A-projective resolution, then $P \otimes_A B_i \longrightarrow M \otimes_A B_i$ is a B_i-projective resolution, since $\otimes_A B_i$ is exact This establishes the opposite inequality.

(iii) *Let* $s = t - 1$. *If* $M \in$ mod-B_i $(i = 0$ or $1)$ *then*

(1) $$hd_{B_i}(M) \leq 1 + hd_A(M),$$

with equality if $M = 0$ *and* $Ms^n = 0$ *for some* $n > 0$.

The last assertion follows immediately from (2.1) if $hd_A(M) < \infty$, and it follows from (i) above if $hd_A(M) = \infty$. Since $B_1 = T^{-1}B_o$, where $T = \{t^n \mid n \geq 0\}$, we have, for $M \in$ mod-B_1, $M = T^{-1}M$, so $hd_{B_1}(M) \leq hd_{B_o}(M)$, since localization is exact. Hence it suffices to establish (1) for $i = 0$; $B_o = A[t]$.

We can write $M = M_f$ (f = multiplication by t), and then we have the characteristic sequence (1.1):

$$0 \longrightarrow M[t] \longrightarrow M[t] \longrightarrow M \longrightarrow 0.$$

According to (ii) $hd_{B_o}(M[t]) = hd_A(M)$. It follows from (I, 6.8) that $hd_{B_o}(M) \leq 1 + hd_{B_o}(M[t])$, thus proving (iii).

Evidently the theorem follows from (iii), since $A = B_i/sB_i$ ($i = 0, 1$).

In the next two results T denotes a free abelian group or monoid on one generator t, and T^n denotes a product of n copies of T. By induction on n, (2.2) implies

(2.3) COROLLARY. rt. gl. dim. $A[T^n]$ = n + rt. gl. dim. A.

(2.4) THEOREM (Swan). _Let_ A _be a right noetherian ring and let_ S _be a central multiplicative set in_ A.

(a) _If_ A _is right regular then_ $A[T^n]$ _is also_.

(b) _If_ S _is regular for_ A _then_ S _is also regular for_ $A[T^n]$.

Proof. Part (a) follows from part (b) in the special case $0 \in S$, and part (b) follows, by induction, from the case n = 1, with the aid of the Hilbert Basis Theorem.

We can write $A[t, t^{-1}] = U^{-1}A[t]$, where $U = \{t^n \mid n \geq 0\}$. Therefore, the result for $A[t, t^{-1}]$ will follow once we know it for A[t], thanks to the following general fact: If S and U are central multiplicative sets in a right noetherian ring B, and if S is regular for B, then S is also regular for $U^{-1}B$. For, given any $M \in \underline{\underline{M}}(U^{-1}B)$, we can write $M = U^{-1}N$ where $N \in \underline{\underline{M}}(B)$, and we can choose $N \subset M$. Therefore if $Ms = 0$ for some $s \in S$ we have $Ns = 0$ also. Assuming S is regular for B the desired conclusion, $hd_{U^{-1}B}(M) < \infty$, now follows from $hd_B(N) < \infty$ and $hd_{U^{-1}B}(U^{-1}N) \leq hd_B(N)$.

Finally, we must show that if $M \in \underline{\underline{M}}(A[t])$ and if $Ms = 0$ for some $s \in S$, then $hd_{A[t]}(M) < \infty$. According to part (iii) of the proof of (2.2) we have $hd_{A[t]}(M) \leq 1 + hd_A(M)$, so it suffices to show that $hd_A(M) < \infty$. Let $M_o \subset M$ be a finitely generated A-submodule which generates M as an A[t]-module: $M = \sum_{i\geq 0} M_o t^i$. Put $M_n = M_o + M_o t + \ldots + M_o t^n$; then $M = \mathrm{colim}(M_n)$ as an A-module. Hence it suffices to prove:

(i) $hd_A(M_n)$ is bounded as $n \longrightarrow \infty$.

(ii) $hd_A(\mathrm{colim}(M_n)) \leq 1 + \sup_{n\geq 0} hd_A(M_n)$.

Proof of (i). Consider the sequence of epimorphisms $M_o \xrightarrow{t} M_1/M_o \xrightarrow{t} M_2/M_1 \longrightarrow \ldots$. Since M_o is a noetherian A-module there is an n_o such that $M_n/M_{n-1} \xrightarrow{t} M_{n+1}/M_n$ is an isomorphism for all $n \geq n_o$. Let $d = \max(hd_A(M_o), hd_A(M_{n_o}/M_{n_o-1}))$. By induction on $n \geq n_o$ we claim that $hd_A(M_n) \leq d$. This is obvious for $n = n_o$. If $n > n_o$ we use the exact sequence $0 \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_n/M_{n-1} \longrightarrow 0$ and the isomorphism $M_n/M_{n-1} \simeq M_{n_o}/M_{n_o-1}$, together with the induction assumption $hd_A(M_{n-1}) \leq d$, to conclude that $hd_A(M_n) \leq d$.

Since each $M_n \in \underline{\underline{M}}(A)$ and $M_n s = 0$ we have $hd_A(M_n) < \infty$ for each n, since S is regular for A. Now $\sup_{0\leq n} hd_A(M_n) \leq \sup(d, hd_A(M_o),\ldots,hd_A(M_{n_o})) < \infty$, by the last paragraph. q.e.d.

Proof of (ii). We have an exact sequence

$$0 \longrightarrow \coprod_{n\geq 0} M_n \xrightarrow{j} \coprod_{n\geq 0} M_n \xrightarrow{f} \mathrm{colim}(M_n) \longrightarrow 0$$

defined by $j(m_o, m_1, m_2,\ldots) = (m_o, m_1 - m_o, m_2 - m_1,\ldots)$ and $f(m_o, m_1, m_2,\ldots) = \Sigma m_i$. (A similar construction can be made for any colimit of a sequence of modules.) Hence $hd_A(\mathrm{colim}(M_n)) \leq 1 + hd_A(\coprod_{n\geq 0} M_n) = 1 + \sup_{n\geq 0} hd_A(M_n)$. q.e.d.

§3. GROTHENDIECK'S THEOREM FOR $K_o(A[T])$: SERRE'S PROOF

The theorem is:

(3.1) THEOREM. Let A be a right regular ring and let T be a free abelian monoid or group. Then $K_0(A) \longrightarrow K_0(A[T])$ is an isomorphism.

Since K_0 commutes with direct limits one can assume, for the proof, that T has a basis of finite cardinality n. Then, thanks to Hilbert's Basis Theorem (III, 3.6) and Syzygy Theorem (2.4), an induction on n reduces this further to the case n = 1, in which case the theorem asserts that i and j in

$$K_0(A) \xrightarrow{i} K_0(A[t]) \xrightarrow{j} K_0(A[t, t^{-1}])$$

are isomorphisms. Since $A[t] \longrightarrow A[t, t^{-1}]$ is a localization of a right regular ring it follows from (IX, 6.5) that j is surjective. Moreover ji has a left inverse, induced by the augmentation $A[t, t^{-1}] \longrightarrow A$. Thus the theorem will follow if i is surjective. This, in turn, follows from the more general:

(3.2) THEOREM. Let $A = \coprod_{n \geq 0} A_n$ be a graded right regular ring. Then the inclusion of A_0 in A induces an isomorphism $K_0(A_0) \longrightarrow K_0(A)$.

The proof of (3.2) requires some preliminary observations. Let $A = \coprod_{n \geq 0} A_n$ be any graded ring. We shall agree that $A_n = 0$ for $n < 0$. A graded right A-module M is a right A-module together with a decomposition $M = \coprod_n M_n$ such that $M_n A_m \subset M_{n+m}$ $(n, m \in \mathbb{Z})$. These are the objects of the category gr mod-A in which a morphism $f: M \longrightarrow N$ must be A-linear and such that $f(M_n) \subset N_n$ for all n. We have the full subcategories

$$\text{gr } \underline{\underline{P}}(A) \subset \text{gr } \underline{\underline{M}}(A) \subset \text{gr mod-}A$$

whose objects are those which belong to $\underline{\underline{P}}(A)$ and $\underline{\underline{M}}(A)$, respectively, as ungraded modules. If $M \in \text{gr } \underline{\underline{M}}(A)$ then, since any generating set contains a finite one, M is generated by a finite set of homogeneous elements. In particular $M_n = 0$ for all sufficiently small n, i.e. M is "bounded below".

Suppose h: F $\longrightarrow$ P is a morphism in gr mod-A and there is a homomorphism g: P $\longrightarrow$ F of ungraded modules such that $hg = 1_P$. If we set g_n' equal to the composite $P_n \subset P \xrightarrow{g} F \longrightarrow F_n$, then $g' = \amalg g_n': P \longrightarrow F$ is easily seen to be a morphism in gr mod-A such that $hg' = 1_P$.

If M ε gr mod-A and h ε $\underline{\underline{Z}}$ then M(h) ε gr mod-A is defined by $M(h)_n = M_{n+h}$. A free graded module is defined to be a direct sum of modules of the form A(h). Every M ε gr $\underline{\underline{M}}(A)$ is a quotient of a free module in gr $\underline{\underline{P}}(A)$, since M has a finite number of homogeneous generators. If P ε gr $\underline{\underline{P}}(A)$ choose an epimorphism h: F $\longrightarrow$ P with F a free module in gr $\underline{\underline{P}}(A)$. Then there is a homomorphism g: P $\longrightarrow$ F of ungraded modules such that $hg = 1_P$. Thus we conclude from the last paragraph that P is a direct summand in gr $\underline{\underline{P}}(A)$ of a free module.

(3.3) PROPOSITION (Swan). Let $A = \amalg_{n\geq 0} A_n$ be a graded ring. Then $\otimes_{A_0} A$: gr $\underline{\underline{P}}(A_0) \longrightarrow$ gr $\underline{\underline{P}}(A)$ induces a bijection on isomorphism classes of objects.

Proof. We view A_0 as a graded ring concentrated in degree zero, and we can identify A_0 with A/A_+, where $A_+ = \amalg_{n>0} A_n$ is a homogeneous ideal. Therefore we have the functor T: gr $\underline{\underline{P}}(A) \longrightarrow$ gr $\underline{\underline{P}}(A_0)$, $T(P) = P \otimes_A A_0 = P/PA_+$. If P_0 ε gr $\underline{\underline{P}}(A_0)$ then evidently there is a natural isomorphism $P_0 \simeq T(P_0 \otimes_{A_0} A)$. Next suppose P ε gr $\underline{\underline{P}}(A)$. The projection f: P $\longrightarrow$ T(P) is an epimorphism in gr mod-A_0, so there is a homomorphism g: T(P) $\longrightarrow$ P, of graded A_0-modules, such that fg: $1_{T(P)}$. This yields $h = g \otimes_{A_0} A: T(P) \otimes_{A_0} A \longrightarrow P$, a morphism in gr $\underline{\underline{P}}(A)$, and clearly T(h) is an isomorphism. Since T is right exact we have T(Coker(h)) = 0. But Coker(h) is bounded below, and evidently $M/MA_+ = 0 \Rightarrow M = 0$ if M is bounded below. Therefore h is surjective. But P is projective so h is split, and by a morphism in gr $\underline{\underline{P}}(A)$, as we saw above. Therefore Ker(h) ε gr $\underline{\underline{P}}(A)$, and $\overline{\overline{T}}(\mathrm{Ker}(h)) = 0$. Just as above this implies Ker(h) $\overline{\overline{=}}$ 0. Thus $P \simeq T(P) \otimes_{A_0} A$.

q.e.d.

With A graded, as above, we grade the polynomial ring, A[t], by

$$A[t]_n = \coprod_{0 \le i \le n} A_i t^{n-i}.$$

We shall identify A with A[t]/(1 - t) A[t]. Note that this projection A[t] ⟶ A is <u>not</u> a homomorphism of graded rings.

(3.4) PROPOSITION. <u>Let</u> A <u>be a right noetherian graded ring. The functor</u>

$$\otimes_{A[t]} A: \mathrm{gr}\ \underline{\underline{M}}(A[t]) \longrightarrow \underline{\underline{M}}(A)$$

<u>is exact, and it induces a surjection on isomorphism classes of objects</u>.

<u>Proof</u>. Given M ε $\underline{\underline{M}}(A)$ write $M = A^n/N$, where N is generated by elements $\alpha_i = (a_{i_1},\dots,a_{in})$ $(1 \le i \le m)$. Let d be an upper bound for the degrees of all non zero homogeneous components of all a_{ij}. If $a = a_0 + a_1 + \dots + a_d \in A$, $(a_i \in A_i)$ set $a' = a_0 t^d + a_1 t^{d-1} + \dots + a_d \in A[t]_d$. Let $N' \subset A[t]^n$ be the submodule generated by $\alpha_i' = (a_{i1}',\dots,a_{in}')$ $(1 \le i \le m)$, and set $M' = A[t]^n/N'$. Then $0 \longrightarrow N' \xrightarrow{g} A[t]^n \xrightarrow{h} M' \longrightarrow 0$ is an exact sequence in gr $\underline{\underline{M}}(A[t])$, which induces

$N' \otimes_{A[t]} A \longrightarrow A^n \longrightarrow M' \otimes_{A[t]} A \longrightarrow 0$. Clearly

$\mathrm{Im}(g \otimes_{A[t]} A)$ is the submodule generated by all $\alpha_i' = \alpha_i'(t)$ at the special value t = 1. But $\alpha_i'(1) = \alpha_i$ so $\mathrm{Im}(g \otimes_{A[t]} A) = N$, and thus $M \simeq M' \otimes_{A[t]} A$. This proves the last assertion of the proposition.

The functor is right exact, so the proof of exactness reduces to showing that if $M' \subset M$ in gr $\underline{\underline{M}}(A[t])$ then the inclusion $M(1 - t) \subset M' \cap M'(1 - t)$ is an equality. Suppose $m = m_i + m_{i+1} + \ldots \in M$ and $m(1 - t) \in M'$. Since the homogeneous components of $m(1 - t) = m_i + (m_{i+1} - m_i t) + (m_{i+2} - m_{i+1} t) + \ldots$ belong to M' we conclude, by induction on j (≥ 0) that each $m_j \in M'$; i.e. $m \in M'$. q.e.d.

<u>Proof of (3.2)</u>. We claim that $i: K_0(A_0) \longrightarrow K_0(A)$ is an isomorphism, where A is graded and right regular. The retraction $A \longrightarrow A_0 = A/A_+$ induces a left inverse for i, so we need only show that i is surjective.

Given $P \in \underline{\underline{P}}(A)$ there is an $M \in$ gr $\underline{\underline{M}}(A[t])$ such that $M \otimes_{A[t]} A \simeq P$, thanks to (3.4); here we identify A with $A[t]/(1 - t) A[t]$. By the Syzygy Theorem (2.4) A[t] is right regular. Say $hd_{A[t]} (M) = n$. Choose an exact sequence E = $(0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0)$ with $P_i \in$ gr $\underline{\underline{P}}(A[t])$ $(0 \leq i < n)$. Then automatically $P_n \in$ gr $\underline{\underline{P}}(A[t])$. According to (3.4), $E \otimes_{A[t]} A$ is an exact sequence in $\underline{\underline{M}}(A)$. Since $M \otimes_{A[t]} A \simeq P$ we conclude that, in K (A), $[P] = \Sigma (-1)^i [P_i \otimes_{A[t]} A]$. Since $A[t]_0 = A_0$ it follows from (3.3) that, for each i, $P_i \simeq Q_i \otimes_{A_0} A[t]$ for some $Q_i \in$ gr $\underline{\underline{P}}(A_0)$. Therefore $P_i \otimes_{A[t]} A \simeq (Q_i \otimes_{A_0} A[t]) \otimes_{A[t]} A \simeq Q_i \otimes_{A_0} A$, so $[P] = \Sigma (-1)^i [Q_i \otimes_{A_0} A] \in \mathrm{Im}(K_0(A_0) \longrightarrow K_0(A))$. q.e.d.

The following corollary of (3.1) is sometimes useful.

(3.5) COROLLARY. <u>Let</u> B <u>be a ring with a two sided nilpotent ideal</u> J <u>such that</u> A = B/J <u>is right regular. Let</u> T <u>be a free abelian group or monoid. Then</u> $K_0(B) \longrightarrow K_0(B[T])$ <u>is an isomorphism</u>.

<u>Proof</u>. The ideal JB[T] is also nilpotent, so (IX, 1.3 (0)) implies that the verticles in the commutative square

$$\begin{array}{ccc} K_0(B) & \longrightarrow & K_0(B[T]) \\ \downarrow & & \downarrow \\ K_0(A) & \longrightarrow & K_0(A[T]) \end{array}$$

are isomorphisms. The corollary now follows by applying (3.1) to the bottom arrow. q.e.d.

This corollary applies notably when B is an Artin ring.

§4. GROTHENDIECK'S THEOREM FOR $G_0(A[T])$: GROTHENDIECK'S PROOF

The theorem is:

(4.1) THEOREM. *Let* R *be a commutative noetherian ring, let* A *be a finite* R-*algebra, and let* T *be a free abelian monoid or group with a finite basis. Then* $G_0(A) \longrightarrow G_0(A[T])$ *is an isomorphism.*

Proof. By induction on the number n of generators of T we reduce, with the aid of the Hilbert Basis Theorem, to the case n = 1, in which case we claim that i and j in

$$G_0(A) \xrightarrow{\ i\ } G_0(A[t]) \xrightarrow{\ j\ } G_0(A[t, t^{-1}])$$

are isomorphisms. Since $A[t, t^{-1}]$ is a localization of $A[t]$ it follows that j is surjective. Let $s = 1 - t$ and consider the augmentation $A[t, t^{-1}] \longrightarrow A$ ($s \longmapsto 0$). The resulting functor $\otimes_{A[t, t^{-1}]} A$ is not exact, but it has an exact restriction to the full subcategory $\underline{\underline{M}}_0(A[t, t^{-1}]) \subset \underline{\underline{M}}(A[t, t^{-1}])$ consisting of modules M on which multiplication by s is a monomorphism. If $M \in \underline{\underline{M}}(A[t, t^{-1}])$ we can take an exact sequence $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$ with $P \in \underline{\underline{P}}(A[t, t^{-1}])$ and then clearly $N \in \underline{\underline{M}}_0(A[t, t^{-1}])$. Therefore it follows from (VIII, 4.2) that

$$K_0(\underline{\underline{M}}_0(A[t, t^{-1}])) \longrightarrow K_0(\underline{\underline{M}}(A[t, t^{-1}]))$$
$$= G_0(A[t, t^{-1}])$$

is an isomorphism. Therefore we can define h: $G_0(A[t, t^{-1}]) \longrightarrow G_0(A)$ via the exact functor $\otimes_{A[t, t^{-1}]} A$: $\underline{\underline{M}}_0(A[t, t^{-1}]) \longrightarrow \underline{\underline{M}}(A)$. Evidently h is a left inverse for j · i. Therefore it will suffice to prove that i is surjective.

If $\underline{a}$ is a two sided ideal in A write

$$i_{\underline{a}}: G(A/\underline{a}) \longrightarrow G_0((A/\underline{a})[t]),$$

so that $i = i_{(0)}$. If i is not surjective choose a maximal $\underline{a}$ (by noetherian induction) so that $i_{\underline{a}}$ is not surjective. Replacing A by $A/\underline{a}$ we can then assume $i_{\underline{a}}$ is surjective for all $\underline{a} \neq 0$.

Suppose $M \in \underline{\underline{M}}(A[t])$, $M \neq 0$. Among the ideals $ann_R(x)$ ($x \in M$, $x \neq 0$) let $\underline{p}$ be a maximal one. One sees immediately that $\underline{p}$ is prime, and that $M_1 = \{x \in M \mid x\underline{p} = 0\}$ is an A-submodule of M which is a torsion free $(R/\underline{p})$-module. Repeating this construction on M/M_1, etc. and using noetherian induction, we conclude that M has a finite filtration with successive quotients which are torsion free $(R/\underline{p})$-modules for various $\underline{p} \in spec(R)$. Therefore, since i is not surjective, there is an $M \in \underline{\underline{M}}(A[t])$ and a $\underline{p} \in spec(R)$ such that $[M] \notin Im(i)$ and such that M is a torsion free $(R/\underline{p})$-module. If $\underline{a} = ann_A(M)$ we must have $\underline{a} = 0$, for otherwise we would contradict the surjectivity of $i_{\underline{a}}$. Thus A is torsion free R/p-algebra, so we can assume R is an integral domain. Let L be the field of fractions of R. Then A is an R-order in $A \otimes_R L$. If this finite dimensional algebra were not semi-simple we would have a nilpotent two sided ideal N in A. But we have a commutative square

$$\begin{array}{ccc} G_o(A) & \xrightarrow{\ i\ } & G_o(A[t]) \\ \downarrow & & \downarrow \\ G_o(A/N) & \xrightarrow[i_N]{} & G_o((A/N)[t]) \end{array}$$

in which i_N is surjective. Moreover (IX, 2.3) implies that the verticals are isomorphisms. Since i is not surjective this is a contradiction. We conclude that $A \otimes_R L$ is semi-simple. Write $L = S^{-1}R$, $S = R - \{0\}$. Then we have a commutative diagram

$$\begin{array}{ccccccc} K_o(\underline{\underline{M}}_S(A)) & \longrightarrow & G_o(A) & \longrightarrow & G_o(S^{-1}A) & \longrightarrow & 0 \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} & & \\ K_o(\underline{\underline{M}}_S(A[t])) & \longrightarrow & G_o(A[t]) & \longrightarrow & G_o(S^{-1}A[t]) & \longrightarrow & 0 \end{array}$$

with exact rows (see (IX, 6.2)). Here $\underline{\underline{M}}_S(A)$ is the category of $N \in \underline{\underline{M}}(A)$ such that $Ns = 0$ for some $s \in S$, and similarly for $\underline{\underline{M}}_S(\bar{A}[t])$. Thus i' is the direct limit, over $s \in S$, of the homomorphism i_{sA}, and hence i' is surjective. If we show that i'' is surjective then the diagram implies i is surjective, thus giving us the required contradiction.

But $S^{-1}A = A \otimes_R L$ is semi-simple, as we saw above. Hence $S^{-1}A$ and $S^{-1}A[t]$ are right regular, and i'' is isomorphic to $K_o(S^{-1}A) \longrightarrow K_o(S^{-1}A[t])$, which, by (3.1), is an isomorphism. q.e.d.

(4.2) COROLLARY. *In the setting of* (4.1) *let* S *be a multiplicative set in* R. *Then*

(1) $G_o(A, S) \longrightarrow G_o(A[T], S)$

is an isomorphism.

Proof. Recall that $G_i(A, S) = K_i(\underline{\underline{M}}_S(A))$ where $\underline{\underline{M}}_S(A)$ is the category of $M \in \underline{\underline{M}}(A)$ such that $S^{-1}M = 0$, and similarly for $G_i(A[T], S)$. Since $\underline{\underline{M}}_S(A)$ is the (directed) union of the subcategories $\underline{\underline{M}}(A/As)$ $(s \in S)$, and similarly for $A[T]$, we conclude that (1) is the direct limit of the homomorphisms

$$G_o(A/As) \longrightarrow G_o((A/As)[T]) \quad (s \in S).$$

Each of these is an isomorphism by (4.1). q.e.d.

(4.3) COROLLARY. (cf. (3.1)) *In the setting of* (4.2), *if* S *is regular for* A, *then*

$$K_o(A, S) \longrightarrow K_o(A[T], S)$$

is an isomorphism. In particular, if A *is right regular, then*

$$K_o(A) \longrightarrow K_o(A[T])$$

is an isomorphism.

Proof. It follows from (2.4) (b) that S is also regular for A[T], so the K_o's above coincide with the corresponding G_o's, and the present assertion reduces to (4.2). The last assertion is just the first one in the special case $S = \{0\}$. q.e.d.

§5. LINEARIZATION IN GL(A[t]).

The title refers to the following well known device, whose origin I can't determine (cf., for example, Higman [1]).

(5.1) PROPOSITION. *Let* R *be a subring of a ring* A *and let* $M \subset A$ *be an* R-*bimodule which, together with* R, *generates* A *as a ring. Let* $A_+ = AMA$ *be the* A-*ideal generated by* M. *Suppose* $\alpha = (\alpha_{ij})$ $i, j \geq i$ *is a "formally infinite"* matrix over A, i.e. one such that $\alpha_{ij} = \delta_{ij}$ for all sufficiently large i and j. Then there exist $\varepsilon_1, \varepsilon_2 \in E(A, A_+)$ such that $\varepsilon_1\alpha\varepsilon_2 = \alpha_o + \alpha_1$, where α_o is a formally infinite matrix over R, and where α_1 has all coefficients in M. If $\alpha \in GL(A)$ we can arrange that $\varepsilon_1 = I$.

Proof. The hypotheses imply that A is a quotient of the tensor algebra, $T_R(M)$. Thus $A = \Sigma M^d$ $(d \geq 0)$, where $M^o = R$ and $M^{d+1} = M^d \cdot M$.

We can write $\alpha = \begin{pmatrix} \beta & 0 \\ 0 & I \end{pmatrix}$ where β is an n x n matrix for some n. Further, $\beta = \beta_o + \ldots + \beta_d$ for some $d \geq 0$, where β_i has coefficients in M^i $(0 \leq i \leq d)$. We shall prove the proposition by induction on d. If $d \leq 1$ we can take $\varepsilon_1 = I = \varepsilon_2$, so assume $d > 1$. Then we can write $\beta_d = \Sigma\, \gamma_j x_j$ $(1 \leq j \leq m)$ where $x_j \in M$ and γ_j has coefficients in M^{d-1} $(1 \leq j \leq m)$. Now in the ring of matrices of size n(m + 1) we multiply $\begin{pmatrix} \beta & 0 \\ 0 & I_{nm} \end{pmatrix}$ first on the left, and then on the right, by elements of $E(A, A_+)$, to achieve the following transformations:

$$\begin{pmatrix} \beta & 0 \\ 0 & I_{nm} \end{pmatrix} \longmapsto \begin{pmatrix} \beta & \gamma_1 \cdot \cdot \gamma_m \\ 0 & I_{nm} \end{pmatrix} \longmapsto$$

$$\beta' = \begin{pmatrix} \beta - \beta_d & \gamma_1 \cdot \cdot \gamma_m \\ -I_n x_1 & \\ \cdot & I_{nm} \\ \cdot & \\ -I_n x_m & \end{pmatrix}$$

In case β is invertible we can achieve the first transformation also by right multiplication. Since β' has "degree" $\leq d-1$ we can complete this procedure by induction. q.e.d.

(5.2) PROPOSITION. *Let* $A = \coprod_{n \geq 0} A_n$ *be a graded ring, and let* $A_+ = \coprod_{n > 0} A_n$.

(a) *If* $a \in A_1$ *then* $1 - a \in U(A) \iff a$ *is nilpotent*.

(b) *Assume* A_o *and* A_1 *generate* A *as a ring*. If $\alpha \in GL(A)$ *then* α *admits a factorization*

$$\alpha = \alpha_o(I + \nu)\varepsilon$$

with $\alpha_o \in GL(A_o)$, $\varepsilon \in E(A, A_+)$, *and with* ν *a matrix having coefficients in* A_1. *Such a matrix* ν *is necessarily nilpotent*.

Proof. (a) If a is nilpotent then $(1 - a)^{-1} = \Sigma_{i \geq 0} a^i \in A$. Conversely, suppose $(1 - a)^{-1} = \Sigma\, b_i \in A$ $(b_i \in A_i)$. Then $1 = (1-a)(\Sigma\, b_i) = b_o + (b_1 - ab_o) + (b_2 - ab_1) + \ldots.$ By induction we see that $b_i = a^i$. Therefore $a^i = b_i = 0$ for large i.

(b) We can apply (5.1) with $R = A_o$ and $M = A_1$. Then we obtain $\varepsilon \in E(A, A_+)$ such that $\alpha\varepsilon^{-1} = \alpha_o + \alpha_1$ where α_i has coefficients in A_i $(i = 0, 1)$. Factoring out the ideal A_+ we see that $\alpha_o \in GL(A_o)$, so we can write $\alpha\varepsilon^{-1} = \alpha_o(I + \nu)$ where $\nu = \alpha_o^{-1}\alpha_1$. Write $\nu = \begin{pmatrix} \nu' & 0 \\ 0 & 0 \end{pmatrix}$ where $\nu' \in M_n(A_1)$ for some n. Since $I_n + \nu' \in GL_n(A) = U(M_n(A))$ it follows from part (a) that ν', and hence also ν, are nilpotent.

(5.3) COROLLARY. *Let* A *be as in* (5.2) (b). *Then*

$$K_1(A) = K_1(A_o) \oplus K_1(A, A_+)$$

and every element of $K_1(A, A_+)$ *is represented by a unipotent*

$I + \nu$ *where* ν *has coefficients in* A_1. *Moreover,*

(a) *If* $nA_1 = 0$ *for some* $n > 0$ *then every element of* $K_1(A, A_+)$ *has finite order dividing some power of* n.

(b) *If* A *is right regular then* $K_1(A, A_+) = 0$.

Proof. The direct sum decomposition follows from (IX, 2.6) because $GL(A) = GL(A, A_+) \times_{s-d} GL(A_0)$ is a semi-direct product. The assertion concerning unipotents follows directly from (5.2) (b).

(a). Suppose $\alpha = I + \beta \in GL_n(A)$ where β has coefficients in A_1, and hence is unipotent. In $M_n(A)$ let R be the (commutative) subring generated by I and β, and let $\underline{a}$ be the nilpotent ideal βR. By assumption we have $n\underline{a} = 0$, and say $\underline{a}^{d+1} = 0$. Then it follows from (X, 3.8 (c)) that $(I + \beta)^{n^d} = I$.

(b) follows immediately from the first assertion, thanks to (IX, 2.2).

(5.4) COROLLARY. *Let* A *be a right regular ring and let* T *be a free abelian monoid. Then*

$$K_1(A) \longrightarrow K_1(A[T])$$

is an isomorphism.

Proof. A[T] is a polynomial ring in several variables, so it has a natural grading. Moreover the Syzygy Theorem (2.4) implies that A[T] is right regular. Therefore the corollary follows from (5.3) (b) above.

Remark. Note that T above is not allowed to be a group. Indeed, the next two sections are devoted to an analysis of $K_1(A[t, t^{-1}])$. By the direct techniques of this section we can obtain partial results, as indicated in (5.6) below.

(5.5) COROLLARY (Gersten). *Let* A *be as in* (5.4) *and let* M *be an* A-*bimodule isomorphic to a coproduct of copies* *of* A. *Let* $B = T_A(M)$ *be the tensor algebra of* M *over* A. *Then*

$$K_1(A) \longrightarrow K_1(T_A(M))$$

is an isomorphism.

Remark. If X is an A-basis for M then $T_A(M)$ is the free associative algebra over A generated by X. Equivalently, it is the monoid algebra over A of the free (non commutative) monoid generated by X.

Proof. $B = A \oplus M \oplus (M \otimes M) \oplus \ldots$ is a graded ring, generated by M over A. Therefore $K_1(B) = K_1(A) \oplus K_1(B, B_+)$ as in (5.3), where B_+ is the ideal generated by M. Moreover every element of $K_1(B, B_+)$ has a representative of the form $I + \nu \in GL_n(B)$ for some n, where ν has coefficients in M. We must show that $I + \nu \in E(B)$. Let $(x_i)_{i \in J}$ be a bimodule basis for M. Thus the x_i commute with elements of A, but not with each other. Write $\nu = \Sigma\, \alpha_i x_i$ $(1 \leq i \leq n)$ where $\alpha_i \in M_n(A)$. Then $\nu^m = 0$ for all large m. The monomials in the x_i are a free monoid whose elements are linearly independent over A. Hence, when we expand ν^m and set coefficients of each monomial in the x_i equal to zero, we find that any product of m of the α_i's is zero. We will show now, by induction on m, that $I + \nu$ is a product of matrices of the form $I + \alpha w$ where α is a nilpotent matrix over A, and w is a monomial in the x_i's. For consider

$$I + \nu' = (I - \alpha_1 x_1)\ldots(I - \alpha_m x_m)\ (I + \nu).$$

We can write $\nu' = \Sigma\, \beta_j y_j$ $(1 \leq j \leq s)$ where each $\beta_j y_j$ is a monomial of degree ≥ 2 in the $\alpha_i x_i$. Hence any product of at least (m/2) of the β_j's is zero. Applying induction to $I + (\Sigma\, \beta_j z_j)$, where the z_j's are new variables which generate a free algebra over A, we deduce that $I + \nu'$ is a product of matrices $I + \gamma w$ where γ is nilpotent over A and w is a monomial in the y_j's, and hence also in the x_i's.

To complete the proof we must show that $I + \alpha w \in E(B)$ if α is a nilpotent matrix over A and w is a monomial in the x_i. But it follows from (5.4) that $I + \alpha w \in E(A[w])$. q.e.d.

In a special case, which will arise in one of our calculations, we can refine (5.3) and (5.4) as follows:

(5.6) PROPOSITION. Let A be a subring of $B = \Pi A_i$, and assume that the projection of A into each A_i is surjective. Assume that B is right regular, and that $NB \subset A$ for some integer $N > 0$. Let T be a free abelian monoid, and let $L_1(A, T) = \mathrm{Ker}(K_1(A[T]) \longrightarrow K_1(A))$, where the homomorphism is induced by the augmentation $A[T] \longrightarrow A$. Then every element of $L_1(A, T)$ has finite order dividing some power of N.

Proof. An induction argument, using Hilbert's Basis and Syzygy Theorems, reduces the problem quickly to the case when T has one generator, say t. Writing $s = t - 1$ we then have $L_1(A, T) = K_1(A[s], sA[s])$. According to (5.3) each element of $L_1(A, T)$ has a representative of the form $\alpha = I + s\nu \in GL_n(A[s])$ where ν is a nilpotent matrix over A. In $M_n(A[s]) = M_n(A)[s]$ let R be the subring generated by I and α. Then R consists of polymonials of degree $\leq d$ in $s\nu$ with integer coefficients, where $\nu^{d+1} = 0$, say. Applying (X, 3.8 (c)) to R/N^dR, and the ideal generated by $s\nu$, we find that $\alpha^{N^r} \equiv I \bmod N^dR$ for some $r > 0$. Hence we can write $\alpha^{N^r} = I + (\alpha_1 s + \ldots + \alpha_d s^d)N^d$, where each α_i is an integer times ν^i. Set $\beta_i = N^{d-i}\alpha_i$ $(1 \leq i \leq d)$ and put $\beta = \alpha^{N^r} = I + \beta_1 Ns + \ldots + \beta_d(Ns)^d$. Let $\gamma = I + \beta_1 s + \ldots + \beta_d s^d$; clearly γ is unipotent. Therefore, since B[s] is right regular, (5.3) (b) implies $\gamma \in E(B[s], sB[s])$. Define $f\colon B[s] \longrightarrow B[s]$ by $f(b) = b$ $(b \in B)$ and $f(s) = Ns$. Then $f(\gamma) = \beta$ so $\beta \in E(B[s], sNB[s])$. Now $sNB[s] \subset A[s]$ so it follows from (IX, 5.8) that

$E(B[s], sNB[s]) = E(A[s], sNB[s])$. Thus $\alpha^{N^r} = \beta \in E(A[s]), sA[s])$. q.e.d.

We close this section now by applying (5.1) to $A[t, t^{-1}]$. The conclusions are somewhat more complicated than in the case of polynomials.

(5.6) PROPOSITION. *Every* $\alpha \in GL(A[t, t^{-1}])$ *admits a factorization*

(1) $$\alpha = \tau_- \varepsilon_1 \omega_- \tau_+ \alpha_o \omega_+ \varepsilon_2$$

where:

$$\varepsilon_i \in E(A[t], (1 - t) A[t]) \quad (i = 1, 2),$$

$$\alpha_o \in GL(A)$$

$$\omega_{\pm} = I + (t^{\pm 1} - 1)\nu_{\pm}, \ \nu_{\pm} \text{ a nilpotent matrix over } A.$$

Moreover $\tau_{\pm} = \sigma_{\pm} \oplus I$ *where* $\sigma_{\pm} \in GL_n(A[t, t^{-1}])$ *is of the form* $\sigma_{\pm} = 1_{P_o[t, t^{-1}]} \oplus t^{\pm 1} 1_{P_1[t, t^{-1}]}$ *for some decomposition* $A^n = P_0 \oplus P_1$. (*In fact* $P_i = \mathrm{Ker}(\sigma_{\pm} - t^{\pm i} I_n) \subset A^n$, $(i = 0, 1)$.) *Moreover we can choose* τ_- *so that it is diagonal and commutes with* α.

Proof. Say $\alpha \in GL_m(A[t, t^{-1}])$. Then for sufficiently large N, $t^N \alpha$ has polynomial coefficients. Set $\tau_- = t^{-N} I_m$ and put $s = 1 - t$. Then we can write

$$\varepsilon_1^{\ 1} \tau_-^{-1} \alpha \, \varepsilon_2^{-1} = \alpha_o + \alpha_1 s$$

as in (5.1), with $\varepsilon_i \in E(A[t], sA[t])$ $(i = 1, 2)$. The augmentation $A[t, t^{-1}] \longrightarrow A$ $(s \longmapsto 0)$ sends α to $\alpha_o \in GL(A)$ so we have

(2) $$\alpha = \tau_- \varepsilon_1 \beta \alpha_o \varepsilon_2, \qquad \beta = I + \gamma(t - 1),$$

where $\gamma = -\alpha_1 \alpha_o^{-1}$ has coefficients in A.

To complete the proof we will show that β admits a factorization

(3) $$\beta = \omega_- \tau_+ \omega_+$$

with factors of the type indicated in the proposition. Then, using (2) above we will have

$$\alpha = \tau_- \varepsilon_1 \omega_- \tau_+ \omega_+ \alpha_o \varepsilon_2 .$$

Since $\alpha_o \in GL(A)$ we can write $\omega_+ \alpha_o = \alpha_o \omega_+^{\alpha_o}$, and $\omega_+^{\alpha_o}$ is of the same type as ω_+. Hence it suffices to establish the factorization (3) above for $\beta = I + \gamma(t - 1)$.

We claim that $\gamma^u (I - \gamma)^v = 0$ for some $u, v \geq 0$. For $\beta = I + \gamma(t - 1) = \delta + \gamma t$ $(\delta = I - \gamma)$ has an inverse $\beta^{-1} = \Sigma \gamma_i t^i$ in $GL(A[t, t^{-1}])$:

$$\Sigma (\delta\gamma_i + \gamma\gamma_{i-1}) t^i = I.$$

Thus

$$\delta \gamma_o + \gamma\gamma_{-1} = I, \text{ and}$$

$$\delta \gamma_i = -\gamma\gamma_{i-1} \quad \text{if } i \neq 0.$$

Since δ and γ commute the latter equations show that, for any $u, v > 0$, we have

$$\delta^v \gamma_{-1} = -\delta^{v-1} \gamma\gamma_{-2} = (-1)^2 \delta^{v-2} \gamma^2 \gamma_{-3}$$
$$= \ldots = (-1)^v \gamma^v \gamma_{-(v+1)},$$

and

$$\gamma^u \gamma_o = -\gamma^{u-1} \delta\gamma_1 = (-1)^2 \gamma^{u-2} \delta^2 \gamma_2 = \ldots = (-1)^u \delta^u \gamma_u .$$

For sufficiently large u and v we have $\gamma_{-(v+1)} = 0 = \gamma_u$, so $\delta^v \gamma_{-1} = 0 = \gamma^u \gamma_o$. Now use the equation $\delta\gamma_o + \gamma\gamma_{-1} = I$ above to obtain

$$\gamma^u\delta^v = \gamma^u\delta^v(\delta\gamma_o + \gamma\gamma_{-1}) = \delta^{v+1}\gamma^u\gamma_o + \gamma^{u+1}\delta\gamma_{-1} = 0.$$

Say $\beta \in GL_n(A[t, t^{-1}])$, so $\gamma \in M_n(A)$. In the subring of $M_n(A) = End_A(A^n)$ generated by γ, the elements γ^u and $(I - \gamma)^v$ generate the unit ideal. It follows, since $\gamma^u(I - \gamma)^v = 0$, that $A^n = P_o \oplus P_1$ where $P_o = Ker(\gamma^u)$ and $P_1 = Ker(I - \gamma)^v$. Write $J_i = 1_{P_i}$ and γ_i = the endomorphism of P_i induced by γ ($i = 0, 1$). We shall identify J_i and γ_i with their respective extensions to $P_i[t, t^{-1}]$ ($i = 0, 1$). Then $\beta = I_n + \gamma(t - 1) = \beta_o \oplus \beta_1$ where $\beta_i = J_i + \gamma_i(t - 1)$. Moreover γ_0 and $J_1 - \gamma_1$ are nilpotent. Let $\beta_o' = \beta_o \oplus J_1$ and $\beta_1' = J_o \oplus \beta_1$, so that $\beta = \beta_o'\beta_1' = \beta_1'\beta_o'$. Now we put

$$\omega_+ = \beta_o' = I_n + \nu_+(t - 1)$$

where $\nu_+ = \gamma_o \oplus 0$ is nilpotent. The factorization (3) will be achieved now by factoring β_1' into $\omega_-\tau_+$ as in (3). First consider β_1; we have

$$\begin{aligned}\beta_1 &= J_1 + \gamma_1(t - 1)\\ &= (\gamma_1 + (J_1-\gamma_1)t^{-1})\ (tJ_1)\\ &= (J_1 + (J_1-\gamma_1)\ (t^{-1}-1))\ (tJ_1).\end{aligned}$$

Therefore we achieve the desired factorization of $\beta_1' = J_o \oplus \beta_1'$ by taking $\tau_+ = J_o \oplus tJ_1$ and $\omega_- = I_n + \nu_-\ (t^{-1}-1)$, where $\nu_- = 0 \oplus (J_1-\gamma_1)$ is nilpotent. q.e.d.

Remark. It will follow from the results of §7 that the factorization (1) above has some strong invariance properties. For application to the projective line (see §9) it would be preferable to have a factorization of the form

$$\alpha = \varepsilon_- \omega_- \tau \alpha_o \omega_+ \varepsilon_+$$

with $\varepsilon_\pm \in E(A[t^{\pm 1}], (t^{\pm 1}-1) A[t^{\pm 1}])$ and with $\omega_\pm$, τ, and α_o like the correspondingly denoted factors in (1) above. Actually, the following related result would suffice for this application: For any $\alpha \in GL(A[t, t^{-1}])$, the coset $\alpha \cdot E(A[t, t^{-1}])$ and the double coset $E(A[t^{-1}])\alpha\, E(A[t])$ coincide. If A is a field this holds (even in each GL_n). Examples of Gersten (unpublished) show that it is not true in general, however.

§6. THE CATEGORY OF NILPOTENT ENDOMORPHISMS

Let $\underline{\underline{C}}$ be an admissible subcategory of an abelian category, in the sense of (VIII, 1.1). We now introduce the category

$$\underline{\underline{Nil}}(\underline{\underline{C}})$$

whose objects are pairs (M, ν), where $M \in \underline{\underline{C}}$, and $\nu \in End_{\underline{\underline{C}}}(M)$ is nilpotent. It is a full subcategory of the category of endomorphisms of objects of $\underline{\underline{C}}$. Moreover we have two exact functors

$$Z: \underline{\underline{C}} \longrightarrow \underline{\underline{Nil}}(\underline{\underline{C}}),\ Z(M) = (M, 0) \quad (\text{"zero"})$$

and

$$F: \underline{\underline{Nil}}(\underline{\underline{C}}) \longrightarrow \underline{\underline{C}},\ F(M, \nu) = M \quad (\text{"forget } \nu\text{"}).$$

Since $FZ = 1_{\underline{\underline{C}}}$ we obtain a split exact sequence

$$0 \longrightarrow K_o(\underline{\underline{C}}) \xrightarrow{Z} K_o(\underline{\underline{Nil}}(\underline{\underline{C}})) \longrightarrow Nil(\underline{\underline{C}}) \longrightarrow 0$$

which defines $Nil(\underline{\underline{C}})$.

(6.1) PROPOSITION. *If $\underline{\underline{C}}$ is an abelian category then* $\mathrm{Nil}(\underline{\underline{C}}) = 0$.

Proof. If $(M, \nu) \in \underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}})$ we have a filtration $M \supset \nu M \supset \nu^2 M \supset \dots$ which induces a filtration of (M, ν) in $\underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}})$. Thus, in $K_0\underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}})$, $[M, \nu] = \Sigma\, [\nu^i M/\nu^{i+1}M, 0] \in \mathrm{Im}(Z)$ above. q.e.d.

(6.2) PROPOSITION. *Let $\underline{\underline{C}}_0 \subset \underline{\underline{C}}$ be admissible subcategories of an abelian category. Assume that each object of $\underline{\underline{C}}$ has a finite $\underline{\underline{C}}_0$-resolution. Then every object of $\underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}})$ has a finite $\underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}}_0)$-resolution, and hence $K_0(\underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}}_0)) \longrightarrow K_0(\underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}}))$ is an isomorphism.*

Proof. The last assertion follows from the first by virtue of (VIII, 4.2).

Given $(M, \nu) \in \underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}})$ let n be the length of a finite $\underline{\underline{C}}_0$-resolution of M. By induction on n we claim (M, ν) has a $\underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}}_0)$-resolution of length n. Assume $n > 0$, for the case $n = 0$ is trivial. Choose an exact sequence $0 \longrightarrow N \longrightarrow P \xrightarrow{f} M \longrightarrow 0$ with $P \in \underline{\underline{C}}_0$ and such that N has a $\underline{\underline{C}}_0$-resolution of length $n - 1$.

Say $\nu^{h+1} = 0$. Let $Q = P_0 \oplus P_1 \oplus \dots \oplus P_h$ with each $P_i = P$, and define $\mu \in \mathrm{End}_{\underline{\underline{C}}}(Q)$ by letting $\mu|P_{i-1}$ be the identity morphism from P_{i-1} to P_i $(1 \leq i \leq h)$, and $\mu(P_h) = 0$. Then $\mu^{h+1} = 0$ so $(Q, \mu) \in \underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}}_0)$. Define $g\colon Q \longrightarrow M$ by $g|P_i = \nu^i f$. Since $P_i = \mu^i P_0$ we see that $g\mu = \nu g$. Hence we have an exact sequence

$$0 \longrightarrow (H, \mu|H) \longrightarrow (Q, \mu) \xrightarrow{g} (M, \nu) \longrightarrow 0$$

in $\underline{\underline{\mathrm{Nil}}}(\underline{\underline{C}})$, with $Q \in \underline{\underline{C}}_0$. The fact that g is an epimorphism follows because $g|P_0 = f$, and f is an epimorphism. This further makes it clear that $H = \mathrm{Ker}(g) \simeq \mathrm{Ker}(f) \oplus P_2 \oplus \ldots \oplus P_h$. Since $N = \mathrm{Ker}(f)$ has a $\underline{\underline{C}}_o$-resolution of length $n - 1$, and since each $P_i \in \underline{\underline{C}}_o$, H has a $\underline{\underline{C}}_o$-resolution of length $n - 1$. Therefore we can complete the resolution of (M, ν) by applying the induction hypothesis to $(H, \mu|H)$. q.e.d.

For a ring A we shall write

$$\mathrm{Nil}(A) = \mathrm{Nil}(\underline{\underline{P}}(A)).$$

(6.3) COROLLARY. *Let* A *be a ring. Then* $\mathrm{Nil}(A) = \mathrm{Nil}(\underline{\underline{H}}(A))$ *and* $\mathrm{Nil}(A) = 0$ *if* A *is right regular*.

Proof. The first assertion follows from (6.2) because, by definition, the objects of $\underline{\underline{H}}(A)$ have finite $\underline{\underline{P}}(A)$-resolutions. If A is right regular then $\underline{\underline{H}}(A) = \underline{\underline{M}}(A)$ is abelian so (6.1) implies $\mathrm{Nil}(\underline{\underline{H}}(A)) = 0$. q.e.d.

(6.4) PROPOSITION. *Let* T_+ *be a free monoid on one generator* t, *and let* A *be a ring. The natural isomorphism of* mod-A[t] *with the category of endomorphisms of right* A-*modules* (see §1) *induces isomorphisms*

$$\underline{\underline{M}}_{T_+}(A[t]) \longrightarrow \underline{\underline{\mathrm{Nil}}}(\underline{\underline{M}}(A))$$

and

$$\underline{\underline{H}}_{T_+}(A[t]) \longrightarrow \underline{\underline{\mathrm{Nil}}}(\underline{\underline{H}}(A)).$$

Hence

$$K_o(\underline{\underline{H}}_{T_+}(A[t])) = K_o(A) \oplus \mathrm{Nil}(A).$$

Remark. We shall later also consider the monoid T_- generated by t^{-1}; hence the notation.

Proof. Recall that $\underline{\underline{M}}_{T_+}(A[t])$ is the category of $M \in \underline{\underline{M}}(A[t])$ such that $T_+^{-1}M = 0$, or, equivalently, such that $Mt^n = 0$ for some $n \geq 0$. The only point in the first assertion that is not entirely obvious is that such an M is finitely generated as an A-module. Each of the modules Mt^{i-1}/Mt^{i} $(i \geq 1)$ is finitely generated over A[t], and hence over A. These are the successive quotients in a finite filtration of M, so $M \in \underline{\underline{M}}(A)$, as claimed.

If $M \in \underline{\underline{H}}_{T_+}(A[t])$ then, since $hd_A(M) \leq hd_{A[t]}(M)$ (see part (i) of the proof of (2.2)), we have $M \in \underline{\underline{H}}(A)$. Conversely, if $M \in$ mod-A[t] and if $hd_A(M) < \infty$ then, since $hd_{A[t]}(M) \leq hd_A(M) + 1$ (part (iii) of the proof of (2.2)), we have $hd_{A[t]}(M) < \infty$. This establishes the second isomorphism of categories above.

Using this and (6.2) we have $K_0(\underline{\underline{H}}_{T_+}(A[t])) = K_0(\underline{\underline{Nil}}(\underline{\underline{H}}(A))) = K_0(Nil(\underline{\underline{P}}(A))) = K_0(\underline{\underline{P}}(A)) \oplus Nil(\underline{\underline{P}}(A)) = K_0(A) \oplus Nil(A)$. q.e.d.

If $(P, \nu) \in \underline{\underline{Nil}}(\underline{\underline{P}}(A))$ we shall write $\partial_1(\nu) = 1_P - \nu \in Aut_A(P)$. Similarly we have $(P[t], t\nu) \in \underline{\underline{Nil}}(\underline{\underline{P}}(A[t]))$ and $\partial_1(t\nu) = 1_{P[t]} - t\nu$. There is a canonical embedding of $End_A(P)$ in $End_{A[t]}(P[t])$ so we can define

$$\partial_+ : \underline{\underline{Nil}}(\underline{\underline{P}}(A)) \longrightarrow \Sigma\, \underline{\underline{P}}(A[t]), \qquad (P, \nu) \rightarrowtail (P[t], \partial_+(\nu)), \tag{1}$$

where $\partial_+(\nu) = \partial_1(\nu)^{-1}\, \partial_1(t\nu)$. The map on objects above clearly defines an exact functor. Moreover the augmentation $t \mapsto 1$, from A[t] to A clearly sends $\partial_+(\nu)$ to 1_P. If $\nu = 0$ then $\partial_+(\nu) = 1_{P[t]}$. Hence the functor (1) induces a homomorphism

$$\partial_+ : \mathrm{Nil}(A) \longrightarrow K_1(A[t], (t-1)A[t]).$$

(2) $\partial_+[P, \nu] = [P[t], \partial_+(\nu)]$, where

$$\partial_+(\nu) = \partial_1(\nu)^{-1}\, \partial_1(t\nu), \text{ and } \partial_1(\nu) = I - \nu.$$

We shall see in the next section that (2) is an isomorphism. For the moment we only prove:

(6.5) PROPOSITION. <u>The homomorphism</u> (2) <u>above is surjective</u>.

<u>Proof</u>. Put $s = t - 1$. It follows from (5.3) that every element of $K_1(A[t], sA[t])$ has a representative of the form $I - s\nu_1 \in GL_n(A[t], sA[t])$ (for some n) where ν_1 is a nilpotent matrix over A, which we can identify with an endomorphism of A^n.

Let ν be any nilpotent endomorphism of A^n. We want to choose ν so that $\partial_+(\nu) = I - s\nu_1$. Recall that

$$\begin{aligned}\partial_+(\nu) &= \partial_1(\nu)^{-1}\ (I - t\nu)\\ &= \partial_1(\nu)^{-1}\ (I - \nu - (t-1)\nu\\ &= I - s\ \partial_1(\nu)^{-1}\ \nu.\end{aligned}$$

We complete the proof now by showing that $\nu_1 = \partial_1(\nu)^{-1}\ \nu$ where $\nu = I - (I + \nu_1)^{-1}$. The last equation implies $\partial_1(\nu) = (I + \nu_1)^{-1}$, i.e. $I + \nu_1 = \partial_1(\nu)^{-1}$, i.e. $\nu_1 = \partial_1(\nu)^{-1} - I$. But $\partial_1(\nu)^{-1} = \Sigma_{i \geq 0}\ \nu^i$ so $\partial_1(\nu)^{-1} - I = \Sigma_{i > 0}\ \nu^i = \nu\ \Sigma_{i \geq 0}\ \nu^i = \nu\ \partial_1(\nu)^{-1}$. q.e.d.

§7. THE FUNDAMENTAL THEOREM

It is a description of $K_1(A[t, t^{-1}])$. Its most important feature is the appearance of $K_0(A)$ as a natural direct summand of $K_1(A[t, t^{-1}])$. As a general principle we conclude that "general theorems" about K_1 imply analogous theorems about K_o. This principle has numerous applications, some of which are explored in later sections of this chapter. The first application, however, is that the fundamental theorem itself has an analogue for K_o. In this analogue there appears a functor which bears the same relation to K_o that K_o bears to K_1; we christen this new functor K_{-1}. We can then iterate this whole procedure, and watch it give birth to K_{-2}, $K_{-3}, \ldots$ The main point is that all of these functors fit into a long exact sequence, extending to (K_1, K_o)-sequence to the right (see §8).

The approach taken here is somewhat axiomatized with the result that the fundamental theorem (7.4) comes only after some of the formalism is established.

We shall write

(rings)

for the category of rings and ring homomorphisms. If A is a ring and G is a monoid, an unspecified arrow $A[G] \longrightarrow A$ will always denote the augmentation, sending the elements of G to 1. Its kernel, the augmentation ideal, will be denoted

$$\overline{A[G]} = \mathrm{Ker}(A[G] \longrightarrow A).$$

The augmentation is a left inverse for the inclusion $A \longrightarrow A[G]$. Therefore, if F: (rings) $\longrightarrow \underline{\underline{Z}}$-mod is a functor, $F(A[G]) \simeq F(A) \oplus \mathrm{Ker}(F(A[G]) \longrightarrow F(\bar{A}))$, canonically.

By an <u>oriented cycle</u> we shall mean an infinite cyclic group T with a designated generator, t. We shall often denote this by $(T, T_{\pm})$, where T_+ is the submonoid generated by t, and T_- is the submonoid generated by t^{-1}.

Let $(T, T_{\pm})$ be an oriented cycle, and let

$$F: (\text{rings}) \longrightarrow \underline{\underline{Z}}\text{-mod}$$

be a functor. We shall associate with F two new functors,

$$NF, LF: \text{(rings)} \longrightarrow \underline{\underline{Z}}\text{-mod},$$

as follows:

(1) $$NF(A) = N_{T_+}F(A) = \mathrm{Ker}(F(A[T_+]) \longrightarrow F(A)).$$

(The arrow is understood, as always, to be induced by the augmentation.) Thus we have an identification

$$F(A[T_+]) = F(A) \oplus N_{T_+}F(A),$$

which is functorial in A.

The inclusions $\tau_{\pm}: A[T_{\pm}] \subset A[T]$ induce a homomorphism

$$F(A[T_+]) \oplus F(A[T_-]) \xrightarrow{\tau = (\tau_+, \tau_-)} F(A[T])$$

and we define

(2) $$LF(A) = L_T F(A) = \mathrm{Coker}(\tau).$$

Note that NF and LF are functorial in F; i.e. a natural transformation $\phi: F \longrightarrow F'$ induces $N\phi: NF \longrightarrow NF'$ and $L\phi: LF \longrightarrow LF'$. With this definition we introduce

$$\mathrm{Seq}F(A) = \mathrm{Seq}_T F(A) =$$

(3) $$(0 \longrightarrow F(A) \xrightarrow{e} F(A[T_+]) \oplus F(A[T_-]) \xrightarrow{\tau} F(A[T]) \xrightarrow{p} L_T F(A) \longrightarrow 0),$$

where $e(x) = (x, -x)$. (We have identified $F(A[T_{\pm}]) = F(A) \oplus N_{T_{\pm}}F(A)$, as above.) It is obvious that SeqF(A) is a complex which is acyclic except, perhaps, at $F(A[T_+]) \oplus F(A[T_-])$, and it is functorial in A. The condition that SeqF(A) is exact is equivalent to the condition that $\tau_{\pm}$ are both monomorphisms, and that $\mathrm{Im}(\tau_+) \cap \mathrm{Im}(\tau_-) = F(A)$. In this case, if we regard $\tau_{\pm}$ as inclusions, we have $\mathrm{Im}(\tau) = F(A[T_+]) + F(A[T_-]) = F(A) \oplus N_{T_+}F(A) \oplus N_{T_-}F(A)$. We shall be interested in functors F for which SeqF(A) is not only acyclic, but even contractible; this amounts to the added requirement that $\mathrm{Im}(\tau)$ is a direct summand of $F(A[T])$.

(7.1) DEFINITION. Let

$$F: (\text{rings}) \longrightarrow \underline{\underline{Z}}\text{-mod}$$

be a functor, and let $(T, T_{\pm})$ be an oriented cycle. A <u>contraction</u> of F is a natural homomorphism

$$h = h_{T,A}: L_T F(A) \longrightarrow F(A[T])$$

which is a right inverse for the canonical projection $p: F(A[T]) \longrightarrow L_T F(A)$. The pair (F, h) will be called a <u>contracted functor</u> if, further, $\mathrm{Seq}_T F(A)$ is acyclic for all A.

The naturality in the definition above is with respect to both A and $(T, T_{\pm})$. Thus, if $(S, S_{\pm})$ is a second oriented cycle, and if $f: T \longrightarrow S$ is a group homomorphism (which is determined by an integer n, since T and S have given generators), then the square

$$\begin{array}{ccc} L_T F(A) & \xrightarrow{\;h_{T,A}\;} & F(A[T]) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ L_S F(A) & \xrightarrow[\;h_{S,A}\;]{} & F(A[S]) \end{array}$$

is required to commute. The f on the right is induced by $A[T] \xrightarrow{\text{"}A[f]\text{"}} A[S]$, and the one on the left is induced by that on the right via the definition of the left hand groups as quotients of those on the right. For the functors we shall deal with below, the map on the left above will be multiplication by n, where n is the integer defining f. In particular, the involution, $t \longmapsto t^{-1}$, of T induces multiplication by -1 on LF, in our examples. In other words, the effect of changing $(T, T_{\pm})$ to $(T, T_{\mp})$ will be to change $h_{T,A}$ to $-h_{T,A}$, in our examples below.

If (F, h) is a contracted functor then we have

(4) $$F(A[T]) = F(A) \oplus N_{T_+}F(A) \oplus N_{T_-}F(A) \oplus \mathrm{Im}(h_{T,A}),$$

the last term being isomorphic to $L_T F(A)$. Moreover this direct sum decomposition is natural in A. For the term $\mathrm{Im}(h_{T,A})$ this follows from the definition of a contraction. The terms $\mathrm{Im}(\tau_{\pm}) = F(A) \oplus N_{T_{\pm}}F(A)$ are each invariant, and so also is F(A), clearly. Finally, $N_{T_{\pm}}F(A)$ is the set of elements in $\mathrm{Im}(\tau_{\pm})$ killed by the augmentation, $F(A[T]) \longrightarrow F(A)$, so each of these terms is natural also in A. Using the direct sum decomposition (4) we can construct a contraction, $c_{T,A}$, of the complex $\mathrm{Seq}_T F(A)$, such that the contraction is natural in A. We define $c_{T,A}$ by the homomorphisms c_2, c_1, c_0 in

$$F(A) \underset{c_2}{\overset{e}{\rightleftarrows}} F(A[T_+]) \oplus F(A[T_-]) \underset{c_1}{\overset{\tau}{\rightleftarrows}} F(A[T]) \underset{c_0}{\overset{p}{\rightleftarrows}} L_T F(A).$$

Of course c_0 is just $h_{T,A}$. We put

$$c_1(a, n_+, n_-, b) = ((0, n_+), (a, n_-))$$

and

$$c_2((a_+, n_+), (a_-, n_-)) = a_+$$

for a, $a_{\pm} \in F(A)$ and $n_{\pm} \in N_{T_{\pm}} F(A)$. Since $e(a) = ((a, 0), (-a, 0))$ and $\tau((a_+, n_+), (a_-, n_-)) = (a_+ + a_-, n'_+, n_-, 0)$ we see that $c_{T, A}$ is, indeed, a contraction. Moreover, the naturality in A of the direct sum decompositions with respect to which we took coordinates shows that $c_{T, A}$ is natural in A.

Let (F, h) and (F´, h´) be two contracted functors, and let ϕ: F ⟶ F´ be a natural transformation. We call ϕ a <u>morphism of contracted functors</u> if the square

$$\begin{array}{ccc} L_T F(A) & \xrightarrow{h_{T,A}} & F(A[T]) \\ \downarrow{\scriptstyle L_T\phi_A} & & \downarrow{\scriptstyle \phi_{A[T]}} \\ L_T F'(A) & \xrightarrow[h'_{T,A}]{} & F'(A[T]) \end{array}$$

commutes for all A. It is then clear that the homomorphism of complexes

(5) $\qquad Seq_T\phi_A\colon Seq_T F(A) \longrightarrow Seq_T F'(A)$

is compatible with the contractions $c_{T, A}$ and its analogue $c'_{T, A}$, of the respective complexes. Consequently Ker(5) = $Seq_T Ker(\phi)(A)$, Coker(5) = Seq_T Coker(ϕ)(A), etc., inherit contractions which are natural in A. In particular (Ker(ϕ), h_1) and (Coker(ϕ), h_1') are contracted functors, where h_1 is induced, via restriction, by h, and h_1' is induced, passing to the quotient, by h´.

(7.2) PROPOSITION. <u>Let</u> ϕ: (F, h) ⟶ (F´, h´) <u>be a morphism of contracted functors. Then there are</u> h´ <u>and</u> h_1' <u>induced by</u> h <u>and</u> h´, <u>respectively, so that</u> (Ker(ϕ), h_1) <u>and</u> (Coker(ϕ), h_1') <u>are contracted functors</u>.

In particular, there are contracted functors (NF, Nh) and (LF, Lh). Further, there is a natural isomorphism (LNF, LNh) $\simeq$ (NLF, NLh) of contracted functors.

Proof. The first assertion was demonstrated above. We have

$$NF(A) = \text{Ker}(F(A[T_+]) \longrightarrow F(A))$$

and

$$LF(A) = \text{Coker}(F(A[T_+]) \oplus F(A[T_-]) \longrightarrow F(A[T])).$$

Both of these arrows are morphisms of contracted functors, clearly, so we can apply the first conclusion to obtain contractions, Nh and Lh, of NF and LF, respectively.

If $(S, S_\pm)$ is an oriented cycle we have, by definition, a split exact sequence of contracted complexes

$$0 \longrightarrow \text{Seq}_T\, N_{S_+} F(A) \longrightarrow \text{Seq}_T F(A[S_+]) \longrightarrow \text{Seq}_T F(A) \longrightarrow 0.$$

The right ends (i.e. the L_T-terms) of these complexes constitute the exact sequence

$$0 \longrightarrow L_T\, N_{S_+} F(A) \longrightarrow L_T\, F(A[S_+]) \longrightarrow L_T\, F(A) \longrightarrow 0.$$

By definition the kernel of the right hand arrow here is $N_{S_+} L_T F(A)$, so we have a canonical isomorphism, $L_T\, N_{S_+} F(A) \simeq N_{S_+} L_T F(A)$. This is evidently compatible with corresponding contractions. q.e.d.

Using (7.2) we can obtain an elegant formula for $F(A[T^n])$ where $T^n = T\times\ldots\times T$ (n times) is a free abelian group of rank n. For this purpose the following notation is convenient. Let $P(X, Y) = \Sigma\, a_{ij}\, X^i Y^i$ be a polynomial in two variables with integer coefficients $a_{ij} \geq 0$. Then if

F: (rings) ⟶ $\underline{\underline{Z}}$-mod is a functor we shall write

$$P(L, N)F = \amalg a_{ij} L^i N^j F,$$

where $a_{ij} L^i N^j F$ denotes the direct sum of a_{ij} copies of $L^i N^j F$.

(7.3) COROLLARY. Let (F, h) be a contracted functor and let (T, T_+) be an oriented cycle. Then, with the notation just introduced, we have

$$F(A[T_+^n]) = (1 + N)^n F(A)$$

and

$$F(A[T^n]) = (1 + 2N + L)^n F(A).$$

Proof. The case n = 1 follows from the definitions. Moreover, it follows from (7.2) that (P(L, N)F, P(L, N)h) is a contracted functor for any polynomial P as above. Therefore the general case follows by induction on n, using the fact that NL = LN, up to natural isomorphism (see (7.2)). q.e.d.

(7.4) THEOREM ("Fundamental Theorem"). Let (T, T_+) be an oriented cycle, and let A be a ring. Let ∂_+: Nil(A) ⟶ $N_{T_+} K_1(A)$ be the homomorphism in (6.5) above. Define

$$h = h_{T,A}: K_0(A) \longrightarrow K_1(A[T])$$

by

$$h[P] = [P[T], t\, 1_{P[T]}], \qquad (P \in \underline{\underline{P}}(A)).$$

(a) ∂_+: Nil ⟶ NK_1 is an isomorphism of functors.

(b) The homomorphism h induces, on passing to the quotient, an isomorphism $K_0 \longrightarrow L_T K_1$. If we use this to identify K_0 with LK_1 then (K_1, h) is a contracted functor.

Notation. We shall sometimes write $K_{-n} = L^n K_o$ $(n \geq 0)$.

Proof. We consider $\tau_+ : A[T_+] \longrightarrow A[T]$ to be a localization with respect to the multiplicative set T_+. This yields the exact K-sequence (IX, 6.3) in which the relative group is $K_o(A[T_+], T_+) = K_o(\underline{\underline{H}}_{T_+}(A[T_+]))$. According to (6.4) we can identify $\underline{\underline{H}}_{T_+}(A[T_+])$ with $\underline{\underline{\mathrm{Nil}}}(\underline{\underline{H}}(A))$. For purposes of computing K_o we can further replace $\underline{\underline{H}}(A)$ by the subcategory $\underline{\underline{P}}(A)$, thanks to (6.2). Since there is a canonical isomorphism

$$K_o(\underline{\underline{\mathrm{Nil}}}(\underline{\underline{P}}(A))) \simeq K_o(A) \oplus \mathrm{Nil}(A)$$

we obtain a diagram

$$
\begin{array}{ccccccccc}
K_1(A[T_+]) & \xrightarrow{\tau_+} & K_1(A[T]) & \xrightarrow{\delta_+} & K_o(\underline{\underline{\mathrm{Nil}}}(\underline{\underline{P}}(A))) & \longrightarrow & K_o(A[T_+]) & \xrightarrow{\tau_+} & K_o(A[T]) \\
 & & \Big\uparrow{\scriptstyle (h,\ \tau_-)} & & \Big\downarrow{\scriptstyle (\simeq)} & & & & \\
 & & K_o(A) \oplus N_{T_-} K(A) & \xleftarrow[1 \oplus \partial_-]{} & K_o(A) \oplus \mathrm{Nil}(A) & & & &
\end{array}
\qquad (6)
$$

Here the top row is the exact sequence of the localization τ_+, with $K_o(\underline{\underline{\mathrm{Nil}}}(\underline{\underline{P}}(A)))$ substituted for $K_o(A[T_+], T_+)$, as indicated above. The map ∂_- is the analogue of ∂_+ in (6.5). Explicitly,

$$\partial_-[P, \nu] = [P[T_-], \partial_-(\nu)]$$

(7) $$\partial_-(\nu) = \partial_1(\nu)^{-1}(1 - t^{-1}\nu)$$

$$\partial_1(\nu) = I - \nu \qquad (I = 1_{P[T_-]}).$$

Finally, h is the homomorphism defined in the theorem, and it is evidently well defined. The commutativity of the square in (6), as well as the proof of the fundamental theorem, will be deduced from the next proposition.

(7.5) PROPOSITION. _There is an exact functor_

$$\Delta_+ : \underline{\underline{\mathrm{Nil}}}(\underline{\underline{P}}(A)) \longrightarrow \Sigma\, \underline{\underline{P}}(A[T])$$

defined by $\Delta_+(P, \nu) = (P[T], (I - \nu)^{-1}(t \cdot I - \nu))$ $(I = 1_{P[T]})$. _The homomorphism_ $\Delta_+ : K_o\underline{\underline{\mathrm{Nil}}}(P(A)) \longrightarrow K_1(A[T])$ _which it induces coincides with the composite_,

(8) $$\Delta_+ = (h, \tau_-) \circ (1_{K_o(A)} \oplus \partial_-)$$

in diagram (6). _Furthermore_,

(9) $$\delta_+\Delta_+ = \text{the identity on } K_o(\underline{\underline{\mathrm{Nil}}}(\underline{\underline{P}}(A)).$$

Before proving this we give the:

Conclusion of the proof of (7.4). We know from (6.5) (or its analogue for ∂_-) that $1 \oplus \partial_-$ is surjective. It follows from (8) and (9) that it is injective, and further that $\begin{pmatrix} h \\ \tau_- \end{pmatrix}: K_o(A) \oplus N_{T_-} K_1(A) \longrightarrow K_1(A[T])$ is also injective. The first of these conclusions shows that $\partial_- : \mathrm{Nil}(A) \longrightarrow$

$N_{T_-} K_1(A)$ is an isomorphism. Since all the homomorphisms here are clearly natural in A this establishes part (a) of (7.4). The second conclusion above implies that $\tau_- : K_1(A[T_-]) = K_1(A) \oplus N_{T_-} K_1(A) \longrightarrow K_1(A[T])$ is a monomorphism. This is because $K_1(A[T]) = K_1(A) \oplus \mathrm{Ker}(K_1(A[T]) \longrightarrow K_1(A))$ and τ_- induces the identity on $K_1(A)$ and a monomorphism of $N_{T_-} K_1(A)$ into the second direct summand. By symmetry, τ_+ is likewise a monomorphism. From (9) we conclude that

$$K_1(A[T]) = \mathrm{Im}(\tau_+) \oplus \mathrm{Im}(\Delta_+)$$

and we have seen that

$$\mathrm{Im}(\tau_+) = K_1(A) \oplus N_{T_+} K_1(A)$$

and

$$\mathrm{Im}(\Delta_+) = \mathrm{Im}(h) \oplus N_{T_-} K_1(A) \simeq K_o(A) \oplus N_{T_-} K_1(A).$$

The theorem follows immediately from these decompositions. q.e.d.

<u>Proof of (7.5)</u>. If $(P, \nu) \in \underline{\underline{\mathrm{Nil}}}(\underline{\underline{P}}(A))$ we identify ν with its extension, $\nu[T]$, to $P[T]$, and write $I = 1_{P[T]}$, as above. With $\partial_1(\nu) = I - \nu$ we have $\Delta_+(P, \nu) = (P[T], \Delta_+(\nu))$, where

$$\begin{aligned}\Delta_+(\nu) &= \partial_1(\nu)^{-1}\,(t \cdot I - \nu)\\ &= (t \cdot I) \cdot \partial_1(\nu)^{-1}\,(I - t^{-1}\,\nu)\\ &= (t \cdot I)\,\partial_-(\nu) \qquad \text{(see (7) above).}\end{aligned}$$

This shows that $\Delta_+(\nu) \in \mathrm{Aut}_{A[T]}(P[T])$, since $t \in U(A[T])$, so Δ_+ does define a functor into $\Sigma\,\underline{\underline{P}}(A[T])$, and it is clearly exact. The calculation above further shows that, in $K_1(A[T])$, we have

$$\Delta_+[P, \nu] = [P[T], t \cdot I] + [P[T], \partial_-(\nu)]$$
$$= h[P] + \tau_-[P[T_-], \partial_-(\nu)].$$

In the latter we consider that $\partial_-(\nu) \in \mathrm{Aut}_{A[T_-]}(P[T_-])$, of course, by restriction. This equation establishes formula (8).

To prove (9) we first recall from (IX, 6.3) that

$$\delta_+\Delta_+[P, \nu] = \delta_+[P[T], \Delta_+(\nu)]$$
$$= [M] \in K_o(\underline{\underline{H}}_{T_+}(A[T_+])),$$

where

$$M = \mathrm{Coker}(P[T_+] \xrightarrow{\Delta_+(\nu)} P[T_+]).$$

Under the identification of $\underline{\underline{H}}_{T_+}(A[T_+])$ with $\underline{\underline{\mathrm{Nil}}}(\underline{\underline{H}}(A))$, M corresponds to the pair (M as A-module, $t \cdot 1_M$). Thus (9) will be established if we show that M and P_ν (see §1) are isomorphic $A[T_+]$-modules.

We have $\Delta_+(\nu) = \partial_1(\nu)^{-1}(t \cdot I - \nu)$, and $\partial_1(\nu) = I - \nu$ induces an automorphism of $P[T_+]$. Hence $M = \mathrm{Coker}(\partial_1(\nu)^{-1}(t \cdot I - \nu)) \simeq \mathrm{Coker}(t \cdot I - \nu)$. But $t \cdot I - \nu$ is the "charactistic endomorphism" of ν, so it follows from (1.1) that

$$\mathrm{Coker}(P[T_+] \xrightarrow{t \cdot I - \nu} P[T_+]) \simeq P_\nu$$

as $A[T_+]$-modules. q.e.d.

Before carrying the formalism further we shall amplify certain aspects of the fundamental theorem. First we record an immediate corollary of the fact that K_o is (i.e. admits the structure of) a contracted functor.

(7.6) COROLLARY. *If we identify* K_0 *with* LK_1 *as in* (7.4) *then* (K_0, Lh) *is a contracted functor. Hence, if* A *is a ring and if* $(T, T_{\pm})$ *is an oriented cycle, then*

$$K_0(A[T_+^n]) \simeq (1 + N)^n K_0(A)$$

and

$$K_0(A[T^n]) \simeq (1 + 2N + L)^n K_0(A).$$

If $P \in \underline{\underline{P}}(A[T_+])$ *then* P *is stably isomorphic to* $P_0[T_+]$ *for some* $P_0 \in \underline{\underline{P}}(A)$ *if and only if there is a* $P' \in P(A[T_-])$ *such that* $P \otimes_{A[T_+]} A[T]$ *and* $P' \otimes_{A[T_-]} A[T]$ *in* $\underline{\underline{P}}(A[T])$ *are stably isomorphic*.

Proof. The first assertions follow immediately from (7.4) and (7.3).

In the last assertion the "only if" is trivial; we take $P' = P_0[T_-]$. For the converse, the hypothesis implies that, in $K_0(A[T])$,

$$[P \otimes_{A[T_+]} A[T]]_{A[T]}$$

$$\in \tau_+ K_0(A[T_+]) \cap \tau_- K_0(A[T_-]) = K_0(A),$$

so $[P]_{A[T_+]} = [P_0[T_+]]_{A[T_+]}$, where $P_0 = P \otimes_{A[T_+]} A$ is the augmentation of P. We have here used, of course, the exactness of

$$0 \longrightarrow K_0(A) \longrightarrow K_0(A[T_+]) \oplus K_0(A[T_-]) \longrightarrow K_0(A[T]),$$

which follows from the first part of the corollary. The equality of $[P]_{A[T]}$ and $[P_0[T_+]]_{A[T_+]}$ implies that P and $P_0[T_+]$ are stably isomorphic. q.e.d.

Remark. Horrocks [1] has shown that, if A is a commutative noetherian local ring, the last part of Corollary (7.6) is valid with the word "isomorphism" replacing "stable isomorphism" throughout. When A is commutative these two notions coincide form invertible modules.

(7.7) COROLLARY. *Let* (T, T_+) *be an oriented cycle and let* A *be a commutative ring. If* $P \in \underline{\underline{Pic}}(A[T_+])$, *and if* $P_0 = P \otimes_{A[T_+]} A \in \underline{\underline{Pic}}(A)$, *then* $P \simeq P_0[T_+]$ *if and only if there is a* $P' \in \underline{\underline{Pic}}(A[T_-])$ *such that* $P \otimes_{A[T_+]} A[T] \simeq P' \otimes_{A[T_-]} A[T]$. *In other words, the sequence*

$$0 \longrightarrow \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(A[T_+]) \oplus \mathrm{Pic}(A[T_-]) \xrightarrow{\tau} \mathrm{Pic}(A[T])$$

is exact.

Proof. As above the "only if" is trivial. If there exists a P' as above then (7.6) implies there is a $P_1 \in \underline{\underline{P}}(A)$ such that $P_1[T_+]$ and P are stably isomorphic. It follows from this that P_1 and P_0 in $\underline{\underline{P}}(A)$ are stably isomorphic, so we conclude that the invertible modules P and $P_0[T_+]$ are stably isomorphic, say $P \oplus A[T_+]^n \simeq P_0[T_+] \oplus A[T_+]^n$. Taking determinants (i.e. Λ^{n+1}) of these modules we conclude that $P \simeq P_0[T_+]$. q.e.d.

I do not know whether Pic is a contracted functor (on commutative rings) though this seems very likely. It would be interesting to find a familiar interpretation of LPic(A) = Coker(τ). This would help understand LK_0, for which we also lack an interpretation.

The functor U(= units) is contracted, and we shall now describe this situation. Let T be an infinite cyclic group with generator t. We define a natural homomorphism, for any commutative ring A,

$$Dh = Dh_{T,A}: H_0(A) \longrightarrow U(A[T])$$

$$Dh(f) = "t^f".$$

Here t^f denotes the unique element in U(A[T]) whose image in $U(A_{\underline{p}}[T])$ is $t^{f(\underline{p})}$ for each $\underline{p} \in \mathrm{spec}(A)$. Once we show that this does define a unique element of $U(A[T])$ it will be clear that Dh is a natural homomorphism. The uniqueness, moreover, is obvious. For the existence, write $1 = \Sigma\, e_n$ ($n \in \underline{\underline{Z}}$) where $\mathrm{supp}(e_n A) = f^{-1}(\{n\})$; then $t^f = \Sigma\, e_n t^n$ works. Evidently Dh is a monomorphism.

In the next proposition all functors are considered as functors from commutative rings to abelian groups.

(7.8) PROPOSITION. *Let* (T, T_+) *be an oriented cycle and let* A *be a commutative ring. The homomorphism* $Dh_{T,\,A}$: $H_0(A) \longrightarrow U(A[T])$ *induces, on passing to the quotient, an isomorphism* $H_0 \longrightarrow L_T U$. *If we use this to identify* H_0 *with* LU *then* (U, Dh) *is a contracted functor. Moreover*

$$\det\colon (K_1, h) \longrightarrow (U, Dh)$$

is a split epimorphism of contracted functors, and the diagram

$$(10) \qquad \begin{array}{ccc} K_1(A[T]) & \xrightarrow{\ \det\ } & U(A[T]) \\ \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle Dh} \\ K_0(A) & \xrightarrow[\ rk\ ]{} & H_0(A) \end{array}$$

commutes. Further, we have

$$N_{T_+} U(A) = 1 + \mathrm{nil}(A) \cdot \overline{A[T_+]},$$

where $\mathrm{nil}(A)$ *is the nil radical of* A *and* $\overline{A[T_+]} = \mathrm{Ker}(A[T_+] \longrightarrow A)$ *is the augmentation ideal.*

Proof. We have an exact sequence of complexes

$$0 \longrightarrow \mathrm{Seq}_T\, SK_1(A) \longrightarrow \mathrm{Seq}_T\, K_1(A) \xrightarrow{\ \det\ } \mathrm{Seq}_T U(A) \longrightarrow 0$$

which is split by a natural homomorphism $Seq_T g$: $Seq_T U(A) \longrightarrow Seq_T K_1(A)$. Here $g = g_A$: $U(A) \longrightarrow K_1(A)$ is the natural transformation, $g(u) = [A, u \cdot 1_A] \in K_1(A)$. It follows that all of these complexes are acyclic, because $Seq_T K_1(A)$ is. Moreover, with the aid of g, h induces a contraction, h′, of U, by the commutative diagram

$$\begin{array}{ccc} L_T K_1(A) & \xleftarrow{\;L_T g\;} & L_T U(A) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} \\ K_1(A[T]) & \xrightarrow[\;det\;]{} & U(A[T]). \end{array}$$

This, in turn, identifies $L_T U(A)$ with $Im(h') = det(Im(h))$. Recall that, for $P \in \underline{\underline{P}}(A)$, we have $h[P] = [P[T], t\, 1_{P[T]}]$. Since $det\, h[P] \in U(A[T])$ we can compute it by localizing A at each $P \in spec(A)$, and we find that

$$\begin{aligned} det\, h[P] &= t^{[P:\, A]} \\ &= Dh(rk[P]). \end{aligned}$$

Since Dh is clearly a monomorphism, this shows that Dh induces an isomorphism of $H_0(A)$ with $L_T U(A)$ such that Dh corresponds to h′, and hence so that diagram (10) commutes.

There remains only the calculation of $N_{T_+} U(A)$. The conclusions above imply it equals $det(N_{T_+} K_1(A))$, and hence clearly contains $1 + (nil\ A)\ \overline{A[T_+]}$. Moreover we have seen that every element of $N_{T_+} K_1(A) = K_1(A[T_+], \overline{A[T_+]})$ is represented by a unipotent. Over a field, and hence over an integral domain, unipotents have all eigenvalues 1, and hence determinant 1. Therefore, in general, if α is unipotent, $1 - det(\alpha)$ lies in every prime ideal, so it is nilpotent, i.e. $det(\alpha)$ is unipotent. We conclude that an element of $N_{T_+} U(A)$ is of the form $1 + \alpha$ with $\alpha \in \overline{A[T_+]}$ and nilpotent. We must further show that $\alpha \in \text{nil}(A) \cdot \overline{A[T_+]}$. If $\underline{p} \in \text{spec}(A)$ then $(A/\underline{p})\ [T_+]$ is an integral domain, so $\alpha \in \underline{p}\ A[T_+]$. Varying $\underline{p}$ we have $\alpha \in \text{nil}(A) \cdot A[T_+] \cap \overline{A[T_+]} = \text{nil}(A) \cdot \overline{A[T_+]}$. q.e.d.

Proposition (7.8) says that SK_1 is the kernel of a morphism of contracted functors. Therefore we conclude that:

(7.9) COROLLARY. $(SK_1, Sh) = \text{Ker}((K_1, h) \xrightarrow{\det} (U, Dh))$ *is a contractor functor, and* $L\ SK_1 = Rk_0$.

Proposition (7.8) describes NU and $LU = H_0$. We round off that discussion with:

(7.10) PROPOSITION. $NH_0 = 0 = LH_0$. *Hence* $L^i N^j U = 0$ *whenever* $i > 1$ *or* $i = 1$ *and* $j > 0$.

Proof. It suffices to show that, if T is an infinite cyclic group, say with generator t, then every idempotent in A[T] lies in A. Let $e = \Sigma\, \alpha_i t^i$ be idempotent; we claim $\alpha_i = 0$ for $i \neq 0$. If A is an integral domain then A[T] is also. Therefore we have $\alpha_i \in \text{nil}(A)$ for $i \neq 0$, and α_0 maps onto an idempotent in A/nil(A). According to (III, 2.10) there is an idempotent $e_0 \in A$ such that $e_0 \equiv \alpha_0 \bmod \text{nil } A$. Now $e - ee_0 \in \text{nil}(A)\ A[T]$, and it is idempotent, so $e = ee_0$. Similarly $e_0 = ee_0$ so $e = e_0$. q.e.d.

(7.11) COROLLARY. *Let* A *be a commutative ring and let* (T, T_+) *be an oriented cycle. Then*

$$U(A[T_+^n]) \cong (1 + N)^n\ U(A)$$

and

$$U(A[T^n]) \cong (1 + 2N)^n\ U(A) \oplus nH_0(A).$$

Proof. Since U is a contracted functor (7.3) implies the first formula, as well as

$$U(A[T^n]) \simeq (1 + 2N + L)^n \, U(A).$$

But $(1 + 2N + L)^n = (1 + 2N)^n + n(1 + 2N)^{n-1} L + \ldots$ and (7.10) implies $(1 + 2N + L)^n U = ((1 + 2N)^n + nL) U$. Since $LU = H_o$ (see (7.8)) this concludes the proof.

§8. THE LONG MAYER-VIETORIS SEQUENCES

We shall write

<u>Cart</u>

for the category whose objects are cartesian squares

$$C: \left(\begin{array}{ccc} A & \xrightarrow{c_2} & A_2 \\ {\scriptstyle c_1}\downarrow & & \downarrow{\scriptstyle c_2'} \\ A_1 & \xrightarrow[c_1']{} & A' \end{array} \right)$$

in the category (rings) such that f_1 or f_2 is surjective. A morphism is just a morphism of diagrams, in the usual sense.

If F: (rings) $\longrightarrow$ $\underline{\underline{Z}}$-mod is a functor then we have the sequence

$$F(A) \xrightarrow{(c_1,\ -\ c_2)} F(A_1) \oplus F(A_2) \xrightarrow{\begin{pmatrix} c_1' \\ c_2' \end{pmatrix}} F(A')$$

associated with F and C. We shall always understand this sequence below, even when writing it with the arrows unlabeled.

(8.4) DEFINITION. A <u>Mayer-Vietoris pair</u> is a triple (F_1, F_o, δ), where F_1, F_o: (rings) $\longrightarrow$ $\underline{\underline{Z}}$-mod are functors, and where δ associates to each $C \in \underline{\underline{\text{Cart}}}$ as above, a homomorphism

$$\delta_C: F_1(A') \longrightarrow F_o(A)$$

which is natural in C and is such that the sequence

$$M - V(F_1, F_o; C) =$$

$$(F_1(A) \longrightarrow F_1(A_1) \oplus F_1(A_2) \longrightarrow F_1(A') \xrightarrow{\delta} F_o(A)$$
$$\longrightarrow F_o(A_1) \oplus F_o(A_2) \longrightarrow F_o(A'))$$

is exact.

(8.1) PROPOSITION. Let $((F_1, h_1), (F_o, h_o), \delta)$ *be a Mayer-Vietoris pair of contracted functors and let* J *denote either* N *or* L. *Then* $((JF_1, Jh_1), (JF_o, Jh_o), J\delta)$ *is again a Mayer-Vietoris pair of contracted functors*. ($J\delta$ will be defined in course of the proof.)

Proof. Let $(T, T_\pm)$ be an oriented cycle, and let $C \in \underline{\underline{\text{Cart}}}$ as above. Then clearly

$$C[T_\pm],\ C[T] \in \underline{\underline{\text{Cart}}}$$

also. We define $N\delta$ so that

$$0 \longrightarrow M - V(NF_1, NF_o; C) \longrightarrow M - V(F_1, F_o; C[T_+])$$
$$\longrightarrow M - V(F_1, F_o; C) \longrightarrow 0$$

is a (split) exact sequence of complexes. In particular the left hand term is acyclic.

Similarly, we define $L\delta$ so that

(*) $$M - V(F_1, F_o; C[T_+]) \oplus M - V(F_1, F_o; C[T\_]) \xrightarrow{\tau}$$
$$M - V(F_1, F_o; C[T]) \longrightarrow M - V(LF_1, LF_o; C) \longrightarrow 0$$

is an exact sequence of complexes. If we replace the left hand term by Im(τ) we obtain an acyclic subcomplex of the middle term. The long homology sequence of the resulting short exact sequence of complexes shows that the right hand term, M - V(LF_1, LF_0; C), is acyclic except, perhaps, in the middle position of

$$(**) \qquad LF_0(A) \longrightarrow LF_0(A_1) \oplus LF_0(A_2) \longrightarrow LF_0(A').$$

But this does not involve δ, and the contractibility of F_0 implies that the exact sequence (*) splits on the right, as a sequence of complexes, at the terms occurring in (**). Hence (**) is also exact. The contractibility of NF_i and LF_i follows from (7.2). q.e.d.

(8.2) COROLLARY. *Let* (F, h) *be a contracted functor, and suppose there is a* δ *such that* ((F, h), (LF, Lh), δ) *is a Mayer-Vietoris pair*. Then if

$$C = \begin{array}{ccc} A & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A' \end{array} \quad \varepsilon \ \underline{\underline{\mathrm{Cart}}}$$

there is a "long Mayer-Vietoris sequence",

$$F(A) \longrightarrow \cdots \longrightarrow L^{n-1}F(A') \longrightarrow L^nF(A) \longrightarrow$$

$$L^nF(A_1) \oplus L^nF(A_2) \longrightarrow L^nF(A') \longrightarrow L^{n+1}F(A) \cdots$$

which is exact. Moreover ((N^iF, N^ih), (LN^iF, LN^ih), δ) *is likewise a Mayer-Vietoris pair, so there is a corresponding long Mayer-Vietoris sequence for the functors* (L^nN^iF) $n \geq 0$, *for each* $i \geq 0$.

Proof. The last assertion follows from (8.1), which also implies that ((LF, Lh), (L^2F, L^2h), Lδ) is a Mayer-Vietoris pair of contracted functors, and similarly for

$((L^nF, L^nh), (L^{n+1}F, L^{n+1}h), L^n\delta)$ for all $n \geq 0$. We obtain the long sequence by splicing $M - V(L^{n-1}F, L^nF, L^{n-1}\delta)$ with $M - V(L^nF, L^{n+1}F, L^n\delta)$ for each $n > 0$.

According to Milnor's Theorem (IX, 5.3) we have a Mayer-Vietoris pair (K_1, K_o, δ). According to the Fundamental Theorem (7.4) we can identify K_o with LK_1 so that (K_1, h) and (K_o, Lh) are contracted functors, where h is the homomorphism in (7.4). Therefore we are in a position to apply (8.2), from which we immediately deduce:

(8.3) THEOREM. Let C be as in (8.2). Then, for each $i \geq 0$, there is a long Mayer-Vietoris sequence

$$F(A) \longrightarrow \cdots \longrightarrow L^{n-1}F(A') \longrightarrow L^nF(A) \longrightarrow$$

$$L^nF(A_1) \oplus L^nF(A_2) \longrightarrow L^nF(A') \longrightarrow L^{n+1}F(A) \longrightarrow \cdots,$$

where $F = N^iK_1$. We recall from (7.4) that

$$L^nN^iK_1 = N^iL^nK_1,$$

$$NK_1 = \text{Nil}, \text{ and}$$

$$LK_1 = K_o.$$

In the case $i = 0$ of this theorem the sequence above becomes

$$K_1(A) \longrightarrow \cdots \longrightarrow K_o(A') \longrightarrow K_{-1}(A) \longrightarrow K_{-1}(A_1)$$

$$\oplus K_{-1}(A_2) \longrightarrow K_{-1}(A') \longrightarrow K_{-2}(A) \longrightarrow \cdots,$$

where we write $K_{-n}(A) = L^nK_o(A) = L^{n+1}K_1(A)$.

§9. K_o OF THE PROJECTIVE LINE OVER A.

The group we propose to study is $K_o(P^1(A))$, where

$P^1(A)$ is the "projective line" over A. This group will be defined by

$$K_i(P^1(A)) = K_i(\underline{\underline{P}}(P^1(A))), \qquad (i = 0, 1)$$

where $\underline{\underline{P}}(P^1(A))$ is the category of "algebraic vector bundles over $P^1(A)$. We shall define none of these terms, but rather directly define the category $\underline{\underline{P}}(P^1(A))$; this is clearly sufficient for our purpose.

As in the last section, T is an infinite cyclic group with generator t, and $T_{\pm}$ denote the submonoids generated by $t^{\pm 1}$, respectively. For any ring A, the square of inclusions

$$\begin{array}{ccc} A & \subset & A[T_-] \\ \cap & & \cap \tau_- \\ A[T_+] & \underset{\tau_+}{\subset} & A[T] \end{array}$$

is cartesian (because $A = A[T_+] \cap A[T_-]$). However, since neither τ_+ nor τ_- is surjective we cannot obtain a Mayer-Vietoris sequence involving the groups $K_i(A)$. Nevertheless we can form the fibre product of the categories

$$\begin{array}{ccc} & & \underline{\underline{P}}(A[T_-]) \\ & & \downarrow \\ \underline{\underline{P}}(A[T_+]) & \longrightarrow & \underline{\underline{P}}(A[T]), \end{array}$$

and it is this fibre product that we denote by $\underline{\underline{P}}(P^1(A))$. Thus we have a cartesian square

$$
(1) \qquad \begin{array}{ccc}
\underline{\underline{P}}(P^1(A)) & \xrightarrow{\;g_-\;} & \underline{\underline{P}}(A[T_-]) \\
\Big\downarrow g_+ & & \Big\downarrow \tau_- \\
\underline{\underline{P}}(A[T_+]) & \xrightarrow[\;\tau_+\;]{} & \underline{\underline{P}}(A[T])
\end{array}
$$

of functors of categories with product ($\oplus$) in the sense of (VII, §3). Recall that the objects of $\underline{\underline{P}}(P^1(A))$ are triples (P_+, α, P_-) where $P_\pm \in \underline{\underline{P}}(A[T_+])$ and $\alpha\colon \tau_+P_+ \longrightarrow \tau_-P_-$ is an $A[T]$-isomorphism. Here we have written $\tau_\pm P_\pm = P_+ \otimes_{A[T_+]} A[T]$. A morphism $(P_+, \alpha, P_-) \longrightarrow (Q_+, \beta, Q_-)$ is a pair of morphisms $f_\pm\colon P_\pm \longrightarrow Q_\pm$ such that $(\tau_-f_-)\alpha = \beta(\tau_+f_+)$. The functors $g_\pm$ are just the left and right coordinate projections.

The functors (τ_+, τ_-) are easily seen to be a "cofinal pair" in the sense of (VII, 3.2). Hence we can apply (VII, 4.3) to (1) to obtain the Mayer-Vietoris sequence

$$
\begin{aligned}
(2) \qquad & K_1(P^1(A)) \xrightarrow{\;G_1\;} K_1(A[T_+]) \oplus K_1(A[T_-]) \xrightarrow{\;\tau_1\;} \\
& K_1(A[T]) \xrightarrow{\;\delta\;} K_o{'}(P^1(A)) \xrightarrow{\;G_o\;} \\
& K_o(A[T_+]) \oplus K_o(A[T]) \xrightarrow{\;\tau_o\;} K_o(A[T]).
\end{aligned}
$$

Here we have $G_i = \begin{pmatrix} g_+ \\ -g_- \end{pmatrix}$ and $\tau_i = (\tau_+, \tau_-)$. The sequence is guaranteed by (VII, 4.3) to be exact except, perhaps, at $K_1(A[T_+]) \oplus K_1(A[T_-])$. We note further that there occurs a group $K_o{'}(P^1(A))$ rather than $K_o(P^1(A))$. The former is a quotient of the latter, defined in (VII, §4). We shall discuss below the possible discrepency between these two groups.

According to the Fundamental Theorem (7.4) K_1 is a contracted functor with $K_o = LK_1$, so we have canonical isomorphisms:

(3) $\mathrm{Ker}(\tau_1) = K_1(A)$

(4) $\mathrm{Coker}(\tau_1) = LK_1(A) \simeq K_o(A)$

(5) $\mathrm{Ker}(\tau_o) = K_o(A)$.

If $\alpha \in GL_n(A)$ then $([\alpha], -[\alpha]) \in \mathrm{Ker}(\tau_1)$ $(\simeq K_1(A))$ is the image under G_1 of $[(A^n[T_+], 1_{A^n[T]}, A^n[T_-]), (\alpha[T_+], \alpha[T_-])]$ $\in K_1(P^1(A))$. This shows that (2) <u>is exact</u>, even at the point not covered by (VII, 4.3).

Having used (3), we now use (4) and (5) to extract a short exact sequence

(6) $$0 \longrightarrow K_o(A) \xrightarrow{d} K_o{}'(P^1(A)) \xrightarrow{e} K_o(A) \longrightarrow 0,$$

where d is induced by δ, using (4), and e is induced by G_o, using (5).

It is convenient now to introduce the additive functors

$$h^n\colon \underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}(P^1(A)),$$

$$h^n(P) = (P[T_+], t^n 1_{P[T]}, P[T_-]),$$

and the corresponding homomorphisms

$$h^n\colon K_o(A) \longrightarrow K_o{}'(P^1(A)),\ h^n[P] = [h^n(P)]'.$$

The homomorphism e in (6) is defined by

$$e[P_+, \alpha, P_-]' = [P_+{}'] = [P_-{}'] \in K_o(A)$$

where $P_\pm{}' \in \underline{\underline{P}}(A)$ is defined by $P_\pm{}' = P_\pm \otimes_{A[T_\pm]} A$, the tensor product being with respect to the augmentation. This follows from (7.6). Thus we see that e is split by each h^n, so the h^n are monomorphisms and we have

$$K_o'(P^1(A)) = \mathrm{Im}(h^o) \oplus \mathrm{Im}(d),$$

with each summand isomorphic to $K_o(A)$.

In order to determine d, it suffices, as we see from the decomposition of $K_1(A[T])$ in (7.4), to evaluate δ (in the Mayer-Vietoris sequence (2) above) on the $K_0(A)$-term, $\mathrm{Im}(h_{T,\ A}) \subset K_1(A[T])$. Recall that $h = h_{T,\ A}\colon K_o(A) \longrightarrow K_1(A[T])$ is defined by $h[P] = [P[T],\ t \cdot 1_{P[T]}]$ for $P \in \underline{\underline{P}}(A)$. It follows from (VII, 4.3) (where we wrote ∂ in place of the present δ) that

$$\begin{aligned}\delta[P[t],\ t \cdot 1_{P[T]}] &= [P[T_+],\ t \cdot 1_{P[T]},\ P[T_-]]' \\ &\quad - [P[T_+],\ 1_{P[T]},\ P[T_-]]' \\ &= h^1[P] - h^o[P].\end{aligned}$$

Thus $d = h^1 - h^o$, and we have

$$\text{(7)}\qquad K_o'(P^1(A)) = h^oK_o(A) \oplus h^1K_o(A).$$

In $K_1(A[T])$ we have $[P[T],\ t^21_{P[T]}] = 2[P[T],\ t\ 1_{P[T]}]$ for $P \in \underline{\underline{P}}(A)$. But a calculation like that above shows that $\delta[P[T],\ t^21_{P[T]}] = h^2[P] - h^o[P]$. Thus we conclude that $h^2 - h^o = 2(h^1 - h^o)$, i.e.

$$\text{(8)}\qquad h^2 - 2h^1 + h^o = 0.$$

If we formally define a product on the h^n's by $h^nh^m = h^{n+m}$ then we can write (8) more suggestively as

$$\text{(8')}\qquad (h^1 - h^o)^2 = 0.$$

Given that the $h^n K_o(A)$ generate $K_o'(P^1(A))$, the relation (8), together with its "translates", $h^{n+1} - 2h^n + h^{n-1} = 0$ ($n \in \underline{\underline{Z}}$), already imply (7) above. Thus (8) and its "translates" are a complete set of relations between the h^n. We now summarize:

(9.1) THEOREM. *Let* A *be any ring, and define*

$$h^n : K_o(A) \longrightarrow K_o'(P^1(A)) \qquad (n \in \underline{\underline{Z}})$$

by $h^n[P] = [P[T_+],\ t^n \cdot 1_{P[T]},\ P[T_-]]'$. *Then the* h^n *are monomorphisms, and a complete set of additive relations between them is*

$$h^{n+1} - 2h^n + h^{n-1} = 0 \qquad (n \in \underline{\underline{Z}}).$$

Moreover

$$K_o'(P^1(A)) = h^0 K_o(A) \oplus h^1 K_o(A).$$

When A is commutative this theorem has a much more satisfactory formulation. Before giving that, however, we must comment on the troublesome fact that we have a K_o' above, rather than the *bona fide* K_o. We recall from (VII, §4) that

$$K_o'(P^1(A)) = K_o(P^1(A))/M$$

where M is the subgroup of $K_o(P^1(A))$ generated by elements of the following type. Suppose $P = (P_+, \alpha, P_-) \in \underline{\underline{P}}(P^1(A))$ and that $\alpha_1, \alpha_2 \in \mathrm{Aut}_{A[T]}(\tau_+ P_+)$. Writing $P\alpha_1 = (P_+, \alpha\alpha_1, P_-) \in \underline{\underline{P}}(P^1(A))$ we put

$$\langle P, \alpha_1, \alpha_2 \rangle = [P\ \alpha_1\alpha_2] + [P] - ([P\ \alpha_2] + [P\ \alpha_2])$$
$$\in K_o(P^1(A)).$$

These elements are the generators of M (for variable P and

and α_i). We had to factor out M in order to define the connecting homomorphism δ in the Mayer-Vietoris sequence. On the other hand (VII, 4.2) gives a criterion for the vanishing of M. The condition is that the cartesian square (1) above be "E-surjective" in the sense of (VII, 3.3). In the present circumstances that condition is easily seen to be equivalent to the following one:

(*) <u>Given</u> $\alpha \in GL(A[T])$ <u>and</u> $\varepsilon \in E(A[T])$, <u>there exist</u> $\varepsilon_{\pm} \in E(A[T_{\pm}])$ <u>such that</u> $\alpha\varepsilon = \varepsilon_- \alpha \varepsilon_+$.

This is precisely the condition discussed at the end of §5. It is valid if A is a field, though not in general. <u>If</u> (*) <u>holds then</u> $K_0'(P^1(A)) = K_0(P^1(A))$, so the latter is then completely determined by (9.1).

A noteworthy feature of Theorem (9.1) is that the $K_0'(P^1(A))$ considered is defined using only $\oplus$, despite the fact that short exact sequences in $\underline{\underline{P}}(P^1(A))$ do not (at least not obviously) split. Nevertheless the class of every object in $K_0'(P^1(A))$ coincides with the class of a direct sum of objects of the form $(P[T_+], t^n 1_{P[T]}, P[T_-])$ $(P \in \underline{\underline{P}}(A), n \in \underline{\underline{Z}})$. Granted condition (*) above, therefore, we would be able to conclude that every object of $\underline{\underline{P}}(P^1(A))$ is stably isomorphic to such a direct sum.

Now assume that A is commutative. Then there is a natural tensor product in $\underline{\underline{P}}(P^1(A))$:

$$(P_+, \alpha, P_-) \otimes (Q_+, \beta, Q_-)$$

$$= (P_+ \otimes_{A[T_+]} Q_+, \alpha \otimes_{A[T]} \beta, P_- \otimes_{A[T_-]} Q_-).$$

Moreover the functor

$$h^0 : \underline{\underline{P}}(A) \longrightarrow \underline{\underline{P}}(P^1(A))$$

introduced above preserves tensor products, and

$$h^o(A) \otimes W \simeq W \quad \text{for all } W \in \underline{\underline{P}}(P^1(A)).$$

Further, we have

$$\left.\begin{array}{l} h^n(P) \otimes h^m(P) \simeq h^{n+m}(P) \\ h^n(P) \simeq h^n(A) \otimes h^o(P) \end{array}\right\} \quad (P \in \underline{\underline{P}}(A)).$$

This shows that $K_o(P^1(A))$ is a commutative ring, that h^o makes it a $K_o(A)$-algebra, and that the subgroup generated by the images of all the h^n: $K_o(A) \longrightarrow K_o(P^1(A))$ is just the subalgebra generated over $K_o(A)$ by

$$\underline{\underline{h}} = [h^1(A)].$$

I have not been able to confirm the analogue of the relation (8) above:

$$(\underline{\underline{h}} - 1)^2 = 0 \ ?$$

We can, however, show that $K_o{}'(P^1(A))$ inherits an algebra structure and then the relation above makes sense and is valid in $K_o{}'$. What is required is to show that the subgroup $M \subset K_o(P^1(A))$ defined above is an ideal. Let $\langle P, \alpha_1, \alpha_2 \rangle$ be one of the generators of M, as above, and let $Q = (Q_+, \beta, Q_-) \in \underline{\underline{P}}(P^1(A))$. We must show that $\langle P, \alpha_1, \alpha_2 \rangle [Q] \in M$. If $\gamma \in \mathrm{Aut}_{A[T]}(j_+P_+)$ put $\gamma' = \gamma \otimes_{A[T]} 1_{j_+Q_+}$. Then clearly

$$P\gamma \otimes Q = (P \otimes Q)\gamma'.$$

It follows from this that $\langle P, \alpha_1, \alpha_2 \rangle [Q] = \langle P \otimes Q, \alpha_1', \alpha_2' \rangle$.

Now we can restate (9.1) in the commutative case, as follows:

(9.2) COROLLARY. Let A be a commutative ring. Then $K_o{}'(P^1(A))$ is (via h^o) a commutative $K_o(A)$-algebra. It is presented, as a $K_o(A)$-algebra, by a single generator, $\underline{\underline{h}}' = [h^1[A]]'$, and the single relation,

$$(\underline{\underline{h}}' - 1)^2 = 0.$$

§10. GROUP RINGS OF ABELIAN GROUPS

In this section we shall use the theory developed above to compute K_o and K_1 in some special cases. These examples will include group rings, $\underline{\underline{Z}}\pi$, where π is a finitely generated abelian group.

We shall fix an oriented cycle (T, T_+). Further, we shall call a ring A quasi-regular if it contains a two sided nilpotent ideal J such that A/J is right regular. Thus any right Artinian ring is quasi-regular. It follows also from Hilbert's Basis and Syzygy Theorems that $A[T_+^n]$ and $A[T^n]$ are quasi-regular for all $n \geq 0$ if A is.

(10.1) PROPOSITION. If the ring A is quasi-regular then $L^nN^iK_o(A) = 0$ whenever $n > 0$ or $i > 0$. If A is also commutative then

$$N^i\det\colon N^iK_1(A) \longrightarrow N^iU(A)$$

is an isomorphism, so $N^iSK_1(A) = 0$, for all $i > 0$.

Proof. Let $F = L^nN^iK_o$ with $n, i \geq 0$. Then NF(A) and LF(A) are both direct summands of F(A[T]), and, as remarked above, A[T] is again quasi-regular. Hence, by an induction on (n, i), it suffices to prove that $NK_o(A) = 0 = LK_o(A)$ whenever A is quasi-regular; this will establish the first assertion of the proposition.

Let J be a nilpotent two sided ideal in A such that B = A/J is right regular. In the commutative diagram

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A[T_+]) & \longrightarrow & K_0(A[T]) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(B) & \longrightarrow & K_0(B[T_+]) & \longrightarrow & K_0(B[T]) \end{array}$$

the bottom arrows are isomorphisms, by Grothendieck's Theorem (3.1). According to (IX, 1.3 (0)), the verticals are all isomorphisms. Therefore the top arrows are isomorphisms, and it follows immediately from the definitions that $NK_0(A) = 0 = LK_0(A)$.

For the last assertion, that $N^i\det\colon N^iK_1(A) \longrightarrow N^iU(A)$ is an isomorphism if A is commutative and quasi-regular, we can argue as above, by induction on i, and reduce to the case i = 1. Then, with the notation above, we have a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(A[T_+], JA[T_+]) & \longrightarrow & K_1(A[T_+]) & \longrightarrow & K_1(B[T_+]) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \\
0 & \longrightarrow & U(A[T_+], JA[T_+]) & \longrightarrow & U(A[T_+]) & \longrightarrow & U(B[T_+]) & \longrightarrow & 0.
\end{array}$$

The zero on the right is present because J is nilpotent. It follows from (5.4) that the right hand vertical is isomorphic to $K_1(B) \longrightarrow U(B)$, and hence $NK_1(B) = 0$. Therefore we can deduce from the diagram that det: $NK_1(A) \longrightarrow NU(A)$ is isomorphic to the kernel of the morphism of homomorphisms,

$$(K_1(A[T_+], JA[T_+]) \xrightarrow{\det} U(A[T_+], JA[T_+]) \longrightarrow$$

$$(K_1(A, J) \xrightarrow{\det} U(A, J)),$$

corresponding to the augmentation $A[T_+] \longrightarrow A$. But since J and $JA[T_+]$ are nilpotent, it follows from (IX, 3.10) that both of these dets are isomorphisms, and hence so also is their kernel. q.e.d.

(10.2) PROPOSITION. *Let*

$$\begin{array}{ccc} A & \longrightarrow & A_2 \\ \downarrow & & \downarrow f_2 \\ A_1 & \xrightarrow[f_1]{} & A' \end{array}$$

be a cartesian square of ring homomorphisms in which f_1 *or* f_2 *is surjective, and assume that* A_1, A_2, *and* A' *are quasi-regular. Then*

(i) $L^nN^iK_o(A) = 0$ *if* $n > 0$ *and* $i > 0$ *or if* $n > 1$ *and* $i \geq 0$;

(ii) $LK_o(A) = \text{Coker}(K_o(A_1) \oplus K_o(A_2) \longrightarrow K_o(A'))$;

and

(iii) $N^iK_o(A) = \text{Coker}(N^iK_1(A_1) \oplus N^iK_1(A_2) \longrightarrow N^iK_1(A'))$ for $i > 0$.

If the rings above are commutative then (iii)′ $N^iK_o(A) = \mathrm{Coker}(N^iU(A_1) \oplus N^iU(A_2) \longrightarrow N^iU(A'))$ for $i > 0$.

Proof. Let $F = N^iK_j$ with $i \geq 0$ and $j = 0$ or 1. Then we have the Mayer-Vietoris sequence,

$$\ldots L^{n-1}F(A_1) \oplus L^{n-1}F(A_2) \longrightarrow L^{n-1}F(A') \longrightarrow L^nF(A) \longrightarrow L^nF(A_1) \oplus L^nF(A_2) \ldots$$

(Theorem (8.3)). If $n > 1$ or if $n > 0$ and $i > 0$, and if $j = 0$, then (10.1) implies all terms surrounding $L^nF(A) = L^nN^iK_o(A)$ vanish, and hence the latter vanishes also, thus proving (i). Similarly we obtain (ii) from the exact sequence and (10.1) when $n = 1$ and $i = 0 = j$, and we obtain (iii) in the same way when $n = 1 = j$, thanks to the fact that $LN^iK_1 = N^iLK_1 = N^iK_o$. Finally, the equivalence of (iii)′ with (iii) in the commutative case follows from (10.1) again. q.e.d.

(10.3) COROLLARY. *In the setting of* (10.2) *we have, for all* $n > 1$,

$$K_o(A[T^n]) = (1 + 2N)^n K_o(A) \oplus nLK_o(A),$$

and

$$K_1(A[T^n]) = (1 + 2N)^n K_1(A) \oplus nK_o(A) \oplus \frac{n(n-1)}{2} LK_o(A).$$

Proof. For $F = K_o$ or K_1 we have $F(A[T^n]) = (1 + 2N + L)^n F(A)$ (see (7.3)), and we have

$$(1 + 2N + L)^n = (1 + 2N)^n + n(1 + 2N)^{n-1}L + \frac{n(n-1)}{2}(1 + 2N)^{n-1}L^2 + \ldots$$

All multiples of NL or L^2 kill $K_o = LK_1$ and all multiples of NL^2 or L^3 therefore kill K_1; this follows from (10.2). The corollary follows immediately from these observations. q.e.d.

Proposition (10.2) and its corollary apply notably in the following case: Let R be a Dedekind ring with field of fractions L, let A be an R-order in a semi-simple L-algebra, let A_2 be a maximal order containing A, and let $\underline{c} = \underline{c}_{A_2/A}$ be the conductor. Put $A_1 = A/\underline{c}$ and $A' = A_{2/\underline{c}}$. These are Artin rings, and hence quasi-regular. Moreover A_2 is hereditary, and therefore also quasi-regular (in fact regular). In case $A = R\pi$, the group ring of a finite group π of order not divisible by char(L), then $A[T^n] = R[\pi \times T^n]$ so we can use (10.3) to reduce the calculation of $K_j(R[\pi \times T^n])$ to calculations in $R\pi$.

(10.4) THEOREM. *Let* A *be a commutative noetherian ring of dimension* ≤ 1 *with* nil (A) = 0. *Assume that the integral closure*, B, *of* A *in its full ring of fractions is a finitely generated* A-*module, and let* $\underline{c} = \underline{c}_{B/A}$ *be the conductor. Put* $A' = A/\underline{c}$ *and* $B' = B/\underline{c}$.

(a) *For all* $n \geq 0$, $\det_o(A[T^n]): Rk_o(A[T^n]) \longrightarrow Pic(A[T^n])$ *is an isomorphism, and so likewise is* $\det_o(A[T_+^n])$.

(b) *We have*

$$K_o(A[T^n]) = (1 + 2N)^n K_o(A) \oplus nLK_o(A)$$

and

$$K_1(A[T^n]) = (1 + 2N) K_1(A) \oplus nK_o(A) \oplus \frac{n(n-1)}{2} \cdot LK_o(A).$$

(c) *The group* $LK_o(A)$ *is isomorphic to*

$$\mathrm{Coker}(H_0(A') \oplus H_0(B) \longrightarrow H_0(B')),$$

which is a free abelian group of rank $h_0(A) - (h_0(A') + h_0(B)) + h_0(B')$.

(d) If $G = T_+^n$ then

$$\mathrm{Ker}(K_0(A[G]) \longrightarrow K_0(A))$$
$$\simeq ((1 + N)^n - 1)\, K_0(A)$$
$$\simeq (1 + \mathrm{nil}(B')\, \overline{B'[G]})/(1 + \mathrm{nil}(A')\, \overline{A'[G]}).$$

It vanishes if and only if $\mathrm{nil}(B') = 0$. The powers of $\mathrm{nil}(B')$ induce a finite filtration on this group whose successive factors are isomorphic to A'- module quotients of $(B'/\mathrm{nil}(B'))[G]$.

Proof. The hypotheses imply that B is a finite product of Dedekind rings, and that A' and B' are Artin rings. Hence these three rings are quasi-regular, so part (b) follows from (10.3).

(a) Let $G = T^n$ or T_+^n. To show that $\det_0(A[G])$ is an isomorphism it suffices, according to (IX, 5.13), to show the following:

(i) $\det_0(A'[G])$ and $\det_0(B[G])$ are isomorphisms; and

(ii) $\det_1(B'[G])$ is an isomorphism.

Since A' and B are quasi-regular it follows from Grothendieck's Theorem (3.1) that $\det_0(B[G])$ is isomorphic to $\det_0(B)$, and similarly for A'. Since A' and B are of dimension ≤ 1 it follows from (IX, 3.8) that $\det_0(A')$ and $\det_0(B)$ are isomorphisms, and this proves (i).

Part (ii) says that $SK_1(B'[G]) = 0$. We have $SK_1(B'[G]) = PSK_1(B')$ where $P = (1 + N)^n$ if $G = T_+^n$, and $P = (1 + 2N)^n$

$+ nL + \frac{n(n-1)}{2} L^2$ if $G = T^n$, the latter formula coming from (10.3). It follows from (10.1) that $N^i SK_1(B') = 0$ for all $i > 0$, and $SK_1(B') = 0$ because B' is an Artin ring. Further $LSK_1(B') = Rk_0(B') = 0$, for the same reason. Finally (10.1) implies $L^2K_1(B') = LK_0(B') = 0$ so $L^2SK_1(B') = 0$ also. This proves (ii), and hence part (a).

(c) It follows from (10.2) (ii) that

$$LK_0(A) \simeq \operatorname{Coker}(K_0(A') \oplus K_0(B) \longrightarrow K_0(B')).$$

Since B' is Artinian, $K_0(B') \longrightarrow H_0(B')$ is an isomorphism, so the above cokernel is unaltered if we replace K_0 by H_0 throughout. Therefore it follows from (IX, 5.11) that there is an exact sequence

$$0 \longrightarrow H_0(A) \longrightarrow H_0(A') \oplus H_0(B) \longrightarrow H_0(B')$$
$$\longrightarrow LK_0(A) \longrightarrow 0,$$

and that $LK_0(A)$ is a torsion free, and hence free, abelian group of rank $h_0(A) - (h_0(A') + h_0(B)) - h_0(B')$.

(d) Let $G = T_+^n$ and consider the morphism of Mayer-Vietoris sequences

$$\begin{array}{ccccccc}
\longrightarrow & K_1(B'[G]) & \longrightarrow & K_o(A[G]) & \longrightarrow & K_o(A'[G]) \oplus K_o(B[G]) \\
& \downarrow & & \downarrow & & \qquad\quad \downarrow (\simeq) \\
\longrightarrow & K_1(B') & \longrightarrow & K_o(A) & \longrightarrow & K_o(A') \oplus K_o(B)
\end{array}$$

The right vertical is an isomorphism because A' and B are quasi-regular. Moreover the left arrow is isomorphic to $U(B'[G]) \longrightarrow U(B')$, via det. Taking kernels of the vertical arrows we deduce that

$$\operatorname{Ker}(K_o(A[G]) \longrightarrow K_o(A))$$

$$= ((1 + N)^n - 1)\, K_o(A)$$

$$\simeq \operatorname{Coker}(U(A'[G], \overline{A'[G]}) \oplus U(B[G], \overline{B[G]}) \longrightarrow U(B'[G], \overline{B'[G]})).$$

However it follows easily from (7.8) that, for any commutative ring C, $U(C[G], \overline{C[G]}) = 1 + \operatorname{nil}(C)\, \overline{C[G]}$. Since $\operatorname{nil}(B) = 0$ we conclude that the group above equals

$$(1 + \operatorname{nil}(B')\, \overline{B'[G]})/(1 + \operatorname{nil}(A')\, \overline{A'[G]}).$$

Let $\underline{b}_i = \operatorname{nil}(B')^i \cdot \overline{B'[G]}$, $(i \geq 1)$. Then $\underline{b}_i$ is a nilpotent ideal in $B'[G]$. Since $\underline{b}_i^{\,2} \subset \underline{b}_{2i} \subset \underline{b}_{i+1}$ we see that, in $U(B'[G]/\underline{b}_{i+1})$, the group $1 + (\underline{b}_i/\underline{b}_{i+1})$ is isomorphic to the additive group of $\underline{b}_i/\underline{b}_{i+1} \simeq (\operatorname{nil}(B')^i/\operatorname{nil}(B')^{i+1}) \otimes_{B'} \overline{B'[G]} \simeq (B'/\operatorname{nil}(B')) \otimes_{B'} \overline{B'[G]} \simeq \overline{(B'/\operatorname{nil}(B'))[G]}$. If we factor out the image in $1 + (\underline{b}_i/\underline{b}_{i+1})$ of $(1 + \operatorname{nil}(A')\overline{A'[G]}) \cap (1 + \underline{b}_i)$ this has the effect, via the isomorphism above, of factoring out an A'-submodule of $\overline{(B'/\operatorname{nil}(B'))[G]}$. This establishes the last assertion of part (d).

Evidently the group above vanishes if and only if $\operatorname{nil}(A')\overline{A'[G]} = \operatorname{nil}(B')\overline{B'[G]}$, and this is clearly the case if $\operatorname{nil}(B') = 0$. Conversely the equality $\operatorname{nil}(A')\overline{A'[G]} = \operatorname{nil}(B')\overline{B'[G]}$ implies $\operatorname{nil}(B') \subset A'$, i.e. that ${}^B\!\sqrt{\underline{c}} \subset A$. Since the conductor $\underline{c}$ is the largest B-ideal in A it follows that $\underline{c} = {}^B\!\sqrt{\underline{c}}$, i.e. $\operatorname{nil}(B') = 0$.

This concludes the proof of (d), and hence of (10.4).

(10.5) EXAMPLE. Let k be a field and let $A = k[s^2, s^3] \subset B = k[s]$ where s is an indeterminate. Then $\underline{c} = s^2B$ so $A' = k$ and $B' = k[s']$, $s'^2 = 0$. Hence

$$K_0(A[t]) \simeq K_0(A) \oplus (1 + s'(t - 1)\, B'[t])$$

$$\simeq K_0(A) \oplus k[t].$$

Thus $K_0(A[t])$ contains an infinite dimensional vector space over k.

(10.6) THEOREM. *Let* π *be a finite abelian group of order* $m = [\pi: 1]$.

(a) *For all* $n \geq 0$, $\det_0(\underline{\underline{Z}}[\pi \times T^n]): Rk_0\underline{\underline{Z}}[\pi \times T^n]) \longrightarrow \mathrm{Pic}(\underline{\underline{Z}}[\pi \times T^n])$ *is an isomorphism.*

(b) *We have*

$$K_0(\underline{\underline{Z}}[\pi \times T^n]) = (1 + 2N)^n\, K_0(\underline{\underline{Z}}\pi) \oplus nLK_0(\underline{\underline{Z}}\pi)$$

and

$$K_1(\underline{\underline{Z}}[\pi \times T^n]) = (1 + 2N)^n\, K_1(\underline{\underline{Z}}\pi) \oplus nK_0(\underline{\underline{Z}}\pi) \oplus \frac{n(n - 1)}{2}\, LK_0(\underline{\underline{Z}}\pi).$$

(c) *The group* $LK_0(\underline{\underline{Z}}\pi)$ *is free abelian of rank*

$$(1 - h_0(\underline{\underline{Q}}\pi)) + \Sigma_{p|m}\, h_0(\underline{\underline{F}}_p\, \pi_p')\, (h_0(\underline{\underline{Q}}\pi_p) - 1),$$

where, for each prime p, $\underline{\underline{F}}_p = \underline{\underline{Z}}/p\underline{\underline{Z}}$, π_p *is a Sylow* p-*subgroup of* π, *and* $\pi = \pi_p \times \pi_p'$. *It vanishes if and only if* m *is a prime power.*

(d) *If* m *is square free (so* π *is then cyclic) then* $N^iK_0(\underline{\underline{Z}}\pi) = 0$ *for all* $i > 0$. *Otherwise there is an integer* $d > 0$ *such that, for all* $i > 0$, $N^iK_0(\underline{\underline{Z}}\pi)$ *is an infinite* *group of exponent* m^d. *For any group* π', $NK_0(\underline{\underline{Z}}\pi')$ *is a direct summand of* $Nil(\underline{\underline{Z}}[\pi' \times T])$.

(e) *Each element of* $Nil(\underline{\underline{Z}}[\pi \times T^n])$ *has order dividing some power of* m. *The same is therefore true of* $N^iK_1(\underline{\underline{Z}}\pi)$ *for all* $i > 0$.

Proof. Let $A = \underline{\underline{Z}}\pi$, and let B and $\underline{c}$ be as in (10.4) above. The hypotheses of (10.4) apply here so parts (a) and (b) follow directly from the corresponding parts of (10.4). Part (c) follows from part (c) of (10.4) together with (XI, 6.7). Since $mB \subset \underline{c}$ (see (XI, 1.2)) it follows that $B' = B/\underline{c}$ has characteristic dividing m, so (10.4) (d) implies that, if $nil(B')^{d+1} = 0$, $N^iK_0(\underline{\underline{Z}}\pi)$ has exponent m^d for all $i > 0$. Moreover these groups all vanish if $\underline{c} = {}^B\sqrt{\underline{c}}$, and (XI, 6.5) implies this happens if and only is m is square free.

We further have from (10.4) (d) that

$$\begin{aligned} &Ker(K_0(\underline{\underline{Z}}[\pi \times T_+^n]) \longrightarrow K_0(\underline{\underline{Z}}[\pi]) \\ &\quad \simeq ((1 + N)^n - 1)\, K_0(\underline{\underline{Z}}\pi) \\ &\quad \simeq (1 + \overline{nil(B')B'[T_+^n]})/(1 + \overline{nil(A')A'[T_+^n]}) \end{aligned}$$

where $A' = A/\underline{c}$. We claim that if $nil(B') \neq 0$, and therefore $nil(A') \neq nil(B')$ then this group is infinite (for $n > 0$). We shall give the argument for $n = 1$ and leave the general case to the reader.

Write $s = t - 1$, where t generates T_+. If the group above were finite there would exist an $n_0 > 0$ such that every element in it is represented by a polymonial in s of degree $< n_0$. But if $b \in nil(B')$, $b \notin nil(A')$, and if $P(s) \in B'[s]$ has degree $< n_0$, then the n_0^{th} coefficient of $P(s) \cdot (1 + bt^{n_0})$ is b. Therefore $(1 + bt^{n_0})$ cannot be represented, modulo $1 + \overline{nil(A')A'[T_+]}$, by a polynomial of degree $< n_0$.

To conclude the proof of (d) we note that, for any ring A, $NK_0(A) = NLK_1(A) = LNK_1(A) = L\,Nil(A)$, and the latter is a direct summand of Nil(A[T]), by the Fundamental Theorem (7.4).

Part (e) follows from (5.6), thanks to the fact that $mB \subset A$ and A projects onto each of the factors of B, which are Dedekind rings. The last assertion follows from the first because $NK_1 = Nil$ and therefore

$$Nil(\underline{\underline{Z}}[\pi \times T^n]) = (1 + 2N + L)^n\, Nil(\underline{\underline{Z}}\pi)$$

$$= (1 + 2N + L)^n\, NK_1(\underline{\underline{Z}}\pi).$$

This concludes the proof of (10.6).

§11. THEOREMS OF GERSTEN AND STALLINGS ON FREE PRODUCTS

In this section R denotes a commutative ring, and we shall consider augmented R-algebra A (see (IV, §5)). Thus we have an exact sequence

$$0 \longrightarrow \overline{A} \longrightarrow A \xrightarrow{\varepsilon_A} R \longrightarrow 0,$$

which splits as a sequence of R-modules, where ε_A is the augmentation and $\overline{A}$ is the augmentation ideal. If F is any functor from rings to abelian groups then we have $F(A) = F(R) \oplus \overline{F}(A)$ where $\overline{F}(A) = Ker(F(A) \longrightarrow F(R))$.

If B is another augmented R-algebra then $A *_R B$ denotes the coproduct (or free product) of A and B (see (IV, §5)). If $M \in$ mod-R then $T_R(M)$ denotes its tensor algebra.

(11.1) THEOREM (Stallings). <u>Let</u> A <u>and</u> B <u>be augmented R-algebras. Then there is an exact sequence</u>

$$\overline{K}_1(T_R(\overline{A} \otimes_R \overline{B})) \longrightarrow \overline{K}_1(A *_R B) \longrightarrow$$

$$\overline{K}_1(A) \oplus K_1(B) \longrightarrow 0$$

which splits on the right.

Proof. Let $C = A *_R B$. There are natural homomorphisms of augmented algebras, $A \longrightarrow C \longrightarrow A$ and $B \longrightarrow C \longrightarrow B$ whose composites are the respective identities. Since $C \longrightarrow A$ kills the image of $\overline{B}$, and $C \longrightarrow B$ kills the image of $\overline{A}$, it follows formally that $\overline{K}_1(C) \simeq \overline{K}_1(A) \oplus \overline{K}_1(B) \oplus (?)$.

We have $\overline{K}_1(C) = K_1(C, \overline{C}) = GL(C, \overline{C})/E(C, \overline{C})$. Since C is generated, as an R-algebra, by $\overline{A} \oplus \overline{B}$, and the latter generate $\overline{C}$, it follows from (5.1) that any element of $\overline{K}_1(C)$ is represented by a matrix of the form $\gamma = I + \alpha + \beta$ where α has coordinates in $\overline{A}$, and β coordinates in $\overline{B}$. The map $C \longrightarrow B$ kills α, and $C \longrightarrow A$ kills β. Therefore $I + \alpha \in GL(A, \overline{A})$ and $I + \beta \in GL(B, \overline{B})$.

Put $I + \delta = (I + \alpha)^{-1} (I + \alpha + \beta) (I + \beta)^{-1}$. Then

$$(I + \alpha) (I + \delta) = I + \delta + \alpha(I + \delta)$$

is equal to

$$(I + \alpha + \beta) (I + \beta)^{-1} = I + \alpha(I + \beta)^{-1}$$

so α is a left factor of δ. Similarly β is a right factor of δ, so δ has coordinates in $\overline{AB}(= \overline{A} \otimes_R \overline{B})$ in C. Now the description of C given in (IV, §5) shows that the homomorphism $T_R(\overline{A} \otimes_R \overline{B}) \longrightarrow C$, induced by $\overline{AB} \subset C$, is a monomorphism. This induces a homomorphism $f: \overline{K}_1(T_R(\overline{A} \otimes_R \overline{B})) \longrightarrow \overline{K}_1(C)$ whose image contains the class of $I + \delta$ above. Since the γ we started with is the product $\gamma = (I + \alpha) (I + \delta) (I + \beta)$ it follows that the class of γ lies in $\overline{K}_1(A) \oplus \mathrm{Im}(f) \oplus \overline{K}_1(B)$. This concludes the proof of Theorem (1.1).

(11.2) COROLLARY. *If* R *is regular, and if* $A \otimes_R B$ *is a free* R-*module, then*

$$\overline{K}_1(A *_R B) \longrightarrow \overline{K}_1(A) \oplus \overline{K}_1(B)$$

is an isomorphism.

Proof. In this case $T_R(\bar{A} \otimes_R \bar{B})$ is a polynomial ring in non commuting indeterminates over R, and it follows from Gersten's Theorem (5.5) that $\bar{K}_1(T_R(\bar{A} \otimes_R \bar{B})) = 0$ when R is regular. Thus the corollary follows from the theorem.

(1.3) COROLLARY (Gersten). Under the hypotheses of (1.2) the natural homomorphism

$$\bar{K}_0(A *_R B) \longrightarrow \bar{K}_0(A) \oplus \bar{K}_0(B)$$

is an isomorphism.

Proof. Let (T, T_+) be an oriented cycle and consider the base changes $R \subset R[T_\pm] \subset R[T]$. Further, write

$$f_i(R) \colon K_i(A) \oplus K_i(B) \longrightarrow K_i(A *_R B) \qquad (i = 0, 1)$$

for the natural homomorphisms (induced by $A \subset A *_B B \supset B$). It follows by a formal argument (cf. proof of (1.1)) that $K_i(A *_R B) = K_i(R) \oplus \bar{K}_i(A) \oplus \bar{K}_i(B) \oplus (?)_i$, so that the corollary, which asserts $(?)_0 = 0$, will follow if we show that $f_0(R)$ is surjective. From (1.2) we know that $f_1(R)$ is surjective whenever R is regular.

Now $R[T_\pm]$ and R[T] are regular, by Hilbert's Syzygy Theorem (2.4), and the coproduct of augmented algebras commutes with base change. Applying the Fundamental Theorem (7.4) we obtain a contractible exact sequence of morphisms

$$0 \longrightarrow f_1(R) \longrightarrow f_1(R[T_+]) \oplus f_1(R[T_-]) \longrightarrow$$
$$f_1(R[T]) \longrightarrow f_0(R) \longrightarrow 0.$$

Since the f_1's are all surjective, as remarked above, it follows that $f_0(R)$ is likewise surjective. q.e.d.

Remark. A result for the functor Nil which is analogous to (1.3) for K_0, can be obtained by a similar argument.

HISTORICAL REMARKS

The elegant proof of Hilbert's Theorem in §2 was taught to me by Kaplansky. He attributed the main idea (use of the characteristic sequence) to Hochschild. Grothendieck's Theorem is treated in Borel-Serre [1], in Serre [1], and in Bass-Heller-Swan [1]. The latter is also the main reference for the Fundamental Theorem, though it is given a much more precise form here. The material on the functor Nil, and the idea for the proof that Nil $\approx NK_1$, grew out of conversations with Tom Farrell and with W. C. Hsiang. (Farrell has applied the functor Nil to the problem of determining when a manifold has the homotopy type of a fibre bundle over a circle.)

The axiomatization of contracted functors, and the operations N and L and their properties, have not before been published. Neither has the theorem on the projective line in §9. The latter is related to a result of Horrocks [1]. It should also be compared with the formulation of the "periodicity theorem" in Atiyah [1] (the reprint on "K-theory and reality").

The results of §10 are taken from Bass-Murthy [1]. Those of §11 are taken from Stallings [1] and from Gersten [1].

Chapter XIII
RECIPROCITY LAWS AND FINITENESS QUESTIONS

If $\underline{q}$ is an ideal in a Dedekind ring A then we saw in Chapter VI that $SK_1(A, \underline{q})$ can be related to what are there called reciprocity laws. By applying the exact sequence of the localization from A to its field of fractions we show (in §1) that $SK_1(A, \underline{q})$ can be computed from automorphisms of certain torsion modules. This makes it rather easy to analyze (in §2) the relationship between reciprocity laws over A and those over the integral closure of A in a finite extension of its field of fractions.

In the third (and last) section we ask whether $G_0(A)$ is finitely generated when A is a (commutative) ring of absolutely finite type. Examples are given which show that the analogous questions for K_0, K_1, and G_1 have negative responses. Evidence for the finite generation of $G_0(A)$ is given by the Mordell-Weil Theorem, which implies that this is so if dim A $\leq$ 1. A method is indicated for handling A of dimension two. While the method hasn't been pushed through it does yield examples of coordinate rings, B, of non singular curves over fields of finite type for which $SK_1(B)$ is a very large group. These correspond to reciprocity laws which seem to be new, i.e. not derivable from that of Weil (see (VI, §8)). We also obtain examples of non trivial reciprocity laws on non singular affine curves over algebraically closed fields. These examples answer negatively a question of Mumford about their existence.

The discussion in this chapter often invokes material from algebraic geometry for which no preparation has been made in these notes. The aim here, however, is mainly to raise some questions, and to indicate how the techniques developed in these notes can be applied to them.

§1. THE LOCALIZATION SEQUENCE FOR DEDEKIND RINGS

Let A be a Dedekind ring with field of fractions $L = S^{-1}A$, $S = A - \{0\}$. The exact sequence of the localization $A \longrightarrow L$ is the sequence

$$(1) \qquad K_1(A) \longrightarrow K_1(L) \longrightarrow K_o(\underline{\underline{M}}_S(A)) \longrightarrow K_o(A) \longrightarrow K_o(L) \longrightarrow 0$$

(see (IX, 6.3)). Here $\underline{\underline{M}}_S(A)$ is the category of finitely generated torsion A-modules, and it follows by "devissage" (VII, §3) that

$$K_i(\underline{\underline{M}}_S(A)) = \coprod_{\underline{p} \in \max(A)} K_i(A/\underline{p}).$$

The sequence (1) leads us to inquire whether or not the sequence

$$K_1(\underline{\underline{M}}_S(A)) \longrightarrow K_1(A) \longrightarrow K_1(L)$$

is also exact. Since the composite is zero this is equivalent to asking whether the natural homomorphism

$$(2) \qquad K_1(\underline{\underline{M}}_S(A)) \longrightarrow SK_1(A)$$

is surjective. This would be of interest particularly in view of the interpretation of $SK_1(A)$ in terms of "reciprocity laws" (see Chapter VI). We shall show that (2) is, indeed, surjective. This will be done by expressing it in terms of Mennicke symbols. In the next section we shall use this information to describe the behavior of reciprocity laws under the passage from A to its integral closure in a finite extension of L.

Let $\underline{q}$ be a non zero ideal in A and write

$$\underline{\underline{M}}(A, \underline{q})$$

for the full subcategory of $\underline{\underline{M}}(A)$ whose objects have no "$\underline{q}$-torsion". I.e. $M \in \underline{\underline{M}}(A, \underline{q})$ if no non zero element of M is annihilated by $\underline{q}$. The importance of this condition for our purposes is that if $E = (0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0)$ is an exact sequence in $\underline{\underline{M}}(A, \underline{q})$ then $E \otimes_A (A/\underline{q})$ is still exact.

We further introduce

$$\underline{\underline{M}}_S(A, \underline{q}) = \underline{\underline{M}}(A, \underline{q}) \cap \underline{\underline{M}}_S(A).$$

By the devissage theorem (VIII, 3.3) we have

$$(3) \qquad K_i(\underline{\underline{M}}_S(A, \underline{q})) = \coprod_{\substack{\underline{p} \in \max(A) \\ \underline{p} \nmid \underline{q}}} K_i(A/\underline{p}),$$

because the category of semi-simple modules in $\underline{\underline{M}}_S(A, \underline{q})$ is the direct sum of the categories $\underline{\underline{M}}(A/\underline{p})$ $(\underline{p} \in \max(A), \underline{p} \nmid \underline{q})$.

We can compute $K_1(A, \underline{q})$ from the category $\Sigma(\underline{\underline{P}}(A), \underline{q})$ whose objects are pairs (P, α) with $P \in \underline{\underline{P}}(A)$ and $\alpha \in \mathrm{Aut}_A(P, \underline{q})$, i.e. $\alpha \otimes_A (A/\underline{q}) = 1_{P \otimes_A (A/\underline{q})}$. We can similarly define the category $\Sigma(\underline{\underline{M}}(A, \underline{q}), \underline{q})$ by allowing P to be any object of $\underline{\underline{M}}(A, \underline{q})$.

(1.1) PROPOSITION. $\Sigma(\underline{\underline{M}}(A, \underline{q}), \underline{q})$ _is an admissible subcategory of_ $\Sigma\, \underline{\underline{M}}(A)$. _The inclusion_ $\Sigma(\underline{\underline{P}}(A), \underline{q}) \subset \Sigma(\underline{\underline{M}}(A, \underline{q}), \underline{q})$ _induces an isomorphism_,

$$(4) \qquad K_1(A, \underline{q}) \longrightarrow K_1(\underline{\underline{M}}(A, \underline{q}), \underline{q}),$$

where the right side is defined by relations analogous to those for the ordinary K_1.

Proof. The only non trivial point of the first assertion is that if

$$0 \longrightarrow (M', \alpha') \longrightarrow (M, \alpha) \longrightarrow (M'', \alpha'') \longrightarrow 0$$

is an exact sequence in $\Sigma\,\underline{\underline{M}}(A)$, and if (M, α), $(M'', \alpha'') \in \Sigma(\underline{\underline{M}}(A, \underline{q}), \underline{q})$, then $(M', \alpha') \in \Sigma(\underline{\underline{M}}(A, \underline{q}), \underline{q})$. First of all $M' \in \underline{\underline{M}}(A, \underline{q})$ because M has no $\underline{q}$-torsion. It remains to check that $\alpha' \equiv I \bmod \underline{q}$. But since $M'' \in \underline{\underline{M}}(A, \underline{q})$ it follows that

$$0 \longrightarrow (M', \alpha') \otimes_A (A/\underline{q}) \longrightarrow (M, \alpha) \otimes_A (A/\underline{q})$$

is exact. Since $\alpha \otimes_A (A/\underline{q}) = I$ the same is true of its restriction, $\alpha' \otimes_A (A/\underline{q})$ to $M' \otimes_A (A/\underline{q})$.

To show now that (4) is an isomorphism it suffices, by (VII, 4.4), to show that, given $M \in \underline{\underline{M}}(A, \underline{q})$, there is an epimorphism $P \longrightarrow M$ with $P \in \underline{\underline{P}}(A)$ such that any $\alpha \in \mathrm{Aut}_A(M, \underline{q})$ lifts to an element of $\mathrm{Aut}_A(P, \underline{q})$.

Let $f\colon Q \longrightarrow M$ be an epimorphism with $Q \in \underline{\underline{P}}(A)$, and set $P = Q \oplus Q$. We shall construct an $\varepsilon \in E(Q, Q\colon \underline{q})$ such that

$$(P, \varepsilon) = (Q \oplus Q, \varepsilon) \xrightarrow{f \oplus f} (M \oplus M, \alpha \oplus \alpha^{-1})$$

$$\xrightarrow{(1_M, 0)} (M, \alpha)$$

is a sequence of morphisms in $\Sigma(\underline{\underline{M}}(A, \underline{q}), \underline{q})$. In particular ε will be the required lifting of α.

Put $h_+ = 1 - \alpha$ and $h_- = 1 - \alpha^{-1}$. These are endomorphisms of M with images in $M\underline{q}$. Moreover we have (cf. proof of (VIII, 4.5))

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -h_- & 1 \end{pmatrix} \begin{pmatrix} 1 & h_+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & h_- \\ 0 & 1 \end{pmatrix}$$

$$= \varepsilon_{21}(1)\ \varepsilon_{21}(-h_-)\ \varepsilon_{12}(h_+)\ \varepsilon_{21}(1)^{-1}\,\varepsilon_{12}(h_-),$$

where $\varepsilon_{ij}(t) = I + te_{ij}$. Via the epimorphism $f\colon Q \longrightarrow M$ we can lift $h_{\pm}$ to an endomorphism $g_{\pm}$ of Q with image in $Q\underline{q}$;

this is because Q is projective. Therefore the lifting, ε $E(Q, Q; \underline{q})$ that we seek is given by

$$\varepsilon = \varepsilon_{21}(1)\ \varepsilon_{21}(-g_-)\ \varepsilon_{12}(g_+)\ \varepsilon_{21}(1)^{-1}\ \varepsilon_{21}(g_-),$$

where now, of course, 1 denotes 1_P, etc. This concludes the proof.

(1.2) PROPOSITION. The inclusion $\underline{M}_S(A, \underline{q}) \subset \underline{M}(A, \underline{q})$ induces a homomorphism

$$(5) \qquad K_1(\underline{M}_S(A, \underline{q})) \longrightarrow K_1(A, \underline{q})$$

whose image is $SK_1(A, \underline{q})$. We have

$$K_1(\underline{M}_S(A, \underline{q})) \simeq \coprod K_1(A/\underline{p})$$
$$\simeq \coprod U(A/\underline{p}),$$

where p ranges over all maximal ideals not dividing $\underline{q}$. With this identification, the homomorphism $U(A/\underline{p}) \longrightarrow SK_1(A, \underline{q})$ induced by (5) is $u \longmapsto \begin{bmatrix} \underline{p}\ \underline{q} \\ \alpha \end{bmatrix}$, where $\alpha \equiv 1 \bmod \underline{q}$, $\alpha \equiv u \bmod \underline{p}$, and the symbol is the Mennicke symbol (cf. (VI, §§2,5)).

Remark. It follows now from the theorems in (VI, §6) that (5) is essentially the universal $\underline{q}$-reciprocity.

Proof. We shall first verify the last assertion. If $u \in U(A/\underline{p})$ then the identification above makes u correspond to $[A/\underline{p}, u]_S \in K_1(\underline{M}_S(A, \underline{q}))$. The fact that $\underline{p}$ doesn't divide $\underline{q}$ guarantees that $A/\underline{p} \in \underline{M}_S(A, \underline{q})$. We have confused u with $u \cdot 1_{A/\underline{p}}$ in the notation.

Choose an ideal $\underline{c}$ prime to $\underline{pq}$ such that $\underline{pc} = bA$, a principal ideal. Then

$$[A/\underline{p}, u]_S = [A/\underline{p}, u]_S + [A/\underline{c}, 1]_S = [A/bA, v]_S$$

for some $v \in U(A/bA)$, $v \equiv u \bmod \underline{p}$, $v \equiv 1 \bmod \underline{c}$. Choose $\alpha \in A$ such that $\alpha \equiv v \bmod bA$ and $\alpha \equiv 1 \bmod \underline{q}$. Then we also have $\alpha \equiv 1 \bmod \underline{cq}$, so

$$\begin{bmatrix} p & q \\ & \alpha \end{bmatrix} = \begin{bmatrix} p & q \\ & \alpha \end{bmatrix} \begin{bmatrix} c & q \\ & \alpha \end{bmatrix} = \begin{bmatrix} b & q \\ & \alpha \end{bmatrix}.$$

Therefore it suffices to show that the image, [A/bA, v], in $K_1(A, \underline{q})$, of $[A/bA, v]_S$, is equal to $\begin{bmatrix} b & q \\ & \alpha \end{bmatrix}$. Let $\alpha' \in A$ be such that $\alpha' \equiv v^{-1} \bmod bA$, $\alpha' \equiv 1 \bmod \underline{q}$. Then, as in the proof of (1.1) above, we have a resolution

$$0 \longrightarrow (A \oplus A, \beta) \xrightarrow{b1_A \oplus 1_A} (A \oplus A, \varepsilon) \xrightarrow{(f,\ 0)} (A/bA, v) \longrightarrow 0,$$

where $f: A \longrightarrow A/bA$ is the canonical projection. Here $\varepsilon \in E_2(A, \underline{q})$ and it is given by

$$\varepsilon = \varepsilon_{21}(1)\ \varepsilon_{21}(-\alpha_-)\ \varepsilon_{12}(\alpha_+)\ \varepsilon_{21}(1)^{-1}\ \varepsilon_{12}(\alpha_-),$$

where $\alpha_+ = 1 - \alpha$ and $\alpha_- = 1 - \alpha'$. Of course β is defined by the exact sequence, which implies now that

(*) $\quad [A/bA, v] = [A \oplus A, \beta^{-1}].$

To evaluate this we first make ε explicit:

$$\begin{aligned} \varepsilon &= \begin{pmatrix} 1 & 0 \\ \alpha' & 1 \end{pmatrix}\begin{pmatrix} 1 & 1-\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1-\alpha' \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1-\alpha \\ \alpha' & 1+\alpha'-\alpha'\alpha \end{pmatrix}\begin{pmatrix} 1 & 1-\alpha' \\ -1 & \alpha' \end{pmatrix} \\ &= \begin{pmatrix} \alpha & 1-\alpha\alpha' \\ \alpha'\alpha - 1 & \alpha'(2-\alpha'\alpha) \end{pmatrix} \end{aligned}$$

Since $\varepsilon \equiv v \oplus v^{-1} \bmod bA$ we can write $\alpha'\alpha - 1 = bc$. Further, $\varepsilon \equiv I \bmod \underline{q}$ and b is prime to $\underline{q}$, so we have $c \in \underline{q}$. If e_1, e_2 is the standard basis for $A \oplus A$, then the matrix representing β is obtained by considering the effect of ε on $e_1bA \oplus e_2A$. Thus a matrix representing β is obtained by conjugating that for ε by the matrix $\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$. Hence, relative to the basis $e_1 b$, e_2, we can write

$$\beta = \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 1 - aa' \\ a'a - 1 & a'(2 - a'a) \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & (1 - aa')b^{-1} \\ (a'a - 1)b & a'(2 - a'a) \end{pmatrix}$$

$$= \begin{pmatrix} a & c \\ (a'a - 1)b & a'(2 - a'a) \end{pmatrix}.$$

Now from (*) we have

$$[A/bA, v] = [A \oplus A, \beta^{-1}] = \begin{bmatrix} (a'a - 1)b \\ a \end{bmatrix} = \begin{bmatrix} b & \underline{q} \\ & a \end{bmatrix} \begin{bmatrix} a'a - 1 \\ a \end{bmatrix}.$$

But $\begin{bmatrix} a'a - 1 \\ a \end{bmatrix} = \begin{bmatrix} a'a - 1 - a(a' - 1) \\ a \end{bmatrix} = \begin{bmatrix} a - 1 \\ a \end{bmatrix} = \begin{bmatrix} a - 1 \\ 1 \end{bmatrix} = 1.$

Thus we have proved the last assertion of the proposition.

It remains only to show that

(**) $\qquad K_1(\underline{\underline{M}}_S(A, \underline{q})) \longrightarrow SK_1(A, \underline{q})$

is surjective. The theory of Chapter VI shows that every element in $SK_1(A, \underline{q})$ is of the $\begin{bmatrix} b \\ a \end{bmatrix}$ with $(a, b) \in W_{\underline{q}}$. We have

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} b(1 - a) \\ a \end{bmatrix} = \begin{bmatrix} (b + ta)\ (1 - a) \\ a \end{bmatrix} = \begin{bmatrix} (b + ta)\underline{q} \\ a \end{bmatrix}$$

for any $t \in A$. Choosing t suitably we can make $b + ta$ prime to $\underline{q}$ and then the formula proved above shows that $\begin{bmatrix} (b + ta)\underline{q} \\ a \end{bmatrix}$ lies in the image of (**). q.e.d.

We can use (1.2) to make explicit, in terms of ideal classes and Mennicke symbols, the $K_0(A)$-module structure of $K_1(A, \underline{q})$.

(1.3) PROPOSITION. *There is a natural isomorphism*

$$K_0(A) \approx \underline{\underline{Z}} \oplus \text{Pic}(A)$$

so that projection on $\underline{\underline{Z}}$ is the rank homomorphism, and Pic(A) *is an ideal of square zero. We further have the natural decomposition*

$$K_1(A, \underline{q}) = U(A, \underline{q}) \oplus SK\ (A, \underline{q}).$$

In these coordinate systems, the $K_0(A)$-*module structure of* $K_1(A, \underline{q})$ *is given by*

$$(n, [\underline{c}]) \cdot (u, \begin{bmatrix} \underline{b} \\ a \end{bmatrix}) = (u^n, \begin{bmatrix} \underline{c}\ \underline{q} \\ u \end{bmatrix}^{-1} \begin{bmatrix} \underline{b} \\ a \end{bmatrix}).$$

Here $\underline{b}$ and $\underline{c}$ are invertible ideals in A, $\underline{b} \subset \underline{q}$, *and* $a \equiv 1$ mod $\underline{q}$ *and is prime to $\underline{b}$. Of course* $n \in \underline{\underline{Z}}$ *and* $u \in U(A, \underline{q})$.

Proof. Since A is Dedekind rk: $Rk_0(A) \longrightarrow \text{Pic}(A)$ is an isomorphism, its inverse being given by $[\underline{c}]_{\underline{\text{Pic}}} \longrightarrow [\underline{c}]_{\underline{\underline{P}}} - [A]_{\underline{\underline{P}}}$ for an invertible ideal $\underline{c}$. The fact that $Rk_0(A)^2 = 0$ follows from (IX, 4.4 (d)). The action described above is clearly bilinear, so, to show it agrees with the usual $K_0(A)$-action, it suffices to check this on additive generators in each variable. Therefore it suffices to treat the case

$$(n, [\underline{c}]) = (1, [\underline{p}]) = [\underline{p}]_{\underline{\underline{P}}} \qquad (\underline{p} \in \max(A)),$$

and we can even assume $\underline{p}$ does not divide $\underline{q}$, since every ideal class has a representative prime to $\underline{q}$.

Let (P, α) represent $\left(u, \begin{bmatrix} \underline{b} \\ a \end{bmatrix}\right)$, so that $u = \det(\alpha)$.

Then

$$[\underline{p}]_{\underline{\underline{P}}} [P, \alpha] = [\underline{p} \otimes P, 1_{\underline{p}} \otimes \alpha] = [P\underline{p}, \beta],$$

where $\beta = \alpha | P\underline{p}$. We have an exact sequence

$$0 \longrightarrow (P\underline{p}, \beta) \longrightarrow (P, \alpha) \longrightarrow (P/P\underline{p}, \gamma) \longrightarrow 0$$

from which we conclude that

$$[\underline{p}]_{\underline{\underline{P}}} [P, \alpha] = [P, \alpha] - [P/P\underline{p}, \gamma]$$

Since $A/\underline{p}$ is a field we can operate on $(P/P\underline{p}, \gamma)$ in $\underline{\underline{M}}(A/\underline{p})$ and find that $[P/P\underline{p}, \gamma] = [A/\underline{p}, \det(\gamma)]$, where $\det(\gamma)$ = the image of $u = \det(\alpha)$ modulo $\underline{p}$. According to (1.2) we have

$$[A/\underline{p}, \det(\gamma)] = \begin{bmatrix} \underline{p}\ \underline{q} \\ u \end{bmatrix},$$

therefore. Since $[\underline{p}]_{\underline{\underline{P}}} = (1, [\underline{p}])$ we have

$$(1, [\underline{p}])\left(u, \begin{bmatrix} \underline{b} \\ a \end{bmatrix}\right) = \left(u, \begin{bmatrix} \underline{b} \\ a \end{bmatrix}\right) - \left(1, \begin{bmatrix} \underline{p}\ \underline{q} \\ u \end{bmatrix}\right)$$

$$= \left(u, \begin{bmatrix} \underline{p}\ \underline{q} \\ u \end{bmatrix}^{-1} \begin{bmatrix} \underline{b} \\ a \end{bmatrix}\right). \text{ q.e.d.}$$

§2. FUNCTORIAL PROPERTIES OF RECIPROCITY LAWS

As in the last section we fix a Dedekind ring A with field of fractions $L = S^{-1}A$, $S = A - \{0\}$, and a non zero ideal $\underline{q}$.

From (1.2) we have an epimorphism

$$\chi(\underline{q}): \coprod_{\underline{p} \nmid \underline{q}} U(A/\underline{p}) \longrightarrow SK_1(A, \underline{q})$$

induced by the inclusion $\underline{\underline{M}}_S(A, \underline{q}) \subset \underline{\underline{M}}(A, \underline{q})$. Here we have identified $U(A/\underline{p})$ with $K_1(A/\underline{p}) = K_1(\underline{\underline{M}}(A/\underline{p}))$. Moreover, if

$$\chi_{\underline{p}}(\underline{q}): U(A/\underline{p}) \longrightarrow SK_1(A, \underline{q})$$

is the $\underline{p}$ component of $\chi(\underline{q})$ then we have, again from (1.2),

$$\chi_{\underline{p}}(\underline{q})\ (u) = \begin{bmatrix} \underline{p}\ \underline{q} \\ \alpha \end{bmatrix} \quad (\alpha \equiv 1 \bmod \underline{q},\ \alpha \equiv u \bmod \underline{p}).$$

Let A' be the integral closure of A in a finite field extension L' of L; thus $L' = S'^{-1}A'$, where $S' = A' - \{0\}$. Further, let $\underline{q}'$ be an ideal of A' containing $\underline{q}$. Then we have a commutative diagram of exact functors

$$\begin{array}{ccc} \underline{\underline{M}}(A, \underline{q}) & \xrightarrow{\otimes_A A'} & \underline{\underline{M}}(A', \underline{q}') \\ \cup & & \cup \\ \underline{\underline{M}}_S(A, \underline{q}) & \xrightarrow[\otimes_A A']{} & \underline{\underline{M}}_{S'}(A', \underline{q}') \end{array}$$

induced by the inclusion $f: A \longrightarrow A'$. This, in turn, induces a commutative diagram

$$\begin{array}{ccc}
SK_1(A, \underline{q}) & \xrightarrow{f_*} & SK_1(A', \underline{q}') \\
\uparrow \chi(\underline{q}) & & \uparrow \chi(\underline{q}') \\
K_1(\underline{\underline{M}}_S(A, \underline{q})) & \longrightarrow & K_1(\underline{\underline{M}}_{S'}(A', \underline{q}')) \\
\| & & \| \\
\coprod_{\underline{p} \nmid \underline{q}} K_1(A/\underline{p}) & \xrightarrow[f_*]{} & \coprod_{\underline{p}' \nmid \underline{q}'} K_1(A'/\underline{p}')
\end{array}$$

To compute f_* on the bottom take a $u \in U(A/\underline{p})$, which is identified with $[A/\underline{p}, u]_S \in K_1(\underline{\underline{M}}_S(A, \underline{q}))$. Then $f_*(u) = f_*[A/\underline{p}, u] = [(A/p) \otimes_A A', u \otimes 1_{A'}]_{S'} = [A'/\underline{p}A', u]_{S'}$. Let

$$\underline{p}A' = \Pi\, \underline{p}'^{\,e_{\underline{p}'/\underline{p}}}$$

be the prime factorization of $\underline{p}A'$. Then $A'/\underline{p}A' = \Pi\,(A'/\underline{p}'^{\,e_{\underline{p}'/\underline{p}}})$, and each $A'/\underline{p}'^{e}$ has a Jordan-Holder series of length e with factors $A'/\underline{p}'$. Hence we conclude that

$$f_*[A/\underline{p}, u] = \Sigma_{\underline{p}'|\underline{p}}\, e_{\underline{p}'/\underline{p}}\, [A'/\underline{p}', u]$$

$$= \Sigma_{\underline{p}'|\underline{p}}\, [A'/\underline{p}', u^{e_{\underline{p}'/\underline{p}}}].$$

As a homomorphism from $\coprod_{\underline{p} \nmid q} U(A/\underline{p})$ into $\coprod_{\underline{p}' \nmid \underline{q}'} U(A'/\underline{p}')$, therefore, f_* is induced by homomorphisms

$$f_{*\underline{p}}: U(A/\underline{p}) \longrightarrow \coprod_{\underline{p}'|\underline{p}} U(A'/\underline{p}')$$

$$f_{*\underline{p}}(u) = (u^{e_{\underline{p}'/\underline{p}}})_{\underline{p}'|\underline{p}}\ .$$

Passing to the corresponding Mennicke symbols we obtain

$$f_*\begin{bmatrix}\underline{p} & \underline{q}\\ \multicolumn{2}{c}{\alpha}\end{bmatrix} = \prod_{\underline{p}'|\underline{p}} \begin{bmatrix}\underline{p}' & \underline{q}'\\ \multicolumn{2}{c}{\alpha}\end{bmatrix}^{e_{\underline{p}'/\underline{p}}} = \begin{bmatrix}\underline{p}A' & \underline{q}'\\ \multicolumn{2}{c}{\alpha}\end{bmatrix} = \begin{bmatrix}\underline{p} & \underline{q}'\\ \multicolumn{2}{c}{\alpha}\end{bmatrix}.$$

Now suppose that $\underline{q}' = \underline{q}A'$. Then the restriction functor $\underline{\underline{M}}(A') \longrightarrow \underline{\underline{M}}(A)$ induces a commutative diagram of exact functors

$$\begin{array}{ccc}
\underline{\underline{M}}(A, \underline{q}) & \xleftarrow{\text{restr.}} & \underline{\underline{M}}(A, \underline{q}) \\
\cup & & \cup \\
\underline{\underline{M}}_S(A, \underline{q}) & \xleftarrow[\text{restr.}]{} & \underline{\underline{M}}_{S'}(A', \underline{q}').
\end{array}$$

Again we obtain induced homomorphisms

$$\begin{array}{ccc}
SK_1(A, \underline{q}) & \xleftarrow{f^*} & SK_1(A', \underline{q}') \\
\uparrow{\scriptstyle \chi(\underline{q})} & & \uparrow{\scriptstyle \chi(\underline{q}')} \\
K_1(\underline{\underline{M}}_S(A, \underline{q})) & \longleftarrow & K_1(\underline{\underline{M}}_{S'}(A', \underline{q}')) \\
\| & & \| \\
\coprod_{\underline{p} \nmid \underline{q}} K_1(A/\underline{p}) & \xleftarrow[f^*]{} & \coprod_{\underline{p}' \nmid \underline{q}'} K_1(A'/\underline{p}')
\end{array}$$

If $u' \in U(A'/\underline{p}')$ then u' is identified with $[A'/\underline{p}', u']_{S'}$ in $K_1(\underline{\underline{M}}_{S'}(A', \underline{q}'))$, and $f^*(u') = f^*[A'/\underline{p}', u']_{S'} = [A'/\underline{p}', u']_S$. In the latter we view $A'/\underline{p}'$ as an A-module and multiplication by u' as an A-automorphism. If $\underline{p} = A \cap \underline{p}'$ then $A'/\underline{p}'$ is a vector space over $A/\underline{p}$, and the class, in $K_1(A/\underline{p})$, of $[A'/\underline{p}', u']_S$ is just the $(A/\underline{p})$-determinant of $u' \cdot 1_{A'/\underline{p}'}$. This is, by definition, the norm of u':

$$[A'/\underline{p}', u']_S = [A/\underline{p}, N_{\underline{p}'/\underline{p}}\, u']_S$$

where $N_{\underline{p}'/\underline{p}}$ denotes the norm from $A/\underline{p}'$ to $A/\underline{p}$. Thus $f^*: \coprod_{\underline{p}' \nmid \underline{q}'} U(A'/\underline{p}') \longrightarrow \coprod_{\underline{p} \nmid \underline{q}} U(A/\underline{p})$ is induced by the norm homomorphisms,

$$f^*_{\underline{p}'} = N_{\underline{p}'/\underline{p}}: U(A'/\underline{p}') \longrightarrow U(A/\underline{p}) \quad (\underline{p}' \cap A = \underline{p}).$$

Passing to the corresponding Mennicke symbols we obtain the following formula: Let $a' \in A'$ represent $u' \in U(A'/\underline{p}')$ and choose $a' \equiv 1 \bmod \underline{q}'$. Then the formulas above imply that

$$f^* \begin{bmatrix} \underline{p}' & \underline{q}' \\ & a' \end{bmatrix} = \begin{bmatrix} \underline{p} & \underline{q} \\ & a \end{bmatrix}$$

where $a \in A$ represents $N_{\underline{p}'/\underline{p}} u'$ and $a \equiv 1 \bmod \underline{q}$. Let $L_{\underline{p}}$ be the $\underline{p}$-adic completion of L, and $L'_{\underline{p}'}$ the $\underline{p}'$-adic completion of L'. Then it follows from basic facts in valuation theory that

$$N_{L'_{\underline{p}'}/L_{\underline{p}}} (a') \equiv N_{\underline{p}'/\underline{p}} (u')^{e_{\underline{p}'/\underline{p}}} \bmod \underline{p}.$$

Consequently we have $N_{L'_{\underline{p}'}/L_{\underline{p}}} (a') \equiv a^{e_{\underline{p}'/\underline{p}}}$, and hence

$$(1) \qquad f^* \begin{bmatrix} \underline{p}' & \underline{q}' \\ & a' \end{bmatrix}^{e_{\underline{p}'/\underline{p}}} = \begin{bmatrix} \underline{p} & \underline{q} \\ N_{L'_{\underline{p}'}/L_{\underline{p}}} & (a') \end{bmatrix}$$

Since

$$N_{L'/L} (a') = \prod_{\underline{p}'|\underline{p}} N_{L'_{\underline{p}'}/L_{\underline{p}}} (a')$$

we can take the product of (1) over $\underline{p}'$ dividing $\underline{p}$ to obtain

$$\begin{bmatrix} \underline{p} & \underline{q} \\ N_{L'/L} & (a') \end{bmatrix} = \prod_{\underline{p}'|\underline{p}} f^* \begin{bmatrix} \underline{p}' & \underline{q}' \\ & a' \end{bmatrix}^{e_{\underline{p}'/\underline{p}}} = f^* \begin{bmatrix} \underline{p} & \underline{q}' \\ & a' \end{bmatrix}.$$

Both sides are multiplicative in all variables so we conclude that:

$$(2) \qquad \left[\begin{matrix} \underline{b} & \underline{q} \\ N_{L'/L} & (\alpha') \end{matrix}\right] = f^* \left[\begin{matrix} \underline{b} & \underline{q}' \\ & \alpha' \end{matrix}\right]$$

whenever $\alpha' \equiv 1 \bmod \underline{q}'$ and $\underline{b} \subset A$ is prime to α'.

We shall close this section by computing an example, due to Milnor. Let $A = \mathbb{R}[x, y]$, where $x^2 + y^2 = 1$, be the real coordinate ring of the circle, $S^1 \subset \mathbb{R}^2$. Then we claim that

$$SK_1(A) \simeq \mathbb{Z}/2\mathbb{Z}.$$

We shall identify S^1 with a subset of max(A), "the real locus."

<u>Step</u> 1. $SK_1(A)$ <u>has exponent</u> 2.

For $\mathbb{C} \otimes_{\mathbb{R}} A = \mathbb{C}[u, u^{-1}]$ where $u = x + y\sqrt{-1}$ and $u^{-1} = x - y\sqrt{-1}$. Since this is a localization of the euclidean ring $\mathbb{C}[u]$ we have $SK_1(\mathbb{C} \otimes_{\mathbb{R}} A) = 0$. The composite $SK_1(A) \longrightarrow SK_1(\mathbb{C} \otimes_{\mathbb{R}} A) \xrightarrow{\text{res}} SK_1(A)$ is multiplication by $[\mathbb{C}]_{\mathbb{R}} = 2 \in K_0(\mathbb{R})$ (see (IX, 1.8)). This establishes the first assertion.

<u>Step</u> 2. <u>Let</u> $\underline{p} \in \max(A)$ <u>and</u> $\alpha \notin \underline{p}$, <u>and let</u> [ ] <u>be a Mennicke symbol</u>. Then $\left[\begin{matrix} \underline{p} \\ \alpha \end{matrix}\right] = 1$ <u>if</u> $\underline{p} \notin S^1$. <u>If</u> $\underline{p} \in S^1$ <u>then</u> $\left[\begin{matrix} \underline{p} \\ \alpha \end{matrix}\right]$ <u>depends only on the sign of</u> $\alpha(\underline{p})$ (= <u>the image of</u> α <u>in</u> $A/\underline{p} \simeq \mathbb{R}$).

For $U(\mathbb{C})$ has no non trivial quotients of exponent two, and the only such quotient of $U(\mathbb{R})$ corresponds to the sign homomorphism. Since $\alpha \longmapsto \left[\begin{matrix} \underline{p} \\ \alpha \end{matrix}\right]$ induces a homomorphism on $U(A/\underline{p})$ step 2 now follows from step 1.

<u>Step</u> 3. <u>Let</u> [ ] <u>be the universal Mennicke symbol with values in</u> $SK_1(A)$. <u>Then</u> $SK_1(A)$ <u>is generated by the elements</u>

$e_{\underline{p}} = \begin{bmatrix} \underline{p} \\ -1 \end{bmatrix}$ $(\underline{p} \in S^1)$.

This follows immediately from step 2.

Step 4. If $\underline{p}_1, \underline{p}_2 \in S^1$ then $\underline{p}_1\underline{p}_2$ is principal. Hence $e_{\underline{p}_1} = e_{\underline{p}_2}$.

Let $d = ax + by - c \in A$ be chosen with $a, b, c \in \underline{\underline{R}}$, so that the line $ax + by = c$ in $\underline{\underline{R}}^2$ passes through $\underline{p}_1$ and $\underline{p}_2$ and is tangent to S^1 if $\underline{p}_1 = \underline{p}_2$. Then d has a zero of order one at each $\underline{p}_i$ if $\underline{p}_1 \neq \underline{p}_2$ and a zero of order two at $\underline{p}_1$ if $\underline{p}_1 = \underline{p}_2$. It is clear that d has no other zeros on the locus $x^2 + y^2 = 1$ in $\underline{\underline{C}}^2$, so it follows that $\underline{p}_1\underline{p}_2 = dA$.

Now we have $e_{\underline{p}_1} e_{\underline{p}_2} = \begin{bmatrix} \underline{p}_1 \\ -1 \end{bmatrix} \begin{bmatrix} \underline{p}_2 \\ -1 \end{bmatrix} = \begin{bmatrix} \underline{p}_1\ \underline{p}_2 \\ -1 \end{bmatrix} = \begin{bmatrix} d \\ -1 \end{bmatrix} = 1.$

Step 1 through 4 clearly imply that $SK_1(A)$ has order at most two. We conclude therefore by showing that $SK_1(A)$ is is not trivial. But this follows from the explicit reciprocity law constructed at the end of Chapter VI, §8.

§3. FINITENESS QUESTIONS; EXAMPLES

We shall discuss the following general question: Under what kinds of finiteness assumptions on a ring A can one deduce corresponding finiteness conditions on the groups $K_i(A)$ and $G_i(A)$ $(i = 0, 1)$?

As an example, we showed in Chapter X that, if A is a finite $\underline{\underline{Z}}$-algebra, then all four groups are finitely generated. These results were deduced more or less directly from theorems on the finiteness of class number and the finite generation of unit groups. The latter results have satisfactory generalizations to dimension > 1, which we shall now describe.

Henceforth in this section all rings will be commutative. We say that a ring A is of absolutely finite type if A is a finitely generated R-algebra where R is either $\underline{\underline{Z}}$ or a field finitely generated (as a field) over the prime field. It is for these rings that we shall investigate finiteness properties for K_i and G_i.

We begin with a trivial counterexample. Suppose A contains a nilpotent ideal $J \neq 0$. Then it contains one for which $J^2 = 0$, in which case $1 + J \simeq J$, and $1 + J \subset U(A)$. If J is not additively finitely generated then U(A), and hence also $K_1(A)$, cannot be finitely generated. Even if J is finitely generated we can obtain a similar example with JA[t] instead of J; here t is an indeterminate so JA[t] is an infinite coproduct of copies of J. Thus, if $A = \underline{\underline{Z}}[s, t]$ with the single relation $s^2 = 0$, then U(A) is not finitely generated even though A is a finitely generated $\underline{\underline{Z}}$-algebra.

In a sense nilpotent elements are the only source of trouble. If A has none then we can pass to the integral closure of A in its full ring of fractions, and thus often reduce this type of question to the case of integrally closed integral domains.

Let A be an integrally closed integral domain of absolutely finite type. Then A is a finitely generated R-algebra for some R as above. By requiring that the field of fractions of R be algebraically closed in that of A we must allow R to be either a field of finite type over the prime field, or else a localization of the ring of algebraic integers in a number field. In the latter case the Dirichlet Unit Theorem says that U(R) is finitely generated, though this certainly will not be true when R is a field unless R is finite. Nevertheless it is true in general that

(1) U(A)/U(R) is finitely generated.

When R is a field A can be viewed as the coordinate ring of an affine open set V in a complete normal variety X over R. The divisor map induces an exact sequence

$$0 \longrightarrow U(R) \longrightarrow U(A) \xrightarrow{\text{div}} D(X)$$

and the divisor of a unit in A has support in the finite set of prime divisors in the complement of V. The general case (i.e. when R is not necessarily a field) can be deduced easily from this one and the Dirichlet Theorem.

A much deeper result is that the divisor class group, and hence also Pic(A), is finitely generated for A as above. This is a direct consequence of the Mordell-Weil Theorem (cf. Roquette [1]). We shall record these conclusions for purpose of reference.

(3.1) THEOREM. *Let* R *be a localization of the ring of integers in a number field, or a field finitely generated over its prime field. Let* A *be an integrally closed integral domain finitely generated over* R *and such that the field of fractions of* R *is algebraically closed in that of* A. *Then* U(A)/U(R) *and the divisor class group* C(A) *are finitely generated abelian groups. Hence* Pic(A) *is finitely generated also.*

(3.2) COROLLARY. *Let* A *be any ring of absolutely finite type and of dimension* ≤ 1. *Then* $G_0(A)$ *is a finitely generated group.*

Proof. Since $G_0(A/\mathrm{nil}(A)) \longrightarrow G_0(A)$ is an isomorphism (VIII, 3.4) we can assume nil(A) = 0. Let B be the integral closure of A in its full ring of fractions. Then B is a finitely generated A-module, and hence also of absolutely finite type. Let $\underline{c} = \underline{c}_{B/A}$ be the conductor. Then (see (IX, 5.9)) there is an epimorphism

$$G_0(A/\underline{c}) \oplus G_0(B) \longrightarrow G_0(A).$$

Since dim A $\leq$ 1 it follows that $A/\underline{c}$ is Artinian and B is a finite product of Dedekind rings. Therefore $G_0(A/\underline{c})$ is free abelian of finite rank, and $G_0(B) = K_0(B) = H_0(B) \oplus \mathrm{Pic}(B)$. The H_0 is free abelian of finite rank, and Pic(B) is finitely generated thanks to (3.1). q.e.d.

(3.3) EXAMPLE. (cf. (XII, 10.5)). Let $A = R[s^2, s^3]$ $B = R[s]$, where s is an indeterminate. Then it follows from (XII, 10.4) that, if t is an indeterminate, $\det_0 : Rk_0(A[t])$

$\longrightarrow$ Pic(A[t]) is an isomorphism. Moreover it follows from (XII, 10.5) that

$$\mathrm{Pic}(A[t]) \simeq \mathrm{Pic}(A) \oplus R[t].$$

This shows that the condition that A be integrally closed in the Mordell-Weil Theorem is vital. On the other hand, it follows from Grothendieck's Theorem (XII, 4.1) that

$$G_o(A[t]) \simeq G_o(A)$$

and it follows from the proof of (3.2) that $G_o(A)$ is finitely generated. In fact $G_o(A) \simeq \underline{\underline{Z}}$. Thus the Cartan homomorphism

$$c_o(A[t]) : K_o(A[t]) \longrightarrow G_o(A[t])$$

is surjective and has a non finitely generated kernel, whereas $G_o(A[t])$ is finitely generated.

Corollary (3.2) suggests the following:

Question. *If* A *is a ring of absolutely finite type then is* $G_o(A)$ *finitely generated?*

The examples above show that the analogous questions for K_o and K_1 have negative answers.

It is more natural to pose the question above for schemes X of absolutely finite type, i.e. of finite type over spec(R) for some R as in (3.1). Here $G_o(X)$ denotes the Grothendieck group of the category of coherent sheaves of $\underline{O}_X$-modules. One can then deduce from Mordell-Weil, just as in (3.2), that the answer is affirmative if dim $X \leq 1$. Arguing by induction on dimension, one can choose an affine open subscheme spec(A) $\subset$ X such that the complement, Y, has strictly smaller dimension than X. The restriction of sheaves to spec(A) is a quotient functor, whose kernel is the category of sheaves with support in Y. Thus there is an exact sequence (see (VIII, §5))

$$(1) \qquad G_o(Y) \longrightarrow G_o(X) \longrightarrow G_o(A) \longrightarrow 0.$$

By induction on dimension one can assume $G_o(Y)$ is finitely generated, and so we see that it suffices to give an affirmative response for affine schemes.

We can further assume, as in the proof of (3.2), that nil(A) = 0. Then we can invert some non zero divisor in A and, with the aid of an exact sequence like (1) above, reduce to the case when A is an integrally closed integral domain. Using a theorem of Zariski on the closedness of the singular locus of a variety we can even arrange that A is regular. But in this case $K_0(A) = G_0(A)$. Thus, to answer the question above affirmatively it suffices to do so for the following special case:

Question. Let A be a regular ring of absolutely finite type. Is $K_0(A)$ finitely generated?

The advantage of the first formulation of the question is that it makes possible the "devissage" arguments indicated above, and it avails a stronger induction hypothesis.

Since the one dimensional case is settled the next one to consider is when A is, say, the coordinate ring of a non singular surface. To indicate the flavor of the problem in this case we consider the following situation, which corresponds to a family of curves parametrized by a base curve.

Modifying our notational conventions slightly, suppose we are given the following:

R = a Dedekind ring with field of fractions L.

A = a finitely generated R-algebra

$B = L \otimes_R A$.

We assume that B and A/pA, for all $\underline{p} \in \max(R)$, are Dedekind rings. Thus spec(A) is like a surface projecting onto the curve spec(R). The generic fibre is spec(B), and the special fibres are spec(A/$\underline{p}$A) ($\underline{p} \in \max(R)$). Under precisely these conditions it was proved in (IX, 6.11) that there is an exact sequence

$$(2) \qquad SK_1(A) \longrightarrow SK_1(B) \longrightarrow \coprod_{\underline{p} \in \max(R)} \mathrm{Pic}(A/\underline{p}A) \longrightarrow$$
$$Rk_o(A) \xrightarrow{\det_o(A)} \mathrm{Pic}(A) \longrightarrow 0.$$

Assume now that R, and hence all the other rings here, are of absolutely finite type. Then Mordell-Weil implies Pic(A) and each $\mathrm{Pic}(A/\underline{p}A)$ is finitely generated. However there are infinitely many of the latter. In order to show that $G_o(A) = K_o(A)$ is finitely generated it is necessary and sufficient to show that the cokernel of

$$SK_1(B) \longrightarrow \coprod_{\underline{p} \in \max(R)} \mathrm{Pic}(A/\underline{p}A)$$

is finitely generated. This problem is also interesting because of the interpretation of $SK_1(B)$ in terms of reciprocity laws. If $K_o(A)$ is to be finitely generated this should force $SK_1(B)$ to be rather large.

While not being able to show that $G_o(A)$ is finitely generated, we can show, by the method above, that the group $SK_1(B)$ can be made quite large in certain cases.

Let F be a field and let A be the coordinate ring of an absolutely irreducible and absolutely non singular curve over F. If R is any F-algebra we shall write

$$A_R = A \otimes_F R.$$

When R is a field A_R is the coordinate ring over R of the same curve. Taking R = F[t], t an indeterminate, we can apply the discussion above to $A_R = A[t]$ and $B = A_{F(t)}$. It follows from Grothendieck's Theorem (XII, 3.1) that $K_o(A[t]) = K_o(A)$, and hence $\det_o(A[t]) : Rk_o(A[t]) \longrightarrow \mathrm{Pic}(A[t])$ is an isomorphism. Further (XII, 5.4) implies that $K_1(A[t]) = K_1(A)$, and hence likewise for SK_1. If we apply this information to the exact sequence (2) we obtain an exact sequence,

(3) $$SK_1(A) \longrightarrow SK_1(A_{F(t)}) \longrightarrow \coprod_{\underline{p}\,\in\,\max(F[t])} \mathrm{Pic}(A_{F(\underline{p})}) \longrightarrow 0,$$

where $F(\underline{p})$ denotes $F[t]/\underline{p}$. We shall now record this conclusion.

(3.4) THEOREM. *Let* F *be a field and let* A *be a finitely generated* F-*algebra such that* $A_{F'} = A \otimes_F F'$ *is a Dedekind ring for all field extension*, F', *of* F. *Let* t *be an indeterminate, and write* $F(\underline{p}) = F[t]/\underline{p}$ *for* $\underline{p} \in \max(F[t])$. *Then there is an exact sequence*

$$SK_1(A) \longrightarrow SK_1(A_{F(t)}) \longrightarrow \coprod_{\underline{p}\,\in\,\max(F[t])} \mathrm{Pic}(A_{F(\underline{p})}) \longrightarrow 0.$$

(3.5) COROLLARY. *If* F *above is an algebraic extension of a finite field then*

$$SK_1(A_{F(t)}) \cong \coprod \mathrm{Pic}(A_{F(\underline{p})}) \qquad (\underline{p} \in \max(F[t])).$$

Proof. It follows from (VI, 8.5) that $SK_1(A) = 0$ when F is a finite field. In general A is a direct limit of Dedekind rings of finite type over finite fields, and SK_1 commutes with direct limits. q.e.d.

Note that the fields $F(\underline{p})$ in (3.4) consist of all finite extensions of F with one generator (with many repetitions) and these include all separable finite extensions. The groups $\mathrm{Pic}(A_{F(\underline{p})})$ can be made to be arbitrarily large, even when F is of finite type over the prime field. For spec(A) is an open set in a complete non singular curve X over F and there is an epimorphism $\mathrm{Pic}(X_{F'}) \longrightarrow \mathrm{Pic}(A_{F'})$ whose kernel is generated by a number of elements which is independent of F' (corresponding to the "points at infinity"). Hence it suffices to know that $\mathrm{Pic}(X_{F'})$ can be made to have a large number of generators. But $\mathrm{Pic}(X_{F'}) \supset J_{F'}$, the group of rational points over F' of an abelian variety J, of dimension g = genus X, and defined over F. If $g > 0$ the $J_{F'}$ become quite large. For example, if $F \subset \underline{\underline{C}}$, $J_{\underline{\underline{C}}}$, as a group, looks like a 2g dimensional real torus $\underline{\underline{C}}^g/\Gamma$ for some lattice Γ, and $J_{\overline{F}}$ contains all elements of finite order in $J_{\underline{\underline{C}}}$.

If we apply the theory of Chapter VI to (3.4) we see that there are reciprocity laws over $A_{F(t)}$ with values in $\mathrm{Pic}(A_{F(\underline{p})})$ for each $\underline{p} \in \max(F[t])$. Since the groups $\mathrm{Pic}(A_{F(\underline{p})})$ may have elements of infinite order these reciprocity laws cannot be obtained from Weil's reciprocity law, by the procedure described in (VI, §8). So far as I know these reciprocity laws have no antecedent.

Let $\overline{F(t)}$ be the algebraic closure of F(t), and consider the homomorphism

$$SK_1(A_{F(t)}) \longrightarrow SK_1(A_{\overline{F(t)}}).$$

It follows from (IX, 4.7) that its kernel consists of elements of finite order; we treat $\overline{F(t)}$ as a direct limit of finite extensions. Thus, by choosing A in (3.4) so that the groups $\mathrm{Pic}(A_{F(\underline{p})})$ have elements of infinite order (for some $\underline{p}$), we conclude that $SK_1(A_{\overline{F(t)}})$ likewise has elements of infinite order. Thus there exist non trivial reciprocity laws on non singular affine curves over algebraically closed fields. This answers negatively a question posed by Mumford.

APPENDIX

Chapter XIV
VECTOR BUNDLES AND PROJECTIVE MODULES

The first three sections of this appendix contain the proof of a theorem of Swan [2] stating that the global section functor is an equivalence from the category of vector bundles ($k = \mathbb{R}$ or $\mathbb{C}$) over a compact space X to the category of finitely generated projective k(X)-modules, where k(X) is the ring of continuous k-valued functions on X. In §4 the basic stability theorems for vector bundles over a finite CW complex are proved. The theorems of Chapter IV are, via Swan's theorem, direct algebraic analogues. The topological counterpart for the algebraic K_1 arises from the Classification Theorem for bundles on a suspension, which we formulate in §5. Specifically, if X is a finite CW complex with base point x_o, write

$$K(X) = \text{Grothendieck group of vector bundles over } X$$

and

$$\tilde{K}(X) = \operatorname{Ker}(K(X) \xrightarrow{\text{"rk"}} K(\{x_o\})).$$

Put $K_1(X) = \tilde{K}(SX)$ where SX is the suspension of X. Then

$$K_1(X) = \pi_o(GL(k(X))) = GL(k(X))/GL(k(X))^o,$$

where $GL(k(X))^o$ is the component of the identity in the topological group GL(k(X)). We prove that

$$E(k(X)) \simeq SL(k(X))^{o},$$

so there is a canonical epimorphism

$$K_1(k(X)) \longrightarrow K_1(X)$$

whose kernel is the group of contractible continuous functions $X \longrightarrow k - \{0\}$.

In §6 we show that Bott's complex periodicity theorem can be formulated to say that

$$\mathrm{Ker}(K_1(X \times S^1) \longrightarrow K_1(X))$$

is naturally isomorphic to K(X). Here X is a finite CW complex, and the arrow is induced by $X \longrightarrow X \times S^1 (x \longmapsto (x, 1))$. Using Swan's Theorem and the results of §5 on K_1 we show that our Fundamental Theorem (Theorem (7.4) in Chapter XII) can be regarded as an algebraic analogue of the periodicity theorem.

The exposition here presumes several basic facts from point set topology, and the later sections quote a number of results, without proof, from the general theory of fibre bundles. All of these results can be found in the union of Steenrod [1] and Husemoller [1].

§1 VECTOR BUNDLES

If $X \in \underline{\underline{Sp}}$, the category of topological spaces and continuous maps, then

$$\underline{\underline{Sp}}/X$$

denotes the category whose objects are "spaces over X," i.e. morphisms $p: E \longrightarrow X$ in $\underline{\underline{Sp}}$, and in which a morphism $f: (E, p) \longrightarrow (E', p')$ is a continuous map $f: E \longrightarrow E'$ such that $p'f = p$. It follows that f induces maps $f_x: E_x \longrightarrow E_x'$ on the "<u>fibres</u>" over each $x \in X$. Here $E_x = p^{-1}(x)$, and similarly for E_x'. A <u>section</u> of E (we shall often suppress p in the notation) is a continuous function $s: X \longrightarrow E$ such that $ps = 1_X$. This is the same as a morphism $(X, 1_X) \longrightarrow$

(E, p). The set of all sections will be denoted $\Gamma(E)$.

If $g: X' \longrightarrow X$ there is an induced functor, "pull-back"

$$g^*: \underline{\underline{Sp}}_{/X} \longrightarrow \underline{\underline{Sp}}_{/X'}$$

defined by

$$g^*(E, p) = (X' \times_X E, X' \times_X p).$$

Thus $g^*(E, p) = (g^*E, g^*p)$ is defined by the cartesian square

$$\begin{array}{ccc} g^*E & \longrightarrow & E \\ {\scriptstyle g^*p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X' & \underset{g}{\longrightarrow} & X \end{array} .$$

It is easily seen that this defines a functor, and that, if $g^*_1: X_1 \longrightarrow X'$, then there is a natural isomorphism $(gg_1)^* \simeq g_1{}^*g^*$.

If $g: A \subset X$ is the inclusion of a subspace then $g^*E = p^{-1}(A)$ and $g^*p = p|g^*E$. In this case we shall sometimes write $g^*E = E|A$.

(1.1) EXAMPLE. Let F be any space and write

$$\tau(F) = \tau_X(F) = (X \times F, pr_1) \in \underline{\underline{Sp}}_{/X}.$$

Of course the fibres can all be canonically identified with F. A morphism $f: \tau(F) \longrightarrow \tau(F')$ must be of the form $f(x, u) = (x, f_x(u))$, and we see that f is determined by a function

$$(1) \qquad X \longrightarrow \underline{\underline{Sp}}(F, F') \qquad (x \longmapsto f_x).$$

For reasonable spaces F and F′ the function space here admits a natural topology so that f is continuous if and only if (1) is continuous. For example we have

$$\Gamma(\tau_X(F)) = \{\text{continuous functions } X \longrightarrow F\}.$$

If $g\colon X' \longrightarrow X$ then it is readily checked that $g^*\tau_X(F) \simeq \tau_{X'}(F)$.

Henceforth we fix a field k which may be either $\underline{\underline{R}}$ or $\underline{\underline{C}}$. Recall that $\underline{\underline{M}}(k)$ is the category of finite dimensional k-modules.

(1.2) DEFINITION. A (k-) vector bundle over X is an object $(E, p) \in \underline{\underline{Sp}}_{/X}$ together with the structure of a k-module on E_x for each $x \in X$. It is further required that each $x \in X$ has a neighborhood U such that there exists a $V \in \underline{\underline{M}}(k)$ and an isomorphism $E|U \longrightarrow \tau_U(V)$ in $\underline{\underline{Sp}}_{/U}$ which is k-linear on each fibre. A trival vector bundle is one of the form $\tau_X(V)$ where $V \in \underline{\underline{M}}(k)$.

The vector bundles over X are the objects of a category,

$$\underline{\underline{B}}(X) = \underline{\underline{B}}_k(X).$$

A morphism of vector bundles is just a morphism in $\underline{\underline{Sp}}_{/X}$ which is k-linear on each fibre. It is then easily checked that a continuous function $g\colon X' \longrightarrow X$ induces an additive functor

$$g^*\colon \underline{\underline{B}}(X) \longrightarrow \underline{\underline{B}}(X').$$

A functor

$$T\colon \underline{\underline{M}}(k)^n \times (\underline{\underline{M}}(k)^o)^m \longrightarrow \underline{\underline{M}}(k)$$

will be called continuous if the maps

$$T\colon \operatorname{Hom}(V, W) \longrightarrow \operatorname{Hom}(TV, TW)$$

are continuous for all objects V, W in the source category. Note that these Hom's are euclidean spaces, so that continuity has a meaning

(1.3) PROPOSITION (cf. Husemoller [1], (Ch 5, (6.2))).

If T is a continuous functor, as above then there is a corresponding functor

$$T_X: \underline{\underline{B}}(X)^n \times (\underline{\underline{B}}(X)^o)^m \longrightarrow \underline{\underline{B}}(X)$$

for each X ε Sp. It is characterized up to natural isomorphism by the following properties.

(i) $\tau(TV)$ *is naturally (in V) isomorphic to* $T_X(\tau(V))$ *for* $V = (V_i, V_j') \in \underline{\underline{M}}(k)^n \times (\underline{\underline{M}}(k)^o)^m$. *Here* $\tau(V)$ *denotes* $(\tau(V_i), \tau(V_j'))$.

(ii) *If* $g: X' \longrightarrow X$ *is a continuous map then* $g^* T_X \simeq T_X \cdot g^*$ *(where the second* g^* *abbreviates* $(g^*)^n \times (g^{*o})^m$).

Outline of Proof. Using local trivializations of the bundles on X, the definition of T_X is forced by (i) locally. Different trivializations lead to compatible definitions thanks to the continuity of T. Moreover the functoriality of T permits a gluing together of the local constructions. Condition (ii), applied to inclusions of open sets on which the bundles are trivialized, shows that T_X must be obtained in this way.

By virtue of (1.3) we can speak of $E \oplus E'$, $E \otimes_k E'$, $\mathrm{Hom}_k(E, E')$, $\Lambda_k^r E$, etc. for $E, E' \in \underline{\underline{B}}_k(X)$.

We now introduce the k-algebra

$$k(X)$$

of continuous functions from X to k. If $E \in \underline{\underline{B}}(X)$ then

$\Gamma(E)$ *is a* $k(X)$-*module*.

This structure is defined as follows: Given $a \in k(X)$, $s, s' \in \Gamma(E)$, and $x \in X$,

$$(as)(x) = a(x)\, s(x)$$

$$(s + s')(x) = s(x) + s'(x).$$

It is easy to see that

$$\Gamma(E) \approx \mathrm{Hom}_{\underline{\underline{B}}(X)}(\tau(k), E)$$

as k(X)-modules. For if $e(x) = (x, 1) \in X \times k$ then a morphism $f: \tau(k) \longrightarrow E$ determines $s = fe \in \Gamma(E)$, and f is determined by s via the formula $f(x, t) = f\, te(x) = ts(x)$. Conversely, given any $s \in \Gamma(E)$, $(x, t) \longmapsto ts(x)$ is clearly a bundle morphism $\tau(k) \longrightarrow E$.

More generally, if $V \in \underline{\underline{M}}(k)$, then, by additivity of the functor

$$\Gamma: \underline{\underline{B}}(X) \longrightarrow k(X)\text{-mod},$$

we see that $\Gamma(\tau(V)) = k(X) \otimes_k V$, and further that

$$\mathrm{Hom}_{\underline{\underline{B}}(X)}(\tau(V), E) = \mathrm{Hom}_{k(X)}(\Gamma(\tau(V)), \Gamma(E))$$

$$= \mathrm{Hom}_k(V, \Gamma(E)).$$

In particular, if $s_1, \ldots, s_n \in \Gamma(E)$ there is a bundle morphism $f: \tau(k^n) \longrightarrow E$ such that $\mathrm{Im}(f_x)$ is spanned by $s_1(x), \ldots, s_n(x)$ for each $x \in X$.

(1.4) DEFINITION. A basis for $E \in \underline{\underline{B}}(X)$ is a set of sections $s_1, \ldots, s_n \in \Gamma(E)$ such that, for all $x \in X$, $s_1(x), \ldots, s_n(x)$ is a basis for E_x. If $x \in X$ a <u>local basis at</u> x for E is a basis for $E|U$ where U is some neighborhood of X.

The local triviality of vector bundles guarantees that local bases exist at every point.

(1.5) PROPOSITION. *Suppose* $E \in \underline{\underline{B}}(X)$ *has a basis* $s_1, \ldots, s_n$. *Then every* $s \in \Gamma(E)$ *can be written uniquely in the form*

$$s(x) = a_1(x)s_1(x) + \ldots + a_n(x)s_n(x) \quad (x \in X),$$

and $\quad a_i \in k(X)\ (i \le i \le n)$.

Proof. We must show that the a_i are continuous. This is a local question so we can assume $E = \tau(k^n) = X \times k^n$ is the trivial k^n-bundle. Let $e_1, \ldots, e_n$ be the standard basis of $\tau(k^n)$, and write $s_i = \Sigma\, a_{ij} e_j$. The a_{ij} are obtained from coordinate projections, so $a_{ij} \in k(X)$ $(1 \le i, j \le n)$. Similarly, since $s = \Sigma_{i,j}\, a_i a_{ij} e_j$ the functions $\Sigma_i\, a_i a_{ij}$ are continuous $(1 \le j \le n)$. If $\alpha = (a_{ij})_{1 \le i, j \le n}$ then $\alpha: X \longrightarrow GL_n(k)$ is continuous, and hence α^{-1} is also continuous, since $GL_n(k) \xrightarrow{\text{inverse}} GL_n(k)$ is continuous. Put $\alpha^{-1} = (b_{jk})$; then $\Sigma_j(\Sigma_i\, a_i a_{ij})b_{jk} = \Sigma_i\, a_i(\Sigma_j\, a_{ij} b_{jk}) = \Sigma_i\, a_i \delta_{ik} = a_k$ is continuous $(1 \le k \le n)$. q.e.d.

It follows from (1.5) that any local basis of E at a point x can be used to define a trivialization of E in a neighborhood of x. A local section of E refers to a section of $E|U$ for some open $U \subset X$.

(1.6) COROLLARY. *Let* $E, E' \in \underline{\underline{B}}(X)$ *and let* $f: E \longrightarrow E'$ *be a map of sets over* X *which is linear on each fibre. Assume that* fs *is a local section of* E′ *whenever* s *is a local section of* E. *Then* f *is continuous, and hence is a vector bundle morphism.*

Proof. The continuity of f is a local question, so we can assume both bundles are trivial, say $f: \tau(k^n) \longrightarrow \tau(k^m)$. Then if $e_1, \ldots, e_n$ and $e_1', \ldots, e_m'$ are the corresponding bases we can write $fe_i = \Sigma_j\, a_{ij}\, e_j'$ $(1 \le i \le n)$. By hypothesis fe_i is a section of $\tau(k^m)$, so (1.5) implies $a_{ij} \in k(X)$. Since

$$f(x, \Sigma\, t_i e_i(x)) = (x, \Sigma\, t_i\, a_{ij}(x)\, e_j(x))$$

it follows that f is continuous.

(1.7) COROLLARY. *Let* $E \in \underline{\underline{B}}(X)$, *let* $x \in X$, *and let*

$t_1, \ldots, t_h$ *be local sections of* E at x *such that* $t_1(x), \ldots, t_h(x)$ *are linearly independent. Then* $t_1(y), \ldots, t_h(y)$ *are linearly independent for all* y *in some neighborhood of* x.

Proof. Let $s_1, \ldots, s_n$ be a local basis at x and write $t_i = \Sigma_i \, a_{ij} \, s_j$. Some $h \times h$ minor of $(a_{ij}(x))$ is not zero. Since, by (1.5), the a_{ij} are continuous, the minor remains non zero near x.

While $\underline{\underline{B}}(X)$ is an additive category it is not abelian. Indeed bundle morphisms need not have kernels in $\underline{\underline{B}}(X)$, as the following simple example shows: Let $X = \{x \in \underline{\underline{R}} \mid 0 \leq x \leq 1\}$, the unit interval, and define $f \colon \tau_X(\underline{\underline{R}}) \longrightarrow \tau_X(\underline{\underline{R}})$ by $f(t, x) = (t, tx)$ $(t \in X, x \in \underline{\underline{R}})$. The problem is that f_t is an isomorphism if $t \neq 0$, while $f_0 = 0$. It turns out that this dimension jump is the only source of difficulty.

(1.8) PROPOSITION. *Let* $f \colon E \longrightarrow E'$ *be a morphism in* $\underline{\underline{B}}(X)$, *and assume that* $\dim \operatorname{Im}(f_x)$ *is a continuous (i.e. locally constant) function of* x. *Then* $\operatorname{Im}(f) = f(E)$ *and* $\operatorname{Ker}(f) = f^{-1}$ *(zero section in* E′*) are sub vector bundles of* E *and* E′, *respectively*.

Proof. Of course Im(f) and Ker(f), with the induced projections and topologies, are spaces over X with fibres in $\underline{\underline{M}}(k)$. Therefore we need only check that they are locally trivial.

We first treat Im(f). If $x \in X$ choose local sections $s_1, \ldots, s_h$ for E at x such that $fs_1(x), \ldots, fs_h(x)$ are a basis for $\operatorname{Im}(f_x)$. Put $t_i = fs_i$ $(1 \leq i \leq h)$ and let $t_1, \ldots, t_n$ be local sections of E′ at x such that $t_1(x), \ldots, t_n(x)$ are a basis for E_x'. According to (1.7) $t_1(y), \ldots, t_n(y)$ are linearly independent for all y near x. This implies that

$$\dim \operatorname{Im}(f_y) \geq \dim \operatorname{Im}(f_x) \quad \text{for all } y \text{ near } x.$$

Our hypothesis of local constancy of $\dim \operatorname{Im}(f_y)$ therefore implies that $t_1(y),\dots,t_h(y)$ are a basis of $\mathrm{Im}(f_y)$ for all y near x. Using $t_1,\dots,t_n$ to define an isomorphism $E|U \longrightarrow \tau_U(k^n)$ for some neighborhood U of x, we obtain an induced isomorphism of $\mathrm{Im}(f)|U \longrightarrow \tau_U(k^h)$.

For Ker(f) we extend $s_1(x),\dots,s_h(x)$ (which are necessarily linearly independent) to a basis $s_1(x),\dots,s_m(x)$, where $s_{h+1},\dots,s_m$ are suitable local sections. Since $fs_1, \dots,fs_h$ is a local basis for Im(f) at x we can, for each $i > h$, write $fs_i = \Sigma\, a_{ij} fs_j$ $(1 \le j \le h)$ where $a_{ij} \in k(U)$, for some neighborhood U of x. Replacing s_i by $s_i' = s_i - \sum_{1\le j\le h} a_{ij}s_j$ for $h < i \le m$, we have $fs_i' = 0$ on U $(h < i \le m)$, and $s_1,\dots,s_h,s_{h+1}',\dots,s_m'$ is still a local basis. By local constancy of dim $\mathrm{Ker}(f_y)$ we conclude, as above, that $s_{h+1}', \dots,s_m'$ is a local basis for x for Ker(f), and that the trivialization $E|U \longrightarrow \tau(k^m)$ defined by $s\ ,\dots,s_h,s_{h+1}', \dots,s_m'$ induces a trivialization $\mathrm{Ker}(f)|U \longrightarrow \tau(k^{m-h})$. q.e.d.

If $V \in \underline{\underline{M}}(k)$ write T(V) for the space of all hermitian forms on V (i.e. quadratic forms if $k = \underline{\underline{R}}$). Then $T\colon M(k)^\circ \longrightarrow M(k)$ is a continuous functor, so we can speak of a <u>hermitian form on a vector bundle</u> $E \in \underline{\underline{B}}(X)$. It is just a section of $T_X(E)$. More explicitly, it is a continuous family of hermitian forms $h_x \in T(E_x)$.

(1.9) PROPOSITION. <u>If</u> X <u>is paracompact then every</u> E $\in \underline{\underline{B}}(X)$ <u>admits an everywhere positive definite hermitian form</u>.

<u>Proof</u>. By paracompactness we can find a locally finite covering $\{U_\alpha\}$ of X such that $E|U_\alpha$ is trivial for each α. Let h_α be a positive definite form on $E|U_\alpha$ (this clearly exists) and let $\{g_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Now we define the form h on E by

$$h_x(e, e') = \Sigma_{x \in U_\alpha} g_\alpha(x) h_{\alpha, x}(e, e')$$

$$(e, e' \in E_x;\ x \in X).$$

By local finiteness this sum is finite. Moreover, if $e \in E_x$, $e \neq 0$, then $h_{\alpha, x}(e, e) > 0$ (if $x \in U_\alpha$) and $g_\alpha(x) \geq 0$ and $\Sigma g_\alpha(x) = 1$. Therefore $h_x(e, e) > 0$, showing that h_x is positive definite. The continuity of $x \longmapsto h_x$ is clear because, locally at x, h is a finite linear combination of the h_α's.

(1.10) COROLLARY. _In the setting of_ (1.9), _if_ E' _is a sub vector bundle of_ E _then_ $E \simeq E' \oplus E''$ _for some_ $E'' \in \underline{\underline{B}}(X)$.

Proof. Choose a positive definite hermitian form on E, and let E_x'' be the orthogonal complement in E_x of E_x', for each $x \in X$. The orthogonal projection of E to E' is clearly an idempotent bundle endomorphism of E with image E' (of locally constant rank) and kernel E''. Therefore (see (1.8)) E'' is a sub bundle of E, and clearly $E = E' \oplus E''$.

§2. BUNDLES ON A NORMAL SPACE HAVE ENOUGH SECTIONS

We shall call the elements of $\Gamma(E)$, for $E \in \underline{\underline{B}}(X)$, _global sections_. They define an additive functor

$$\Gamma: \underline{\underline{B}}(X) \longrightarrow k(X)\text{-mod}.$$

We will show that if X is a normal space this functor is fully faithful.

Throughout this section X _will denote a normal space_. This means that if U is a neighborhood of $x \in X$ then there is a continuous function f from X to the unit interval such that f vanishes outside some closed neighborhood of x contained in U, and f takes the constant value 1 in some (smaller) neighborhood of x.

Suppose $E \in \underline{\underline{B}}(X)$ and $s \in \Gamma(E, U)$. Then if we define s' by $s'(x) = f(x)\, \bar{s}(x)$ for $x \in U$ and $s'(x) = 0$ if $x \in U$ it is easy to see that $s' \in \Gamma(X)$ and $s' = s$ in a neighborhood of x. We now record this conclusion.

(2.1) PROPOSITION. *If* $E \in \underline{\underline{B}}(X)$ *and if* s *is a local section of* E *near* x *then there is an* $s' \in \Gamma(X)$ *such that* s' *and* s *coincide in a neighborhood of* x.

(2.2) COROLLARY. *Given* $x \in X$ *there exist global sections of* E *which are a local basis for* E *at* x.

(2.3) COROLLARY. *If* $f, g: E \longrightarrow E'$ *are morphisms in* $\underline{\underline{B}}(X)$ *and if* $\Gamma(f) = \Gamma(g)$ *then* $f = g$.

Proof. If $e \in E_x$ choose an $s \in \Gamma(X)$ such that $s(x) = e$. Such an s exists locally, and therefore globally by (2.1). Then $f(e) = fs(x) = \Gamma(f)(s)(x) = \Gamma(g)(s)(x) = gs(x) = g(e)$. q.e.d.

If $x \in X$ we have the ring homomorphism

$$\phi_x: k(X) \longrightarrow k$$

$$\phi_x(f) = f(x).$$

We shall write $\underline{m}_x = \mathrm{Ker}(\phi_x)$, which is a maximal ideal in k(X).

(2.4) PROPOSITION. *The homomorphism* $\Gamma(E) \longrightarrow E_x$, $s \longmapsto s(x)$, *induces an isomorphism*

$$\Gamma(E)/\underline{m}_x\Gamma(E) \longrightarrow E_x.$$

Proof. Surjectivity follows from (2.1), and the map clearly kills $\underline{m}_x\Gamma(E)$. It remains to show that, if $s(x) = 0$, then $s \in \underline{m}_x\Gamma(E)$. Choose $s_1,\ldots,s_n \in \Gamma(E)$ which are a local basis at x (see (2.2)). Then there exist $b_i \in k(X)$ such that $\Sigma\, b_i s_i$ and s coincide near x. (We first get the b_i locally at x and then globalize them as in (2.1).) Since $0 = s(x) = \Sigma\, b_i(x)\, s_i(x)$ it follows that $b_i(x) = 0$, i.e. $b_i \in \underline{m}_x$

$(1 \leq i \leq n)$. Put $s' = s - \Sigma\, b_i s_i$. Then s' vanishes in a neighborhood, say U, of x. Choose $r \in k(X)$ such that $r = 1$ outside U and $r = 0$ in a neighborhood of x. Then $r \in \underline{m}_x$ and $s' = rs'$, so

$$s = s' + \Sigma\, b_i s_i = rs' + \Sigma\, b_i s_i \in \underline{m}_x \Gamma(E). \quad \text{q.e.d.}$$

(2.4) THEOREM. <u>The functor</u>

$$\Gamma\colon \underline{\underline{B}}(X) \longrightarrow k(X)\text{-mod}$$

<u>is fully faithful</u>.

<u>Proof</u>. The assertion is that

$$\Gamma\colon \mathrm{Hom}_{\underline{\underline{B}}(X)}(E, E') \longrightarrow \mathrm{Hom}_{k(X)}(\Gamma(E), \Gamma(E'))$$

is an isomorphism for $E, E' \in \underline{\underline{B}}(X)$. The injectivity is just (2.3).

Let $\bar{f}\colon \Gamma(E) \longrightarrow \Gamma(E')$ be a $k(X)$-homomorphism. Using (2.4) we can define $\bar{f}_x\colon E_x \longrightarrow E_x'$ by the commutative diagram

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\ \bar{f}\ } & \Gamma(E') \\ \downarrow & & \downarrow \\ \Gamma(E)/\underline{m}_x\Gamma(E) & \longrightarrow & \Gamma(E')/\underline{m}_x\Gamma(E') \\ (\simeq)\ \downarrow & & \downarrow\ (\simeq) \\ E_x & \xrightarrow[\ \bar{f}_x\ ]{} & E_x' \end{array}$$

The maps $\bar{f}_x$ define a set map $f\colon E \longrightarrow E'$ over X which is linear on each fibre $(f_x = \bar{f}_x)$. If $s \in \Gamma(E)$ then $(fs)(x) = \bar{f}_x(s(x)) = \bar{f}(s)(x)$, so

$$(*) \qquad fs = \bar{f}(s) \qquad (s \in \Gamma(E)).$$

This shows that f carries global sections to global sections. By (2.1) it carries local sections to local sections. Therefore (1.6) implies f is a bundle morphism, and (*) says $\Gamma(f) = \overline{f}$. q.e.d.

§3. Γ: $\underline{\underline{B}}(X) \longrightarrow \underline{\underline{P}}(k(X))$ IS AN EQUIVALENCE FOR COMPACT X.

(3.1) THEOREM (Swan). Let X be a compact space. Then

$$\Gamma: \underline{\underline{B}}(X) \longrightarrow \underline{\underline{P}}(k(X))$$

is an equivalence from the category of vector bundles over X to the category of finitely generated projective k(X)-modules.

The following corollary is the main step of the proof.

(3.2) COROLLARY. Every $E \in \underline{\underline{B}}(X)$ is a direct summand of a trivial bundle.

Proof that (3.2) ⇒ (3.1). It follows from (2.5) that Γ is a fully faithful functor into k(X)-mod. Since $\Gamma(\tau(k^n)) = k(X)^n$ it follows from (3.2) that $\Gamma(E)$ is a direct summand of $k(X)^n$ for some n for each $E \in \underline{\underline{B}}(X)$. Conversely, if $P \in \underline{\underline{P}}(k(X))$ then there is an idempotent $\overline{e} \in \mathrm{End}_{k(X)}(k(X)^n)$ for some n such that $P \simeq \mathrm{Im}(\overline{e})$. Since Γ is fully faithful we have $\overline{e} = \Gamma(e)$ for some $e \in \mathrm{End}_{\underline{\underline{B}}(X)}(\tau(k^n))$ which is also idempotent. If we show that $\mathrm{Im}(e_x)$ has locally constant dimension then it will follow from (1.8) that $\mathrm{Im}(e)$ is a subbundle of $\tau(k^n)$, and clearly then $P \simeq \Gamma(\mathrm{Im}(e))$. The proof of (1.8) showed, in any case, that if $x \in X$, $\dim \mathrm{Im}(e_y) \geq \dim \mathrm{Im}(e_x)$ for all y near x. But since e is idempotent we obtain the opposite inequality by applying the analogue of this one to $I - e$.

Thus an $M \in k(X)$-mod is isomorphic to $\Gamma(E)$ for some $E \in \underline{\underline{B}}(X)$ if and only if $M \in \underline{\underline{P}}(k(X))$. q.e.d.

Proof of (3.2). Let $E \in \underline{\underline{B}}(X)$. If $x \in X$ then (2.2) allows us to choose $s_1, \ldots, s_{n_x} \in \Gamma(E)$ which are a local basis for E in some neighborhood U_x of x. These sections define a bundle morphism $f^x: \tau(k^{n_x}) \longrightarrow E$ such that $f^x | U_x$ is surjective. By compactness a finite number of these U_x's cover X. If $\tau(k^n)$ is the direct sum of the corresponding k^{n_x}'s, then the f^x's define a surjective bundle morphism $f: \tau(k^n) \longrightarrow E$. If $E' = \text{Ker}(f)$ then it follows from (1.8) that E′ is a sub bundle of $\tau(k^n)$, and from (1.10) that $\tau(k^n) = E' \oplus E''$ for some sub bundle E″. Evidently f induces an isomorphism from E″ to E. q.e.d.

Theorem (3.1) is the basis for translating much of the language of vector bundles into that of projective modules. The topological point of view emphasizes X, while the algebraic one emphasizes k(X). We shall now indicate a well known algebraic method for reconstructing X from k(X).

If $x \in X$ recall that

$$\underline{m}_x = \text{Ker}(k(X) \longrightarrow k) \qquad (f \longmapsto f(x)).$$

This defines a map

$$\phi: X \longrightarrow \max(k(X)), \ \phi(x) = \underline{m}_x.$$

Recall (III, §3) that the closed sets of max(k(X)) are of the form $V(\underline{a})$, where $\underline{a} \subset k(X)$ and where

$$V(\underline{a}) = \{\underline{m} \in \max(k(X)) \mid \underline{a} \subset \underline{m}\}.$$

Evidently $\phi^{-1}(V(\underline{a})) = \{x | f(x) = 0 \text{ for all } f \in \underline{a}\} = \bigcap_{f \in \underline{a}} Z(f)$, where Z(f) is the set of zeros of f, a closed set. Thus ϕ is continuous. A completely regular space is a Hansdorff space whose closed sets are all of the form Z(f). In particular its continuous k-valued functions separate points (so ϕ is injective) and they define the topology on X. Therefore, if X is completely regular, ϕ is a homeomorphism onto its image.

(3.3) THEOREM. If X is compact then

$$\phi: X \longrightarrow \max(k(X))$$

is a homeomorphism.

Proof. A compact space is completely regular so we need only show that ϕ is surjective.

Let $\underline{m} \in \max(k(X))$. If we show that the functions in $\underline{m}$ have a common zero, x, then we will have $\underline{m} \subset \underline{m}_x$, and hence $\underline{m} = \underline{m}_x$. If not then $\bigcap_{f \in \underline{m}} Z(f) = \phi$ so, by compactness, there is a finite set $f_1, \ldots, f_n \in \underline{m}$ having no common zero. Put $f = \Sigma f_i \overline{f}_i$ where $\overline{f}_i$ is the complex conjugate of f_i (equal to f_i if $k = \underline{\underline{R}}$). Then evidently $f \in \underline{m}$ and $f(x) > 0$ for all $x \in X$. Therefore f is a unit in $k(X)$, so $\underline{m} = k(X)$; contradiction.

§4. STABILITY THEOREMS FOR VECTOR BUNDLES.

The following theorem, which we quote without proof, is elementary but slightly technical (see Steenrod [1], §11).

(4.1) THEOREM ("Homotopy Theorem"). *Let* $g_0, g_1: X' \longrightarrow X$ *be homotopic maps in* $\underline{\underline{Sp}}$, *and let* $E \in \underline{\underline{B}}(X)$. *Then* $g_0^*E \simeq g_1^*E$.

(4.2) COROLLARY. *Let* T *be a contractible space and let* $E \in \underline{\underline{B}}(X \times T)$. *Define* $g_t: X \longrightarrow X \times T$ *by* $g_t(x) = (x, t)$. *Then, writing* $E_t = g_t^*E \in \underline{\underline{B}}(X)$, *we have* $E_t \simeq E_0$ *for all* $t \in T$, *and* $E = p^*E_t = E_t \times T$, *where* $p: X \times T \longrightarrow X$ *is the projection. In particular all bundles on* T *are trivial.*

Proof. All the g_t's are homotopy inverses to p.

Let Δ^n denote the standard n cell (unit ball in $\underline{\underline{R}}^n$) with interior Int Δ^n and boundary S^{n-1}, the $n - 1$ sphere. (For $n = 0$, Δ^n is a point and $S^{-1} = \phi$.) If X is a space an *n-cell in* X is a continuous function $c: \Delta^n \longrightarrow X$ whose restriction to Int Δ^n is a homomorphism onto its image.

Let $c_i: \Delta^{n_i} \longrightarrow X$ $(i \in I)$ be a finite family of cells in X, and put

$$X^{(n)} = \bigcup_{n_i \leq n} c_i(\Delta^{n_i}),$$

for each $n \geq 0$. We say this family of cells gives X the structure of a *finite* CW *complex* if (i) $X = \bigcup_{n \geq 0} X^{(n)}$ and (ii) For each i, $c_i(S^{n_i - 1}) \subset X^{(n_i - 1)}$. It follows easily from these conditions that X is the disjoint union of the open cells, $c_i(\text{Int } \Delta^{n_i})$. Moreover, a function on X is continuous if and only if its restriction to each $c_i(\Delta^{n_i})$ is continuous. The definition of a CW complex in general allows infinitely many cells, but then the last condition is not automatic, and it must be added to (i) and (ii) above. There is a further condition of "local finiteness" as well.

By abuse of language we shall call X a finite CW complex if it admits such a structure, and we shall then define dim X to be the least d such that $X = X^{(d)}$.

(4.3) THEOREM ("Stability Theorem"). *Let* X *be a connected finite* CW *complex of dimension* d, *and let* E *be a vector bundle over* X *of fibre dimension* n. (*The fibre dimension is constant because* X *is connected*.)

(a) *If* $n > d$ *then* E *has a non-vanishing global section.* (*I.e. there is an* $s \in \Gamma(E)$ *such that* $s(x) = 0$ *for all* $x \in X$.)

(b) *If* $n > d + 1$ *any two such sections are homotopic in the space of non-vanishing global sections.*

The proof requires the following elementary fact:

(4.4) PROPOSITION. *If* $0 \leq h < n$ *then* $\pi_h(S^n) = 0$.

Recall that this is proved by taking a simplicial approximation, g, to a continuous f: $S^h \longrightarrow S^n$. Then g is homotopic to f (for a fine enough approximation) and g is not surjective (there are no n cells in the simplicial subdivision of S^h being used.) If $p \in S^n - g(S^h)$ then we can deform $S^n - \{p\}$ to the base point of S^n, thus deforming g to a constant map.

<u>Proof of (4.3)</u>. Let A be a sub CW complex of X (i.e. a union of some of the closed cells of X). Suppose we have a non-vanishing section $s \in \Gamma(E|A)$. We claim that, if $n > d$, then s can be extended to a non-vanishing global section of E. Part (a) of (4.3) follows from this in the special case $A = \phi$.

We shall extend s to $A \cup X^{(h)}$ by induction on h ($h \leq d$). If $h = 0, A \cup X^{(o)}$ is the disjoint union of A and a finite set of points. Since $n > 0$ we can extend s by picking a non zero vector in the fibre of E over each of these points. Now suppose s is defined in $A \cup X^{(h)}$ and we propose to extend it to $A \cup X^{(h+1)}$ ($h < d$). It suffices to extend s continuously over each (h + 1)-cell, $c(\Delta^{h+1})$, in the CW complex structure of X. Since $c(S^h) \subset X^{(h)}$ the section s is already defined on the boundary. By considering $c^*E \in \underline{\underline{B}}(\Delta^{h+1})$, therefore, we are reduced to showing that a non-vanishing section $t \in \Gamma(c^*E \mid S^h)$ extends to a non vanishing section over all of Δ^{h+1}. Since Δ^{h+1} is contractible it follows from (4.2) that c^*E is trivial, so $c^*E \simeq \tau(k^n)$. Then a non vanishing section is a continuous function into $k^n - \{0\}$. The extendibility of a continuous function $t: S^h \longrightarrow k^n - \{0\}$ to Δ^{h+1} is equivalent to the condition that t is homotopic to a constant, clearly. But, up to homotopy, $k^n - \{0\}$ is equivalent to S^{n-1} if $k = \underline{\underline{R}}$ and S^{2n-1} if $k = \underline{\underline{C}}$. In either case, since $h < n$, the desired conclusion follows from (4.4).

To prove part (b) of (4.3) let $s_o, s_1 \in \Gamma(E)$ be non-vanishing sections, and assume $n > d + 1$. The sections s_o

and s_1 define a single section of $E \times [0, 1] \in \underline{\underline{B}}(X \times [0, 1])$ over the subset $A = X \times \{0\} \cup X \times \{1\}$ of $X \times [0, 1]$. We can clearly extend the CW complex structure on X to one on $X \times [0, 1]$ having dimension d + 1 and so that A is a subcomplex. Therefore the proof above gives us a non-vanishing section $s \in \Gamma(E \times [0, 1])$ restricting to s_o and s_1, respectively, at the two ends. If $g_t: X \longrightarrow X \times [0, 1]$ by $g_t(x) = (x, t)$ then there is a natural isomorphism $E \simeq g_t^*(E \times [0, 1])$ for all t, and then the sections s_t obtained from s by pullback with g_t describe the required homotopy between s_o and s_1. q.e.d.

(4.5) COROLLARY. Keep the notation and hypotheses of (4.3), and assume $n > d$.

(a) $E \simeq \tau(k) \oplus E'$ for some $E' \in \underline{\underline{B}}(X)$.

(b) If $\tau(k) \oplus E \simeq \tau(k) \oplus E'$ for some $E' \in \underline{\underline{B}}(X)$ then $E \simeq E'$.

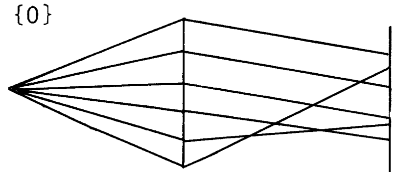
Proof. (a) A non-vanishing global section defines a monmorphism $\tau(k) \longrightarrow E$, and this splits, by (1.10). Therefore part (a) here follows from (4.3) (a).

(b) Changing notation so that E now denotes the $\tau(k) \oplus E$ in the statement, we are reduced to showing that if $E = \tau_o \oplus E_o = \tau_1 \oplus E_1$ where τ_o and τ_1 are trivial line bundles (i.e. $\simeq \tau(k)$) defined by non-vanishing sections s_o and s_1 then $E_o \simeq E_1$. But the proof above showed that we could write $E \times [0, 1] = \tau \oplus E'$ where τ is defined by a non-vanishing section s of $E \times [0, 1]$ over $X \times [0, 1]$ which induces s_o and s_1 on the ends. Pulling back along $g_t: X \longrightarrow X \times [0, 1]$ $(g_t(x) = (x, t))$ we obtain decompositions $E = \tau_t \oplus E_t'$ $(0 \leq t \leq 1)$. According to (4.2) we have $E_o' = g_o^*E' \simeq g_1^*E' = E_1'$. On the other hand $E_i \simeq E/\tau_i \simeq E_i'$ $(i = 0, 1)$, so $E_o \simeq E_1$. q.e.d.

Corollary (4.5) is the topological precursor of the stability theorems of Chapter IV.

§5. BUNDLES ON THE SUSPENSION, AND THE GENERAL LINEAR GROUP

Let f: Y ⟶ X be a morphism in Sp. The mapping cone, Cf, of f is the quotient of (Y × [0, 1]) ⨿ X (⨿ here means disjoint union) defined by collapsing Y × {0} to a point and by identifying (y, 1) with f(y) ε X for y ε Y. Schematically, it looks like

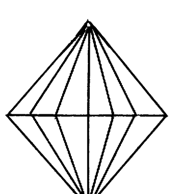


When f is a morphism in the category of spaces with base points then the reduced mapping cone, C′f, is obtained by further collapsing {b} × [0, 1] to a point (b = base point of Y) and this becomes the base point of C′f. The projection Cf ⟶ C′f is a homotopy equivalence. We call CX = Cl_X the cone of X. The maps f_s: X × [0, 1] ⟶ X × [0, 1], $f_s(x, t)$ = (x, st) ($0 \leq s \leq 1$) induce a deformation of CX to a point, so CX is contractible.

The suspension of X is

SX = C(X ⟶ {pt.}).

It looks like



Thus we can think of it as two cones, the upper and lower, glued along the equator, X × {1/2}. Suppose E ε $\underline{\underline{B}}$(SX). Then E has a trivial restriction to each cone (the cones are contractible). If we take two such trivializations and compare them on the equator we obtain an automorphism of a trivial bundle on X which, in turn, clearly determines E up to equivalence. An automorphism of a trivial bundle $\tau_X(V)$ is defined by a continuous function $X \longrightarrow \mathrm{Aut}(V)$. Thus bundles on SX arise from continuous functions $X \longrightarrow \mathrm{Aut}(V)$ (for various V). When do two such functions define isomorphic bundles on SX? The answer is given by the next theorem which we quote without proof (cf. Steenrod [1], §18).

(5.1) THEOREM ("Classification Theorem"). The isomorphism classes of vector bundles on SX with fibre k^n are in natural bijective correspondence with $[X, GL_n(k)]$, the homotopy classes of continuous functions from X to $GL_n(k)$.

If $\alpha: X \longrightarrow GL_n(k)$ is continuous then we can write $\alpha(x) = (a_{ij}(x))$ for each $x \in X$ and clearly $a_{ij} \in k(X)$ $(1 \leq i, j \leq n)$. The same observation applied to α^{-1} shows that $(a_{ij}) \in GL_n(k(X))$. Thus $[X, GL_n(k)]$ is a quotient of the topological group $GL_n(k(X))$. (Assume X is compact and use the uniform topology on $GL_n(k(X))$. Two elements of $GL_n(k(X))$ are homotopic as functions into $GL_n(k)$ if and only if they can be joined by a path, as elements of $GL_n(k(X))$. Thus

$$[X, GL_n(k)] = \pi_o(GL_n(k)) \quad \text{(set of arc components of } GL_n(k(X)))$$

$$= GL_n(k(X))/GL_n(k(X))^o,$$

where the denominator is the connected component of I, a normal subgroup of $GL_n(k(X))$.

Recall that

$$(1) \qquad GL_n(k(X)) = SL_n(k(X)) \underset{s\text{-}d}{\times} U(k(X)),$$

where $U(k(X)) = \{$continuous functions from X to $U(k) = k - \{0\}\}$ is the group of units of $k(X)$, and we identify $u \in U(k(X))$ with $\mathrm{diag}(u, 1,\ldots,1) \in GL_n(k(X))$. The projection $GL_n \longrightarrow U$ is the determinant. The semi-direct product decomposition is a topological one, so it follows that

$$GL_n(k(X))^o = SL_n(k(X))^o \underset{s-d}{\times} U(k(X))^o,$$

and hence

$$[X, GL_n(k(X))] \cong \pi_o(SL_n(k(X))) \underset{s-d}{\times} \pi_o(U(k(X))).$$

If $k = \underline{\underline{R}}$ then $\pi_o(U(\underline{\underline{R}}(X))) = [X, \underline{\underline{R}} - \{0\}] = [X, \{\pm 1\}]$, and if X is connected this is isomorphic to $\{\pm 1\}$. If $k = \underline{\underline{C}}$ then $\pi_o(U(\underline{\underline{C}}(X))) = [X, \underline{\underline{C}} - \{0\}] = [X, S^1]$. The latter is known to be isomorphic to the first cohomology group, $H^1(X, \underline{\underline{Z}})$. There is no apparent way to obtain a purely algebraic description of $\pi_o(U(k(X)))$. In contrast we have a very satisfactory description of $\pi_o(SL_n(k(X)))$.

(5.2) THEOREM. _Assume_ X _is compact. The connected component_, $SL_n(k(X))^o$, _of_ I _in_ $SL_n(k(X))$ _is equal to_ $E_n(k(X))$, _the group generated by elementary matrices. Further, it contains all unipotents._

The proof will be based on the following lemmas.

(5.3) LEMMA. _Let_ G _be a topological group and let_ H _be a subgroup which contains a neighborhood of_ 1 _in_ G. _Then_ H _is open, and therefore also closed, in_ G.

Proof. Say $1 \in U \subset H$ and U is open in G. If $x \in H$ then xU is a G-neighborhood of x in H, so H is open. Since G-H is a union of cosets of H it is likewise open.

(5.4) COROLLARY. _If_ G _is connected then any neighborhood of_ 1 _generates_ G.

Suppose $\alpha = I + \nu \in GL_n(k(X))$ is unipotent. Then clearly $\det(\alpha) = 1$. Moreover $\alpha_t = I + t\nu \in SL_n(k(X))$ $(0 \leq t \leq 1)$ so $\alpha \in SL_n(k(X))^o$. Since elementary matrices are unipotent this implies that

$$E_n(k(X)) \subset SL_n(k(X))^\circ.$$

Let N_+ denote the group of upper triangular unipotent matrices

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

and let N_- denote the lower triangular ones (i.e. the transpose of N_+). Clearly $N_\pm \subset E_n(k(X))$.

Suppose $\delta = \text{diag}(d_1,\ldots,d_n) \in GL_n(k(X))$. It follows from the Whitehead Lemma (see (V, 1.8 (a))) that, modulo $E_n(k(X))$, δ is congruent to $\text{diag}(d, 1,\ldots,1)$ where $d = d_1 \ldots d_n = \det(\delta)$. Therefore if D denotes the group of diagonal matrices in $SL_n(k(X))$ we have

$$N_-,\ D,\ N_+ \subset E_n(k(X)).$$

By virtue of (5.4), therefore, Theorem (5.2) will be proved once we establish,

(5.5) PROPOSITION. _The set_

$$N_- \cdot D \cdot N_+$$

contains a neighborhood of I _in_ $SL_n(k(X))$.

Proof. Let $|a| = \sup_{x \in X} |a(x)|$ for $a \in k(X)$, and write $V_n(\varepsilon)$ for the set of $\alpha = (a_{ij}) \in GL_n(k(X))$ such that $|a_{ij} - \delta_{ij}| < \varepsilon$ $(1 \le i, j \le n)$. We will show, by induction on n, that $\alpha = \tau_-\delta\tau_+$ with $\tau_\pm \in N_\pm$ and δ diagonal, provided ε is sufficiently small.

Let $\alpha \in V_n(\varepsilon)$ with $0 < \varepsilon < 1/2$. Then $|a_{11} - 1| < 1/2$ so $a_{11} \in U(k(X))$. Multiplying α on the left by $\tau = I - \Sigma_{1 < i \le n} a_{11}{}^{-1} a_{i1} e_{i1}$ we have $\tau\alpha = \begin{pmatrix} a_{11} & * \\ 0 & \beta \end{pmatrix}$ where

$\beta = (b_{ij})_{1 < i, j \leq n}$ and $b_{ij} = a_{ij} - a_{11}^{-1}a_{i1}a_{1j}$. Hence $|b_{ij} - \delta_{ij}| \leq |a_{ij} - \delta_{ij}| + |a_{11}^{-1}a_{i1}a_{1j}| < \varepsilon + |a_{11}^{-1}|\ \varepsilon^2 = \varepsilon(1 + \varepsilon\ |a_{11}^{-1}|)$. Since $|a_{11}(x)|$ is never smaller than $1 - \varepsilon$ we have $|a_{11}^{-1}| \leq (1 - \varepsilon)^{-1}$. Since $\varepsilon < 1/2, (1 - \varepsilon)^{-1} < 2$ so $\varepsilon(1 + \varepsilon\ |a_{11}^{-1}|) < \varepsilon\ (1 + 2\varepsilon) < 2\varepsilon$. Thus $\beta \in V_{n-1}(2\varepsilon)$. Therefore if we take $\varepsilon = 2^{-n}$, for example we can apply induction to β and make $\sigma'\beta$ upper triangular for some lower triangular unipotent $\sigma' \in GL_{n-1}(k(X))$. If $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & \sigma' \end{pmatrix}$ then $\sigma\tau\alpha = \begin{pmatrix} d_1 & & & * \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} = \delta\tau_+$ where $\delta = \mathrm{diag}(d_1, \ldots, d_n)$ and $\tau_+ \in N_+$. Put $\tau_- = (\sigma\tau)^{-1} \in N_-$, and the proof is complete.

Combining the Stability Theorem (4.3) with the results of this section one can easily deduce the following corollary.

(5.6) COROLLARY. _Let_ X _be a finite_ CW _complex of dimension_ d, _and consider the natural homomorphisms_

$$h_n : SL_n(k(X))/E_n(k(X)) \longrightarrow SL_{n+1}(k(X))/E_{n+1}(k(X)).$$

Then h_n _is surjective for_ $n \geq \dim SX$ $(= d + 1)$, _and_ h_n _is an isomorphism for_ $n > \dim SX$.

This result corresponds to the stability theorem of Chapter V.

§6. K-THEORY.

In this section we shall quote, without proof, a number of results from topology (cf. Husemoller [1]) leading up to a formulation of Bott's complex periodicity theorem.

Let $\underline{\underline{Sp}}_o$ denote the category of topological spaces with base points, and base point preserving continuous maps. In this category the coproduct of two objects, X and Y, exists, and it is denoted X v Y. It is the quotient of the disjoint union (i.e. coproduct in $\underline{\underline{Sp}}$) obtained by identifying the two base points, x_o and y_o, respectively. We give $X \times Y$ the base point (x_o, y_o), and then there is a canonical sequence

$$(1) \qquad X \vee Y \xrightarrow{\ i\ } X \times Y \longrightarrow X \wedge Y$$

where i is defined by $x \longmapsto (x, y_o)$ $(x \in X)$ and $y \longmapsto (x_o, y)$ $(y \in Y)$, and where $X \wedge Y$ is the space obtained by collapsing Im(i) to a point (the base point of $X \wedge Y$). Both v and $\wedge$ are (up to natural equivalence) associative and commutative operations, and $\wedge$ distributes over v. Moreover

$$S^1 \wedge X \simeq SX,$$

where SX here denotes the reduced suspension. We shall identify S^1 with the unit circle in $\underline{\underline{C}}$. It is easy to see that

$$S^1 \wedge S^n = S^{n+1}.$$

K-theory is the theory of the functor

$$K\colon (\underline{\underline{Sp}}_o)^o \longrightarrow \underline{\underline{Z}}\text{-mod}$$

defined by

$$K(X) = K_o(\underline{\underline{B}}(X)),$$

the Grothendieck group of the category of vector bundles over X (with respect to $\oplus$). According to Theorem (3.1) we have

$$K(X) = K_o(k(X)) \text{ \underline{if} X \underline{is compact}.}$$

Starting from this we can try to algebraisize as much of the theory as possible, and then apply it to rings which are no longer of the form k(X).

The inclusion of the base point x_o in X induces a homomorphism

$$K(X) \longrightarrow K(\{x_0\}) \simeq \underline{\underline{Z}}$$

whose kernel is denoted by

$$\widetilde{K}(X).$$

Most of the results below will be formulated for $\widetilde{K}$, but we can recover K by the isomorphism

(2) $$K(X) \simeq \widetilde{K}(S^0 \vee X)$$

The complex periodicity theorem is:

(6.1) THEOREM (Bott). _Let_ $k = \underline{\underline{C}}$. _Then for compact_ X _there is a natural isomorphism_

$$\widetilde{K}(S^2 X) \simeq \widetilde{K}(X).$$

We shall show below that our Fundamental Theorem (XII, 7.4) is an algebraic analogue of Bott's Theorem.

Let X be compact and let A be a closed subspace of X, containing the base point. We write X/A for the quotient with A collapsed to a point. Then the sequence

$$A \longrightarrow X \longrightarrow X/A$$

induces sequences

$$\widetilde{K}(S^n(X/A)) \longrightarrow \widetilde{K}(S^n X) \longrightarrow \widetilde{K}(S^n A)$$

for all $n \geq 0$.

(6.2) THEOREM. _Let_ X _be a finite_ CW _complex and let_ A _be a subcomplex. Then there are natural connecting homomorphisms_ δ _such that the sequence_

$$\xrightarrow{\delta} \widetilde{K}(S^n(X/A)) \longrightarrow \widetilde{K}(S^n X) \longrightarrow \widetilde{K}(S^n A) \xrightarrow{\delta}$$
$$\widetilde{K}(S^{n-1}(X/A)) \longrightarrow \cdots \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(A)$$

is exact.

Bott's theorem implies that, when $k = \underline{\underline{C}}$, this sequence has period six (i.e. it repeats after every interval of six terms).

With X and A as above, let $\underline{q}_A \subset k(X)$ denote the ideal of functions vanishing on A. Then we have an exact sequence

$$0 \longrightarrow \underline{q}_A \longrightarrow k(X) \longrightarrow k(A) \longrightarrow 0.$$

We can identify k(X/A) with the set of functions in k(X) which are constant along A. Thus

$$k(X/A) = k + \underline{q}_A \subset k(X).$$

If we write

$$K_1(X) = \widetilde{K}(SX)$$

then it can be deduced from the results of §5 that

$$K_1(X) \simeq [X, GL(k)]_o,$$

the group of homotopy classes of base point preserving continuous functions $\alpha: X \longrightarrow GL(k)$ (where I is the base point of GL(k)). Hence $K_1(X/A)$ corresponds to the classes of such α for which $\alpha(A) = \{I\}$. If, as in §5, we identify α with an element of GL(k(X)) then the latter condition translates into the condition that α belongs to the congruence subgroup $GL(k(X), \underline{q}_A)$. Thus, applying the Classification Theorem (5.1) we find that

$$K_1(X/A) \simeq \pi_o(GL(k(X), \underline{q}_A))$$

It follows now from Theorem (5.2) that:

<u>There is a canonical epimorphism</u>

$$K_1(k(X), \underline{q}_A) \longrightarrow K_1(X/A)$$

<u>which induces an isomorphism</u>

$$SK_1(k(X), \underline{q}_A) \longrightarrow SK_1(X/A)$$

<u>on the direct summands corresponding to</u>

$SL(k(X), \underline{q}_A)$.

Next we consider the long exact sequence of (6.2) associated with

(1) $X \vee Y \longrightarrow X \times Y \longrightarrow X \wedge Y,$

where X and Y are finite CW complexes.

(6.3) PROPOSITION. _For every_ $n \geq 0$

$$0 \longrightarrow \tilde{K}(S^n(X \wedge Y)) \longrightarrow \tilde{K}(S^n(X \times Y)) \longrightarrow \tilde{K}(S^n(X \vee Y)) \longrightarrow 0$$

is a (naturally) split short exact sequence.

The splitting

$$\tilde{K}(S^n(X \vee Y)) = \tilde{K}(S^n X) \oplus \tilde{K}(S^n Y) \longrightarrow \tilde{K}(S^n(X \times Y))$$

is induced by the projections

$$X \longleftarrow X \times Y \longrightarrow Y.$$

Hence, if $X \longrightarrow X \times Y$ is the map $x \rightarrowtail (x, y_0)$ then the kernel of $\tilde{K}(S^n(X \times Y)) \longrightarrow \tilde{K}(S^n X)$ is naturally isomorphic to $\tilde{K}(S^n Y) \oplus \tilde{K}(S^n(X \wedge Y))$.

Consider now the special case $Y = S^1$ and $n = 1$. Then we have

$$\mathrm{Ker}(K_1(X \times S^1) \longrightarrow K_1(X))$$
$$\simeq \tilde{K}(SS^1) \oplus \tilde{K}(S(X \wedge S^1))$$
$$\simeq \tilde{K}(S^2) \oplus \tilde{K}(S^2 X)$$
$$\simeq \tilde{K}(S^2(S^0 \vee X)).$$

Therefore we can reformulate the Periodicity Theorem in this case as follows, using (6.1) and (2):

(6.4) THEOREM. _Let_ $k = \underline{\underline{C}}$, _and let_ X _be a finite_ CW _complex. Define_ $f: X \longrightarrow X \times S^1$ _by_ $f(x) = (x, 1)$. _Then_ *there is a natural isomorphism*

$$K(X) \simeq \mathrm{Ker}(K_1(X \times S^1) \longrightarrow K_1(X)),$$

where $K_1(X) = \widetilde{K}(SX)$, *and similarly for* $X \times S^1$.

This formulation of periodicity admits a reasonable algebraic translation, as follows: Put $A = \underline{\underline{C}}(X)$ and $B = \underline{\underline{C}}(X \times S^1)$. The projection $X \times S^1 \longrightarrow X$ induces an embedding $A \subset B$. If $t: X \longrightarrow S^1$ is the other projection then $t \in B$, because $S^1 \subset \underline{\underline{C}}$. Moreover t never vanishes so $t^{-1} \in B$ also, and we have

$$A \subset A[t, t^{-1}] \subset B.$$

The function f in (6.4) induces a homomorphism $B \longrightarrow A$ obtained by restricting functions on $X \times S^1$ to $X \times \{1\}$. Thus it is the identity on A and it sends t to 1. In other words, f induces the unit augmentation on the group ring $A[t, t^{-1}]$. Therefore we obtain a commutative diagram

$$\begin{array}{ccc}
K_1(A[t, t^{-1}] & \longrightarrow & K_1(A) \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle (=)} \\
K_1(B) & \longrightarrow & K_1(A) \\
\downarrow & & \downarrow \\
K_1(X \times S^1) & \longrightarrow & K_1(X)
\end{array} \qquad (3)$$

where j is induced by the inclusion $A[t, t^{-1}] \subset B$, and where the bottom verticals exist because of Theorem (5.2). For example, the right bottom vertical is the natural projection

$$K_1(A) = GL(A)/E(A) \longrightarrow K_1(X) = GL(A)/GL(A)^{\circ},$$

where $GL(A)^{\circ}$ is the component of the identity in the topological group $GL(A) = GL(\underline{\underline{C}}(X)) = \{$continuous functions $X \longrightarrow GL(\underline{\underline{C}})\}$.

The top arrow is defined purely algebraically. We apply K_1 to the unit augmentation $A[t, t^{-1}] \longrightarrow A$ ($t \longmapsto 1$). The diagram (3) makes this a possible candidate for an algebraic analogue of the topologically defined arrow on the bottom. According to the Periodicity Theorem (6.4) the kernel of the bottom arrow is naturally isomorphic to K(X), which, by (3.1), is in turn isomorphic to $K_o(A)$. Therefore the following is an algebraic analogue of the Periodicity Theorem:

> Let A be a ring, let T be an infinite cyclic group with generator t, and let $A[T] \longrightarrow A$ be the unit augmentation ($t \longmapsto 1$). Then there is a natural isomorphism of
>
> $$\mathrm{Ker}(K_1(A[T]) \longrightarrow K_1(A))$$
>
> with $K_o(A)$.

Our Fundamental Theorem (7.4) implies that this kernel is naturally isomorphic to

$$K_o(A) \oplus 2\,\mathrm{Nil}(A).$$

If A is right regular (e.g. the coordinate ring of a non singular affine algebraic variety) then (see (XII, 6.3)) Nil(A) = 0, so we obtain a perfect analogue in this case.

This situation remains rather mysterious. In our rarified algebraic setting why does the complex periodicity theorem appear so naturally rather than, for example, the real one? Why isn't there a similar analogue with K_o in place of K_1? What, if anything, does the algebraic analogue have to do with periodicity phenomena?

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