SS 2021
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Exercises for Functional Analysis

Bonus
Submission date: Saturday, 19.07.2021
Digital submission via the E-Learning site of the tutorial

Exercise 1.
Let (X_i, d_i), $i = 1, 2$ be metric spaces and let (X_2, d_2) be complete. Let $A \subseteq X_1$ and $f : A \longrightarrow X_2$ be uniformly continuous. Prove that there is one and only one continuous function $\bar{f} : \bar{A} \to X_2$ with $f|_A = f$. This function $\bar{f}$ is uniformly continuous on $\bar{A}$. *(4 Points)*

Exercise 2.
Let $n \in \mathbb{N}$. On $[-1, 1]$ consider the measures $\mu_n := n1_{[0,1/n]}$. For each non-negative measurable function f we have $\mu_n(f) := \int_{-1}^{1} f(x)\, \mu_n(\mathrm{d}x) = n \int_0^{1/n} f(x)\, \mathrm{d}x$. Prove that the sequence $(\mu_n)_{n\in\mathbb{N}}$ (as a sequence of linear functionals on $\mathcal{C}([-1, 1])$) converges weakly. *(4 Points)*

Exercise 3.
Let X be a real Hilbert space and $x_n \in X$, $n \in \mathbb{N}$ such that $x_n \to 0$ weakly in X. Prove that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that the Cesaro summation $y_N := \frac{1}{N}\sum_{k=1}^{N} x_{n_k}$ converges strongly to 0. (4 Points)
Hint: Show the existence of subsequence $(n_k)_{k\in\mathbb{N}}$ with $|(x_{n_1}, x_{n_{k+1}})| \leqslant \frac{1}{k+1}$, ..., $|(x_{n_k}, x_{n_{k+1}})| \leqslant \frac{1}{k+1}$ and use that weakly convergent subsequences are bounded.

Exercise 4.
We call a normed vector space X uniformly convex if the following holds

$$\|x_n\| \leqslant 1, \quad \|y_n\| \leqslant 1 \quad \text{and} \quad \|x_n + y_n\| \to 2 \text{ implies that } \|x_n - y_n\| \to 0.$$

Let X be a uniformly convex Banach space X. Prove that the following statements are equivalent:

(i) $\|x_n - x\| \to 0$

(ii) $x_n \to x$ *weakly and* $\|x_n\| \to \|x\|$.

(4 Points)