SS 2021
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Exercises for Functional Analysis

Exercise 1
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Exercise 1.
Let (X_n, d_n) be a family of metric spaces and

$$X := \prod_{n\in\mathbb{N}} X_n = \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X_n \textit{ für } n \in \mathbb{N}\}$$

the cartesian product of the sets X_n, $n \in \mathbb{N}$.
a) Set

$$d\colon X \times X \to \mathbb{R}, \quad (x, y) \mapsto \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

Prove that (X, d) is a metric space. *(2 Points)*
b) Prove that (X, d) is complete if and only if (X_n, d_n) is complete for all $n \in \mathbb{N}$.
(2 Points)

Exercise 2.
The space $\ell^1_{\mathbb{R}}$ is defined by:

$$\ell^1_{\mathbb{R}} := \{(x_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n| < \infty\}.$$

For $x = (x_n)_{n\in\mathbb{N}} \in \ell^1_{\mathbb{R}}$ set

$$\|x\| := \sup_{n\in\mathbb{N}} \left| \sum_{k=1}^{n} x_k \right|.$$

Prove that $(\ell^1_{\mathbb{R}}, \|\cdot\|)$ is a normed space. *(2 Points)*
Is $(\ell^1_{\mathbb{R}}, \|\cdot\|)$ a Banch space? Prove it or construct a counter-example *(2 Points)*

Exercise 3.
The spaces $\ell^p_{\mathbb{R}}$ are defined by:

$$\ell^p_{\mathbb{R}} := \{(x_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}, \quad p \in [1, \infty)$$

and

$$\ell^\infty_{\mathbb{R}} := \{(x_n)_{n\in\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |x_n| < \infty\}.$$

For which $s \in \mathbb{R}$ and $p \in [1, \infty]$ does $(n^s)_{n\in\mathbb{N}} \in \ell^p_{\mathbb{R}}$ hold? *(4 Points)*

Exercise 4.
Let $(X, \mathcal{B}, \mu)$ be a mesure space with a finite measure μ. Let (Y, d) be a metric space. We set:

$$M(\mathcal{B}, d) := \{f\colon X \to Y \mid f \text{ is } \mathcal{B}/\mathcal{B}(Y)\text{-measurable}\}$$

and

$$D_\mu\colon M(\mathcal{B}, d) \times M(\mathcal{B}, d) \to \mathbb{R}, \qquad D_\mu(f, g) := \int \frac{d(f, g)}{1 + d(f, g)} \, \mathrm{d}\mu.$$

a) Prove that D_μ is a pseudometric on $M(\mathcal{B}, d)$.

(1 Point)

b) Prove that the sequence $(f_n)_{n\in\mathbb{N}}$ in $M(\mathcal{B}, d)$ converges in measure μ to a $f \in M(\mathcal{B}, d)$ (i.e. $\mu(d(f, f_n) > \varepsilon) \to 0$) if and only if $\lim_{n\to\infty} D_\mu(f, f_n) = 0$ holds.
Hint: Consider

$$\frac{\varepsilon}{1+\varepsilon}\mu(d(f_n, f) > \varepsilon) = \int_{\{d(f_n,f)>\varepsilon\}} \frac{\varepsilon}{1+\varepsilon} \, \mathrm{d}\mu$$

and use the fact that the mapping $x \mapsto \frac{x}{1+x}$ is increasing.

(2 Points)

c) Consider the equivalence relation

$$f \sim g :\Leftrightarrow f = g \ \mu\text{-almost everywhere.}$$

Then, $((M(\mathcal{B}, d)/\sim, D_\mu)$ is a metric space (No proof necessary!). Show that under the additional assumption (Y, d) being complete, that $((M(\mathcal{B}, d)/\sim, D_\mu)$ is complete as well.

(1 Point)