Exercises for Functional Analysis

Exercise 4
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Digital submission via the E-Learning site of the tutorial

Exercise 1.
Let $\Omega \subseteq \mathbb{R}$ be open and bounded, $x_0 \in \Omega$. Let $\alpha < 0$. Let $f(x) := |x - x_0|^{-\alpha}$.
a) Prove that f is an element of the Sobolev space $H^{1,1}(\Omega)$. *(2 Points)*
b) Calculate the weak derivative of f. *(2 Points)*

Exercise 2.
Let X be a non-empty set equipped with two metrics d_1 and d_2. Let $f, g\colon [0,\infty) \to [0,\infty)$ be functions that are continuous in 0 and with $f(0) = g(0) = 0$ such that

$$d_1(x,y) \leqslant f(d_2(x,y)), \qquad d_2(x,y) \leqslant g(d_1(x,y)), \qquad \forall x, y \in X$$

holds. Prove that the two metrics are equivalent. *(4 Points)*

Exercise 3.
Let $\Omega \subseteq \mathbb{R}$ be open and bounded. Let $0 < \alpha \leqslant 1$. Prove that $C^{0,\alpha}(\bar{\Omega})$ is not separable. *(4 Points)*
Hint: Consider in the case $\Omega = (0,1)$ the set of functions $f_t(x) = (\max\{x - t, 0\})^\alpha$. Show that $\|f_t - f_t'\|_{C^{0,\alpha}(\bar{\Omega})} \geqslant 1$ for $t \neq t'$. Follow from this that the set $\{f_t \colon t \in (0,1)\}$ and hence $C^{0,\alpha}(\bar{\Omega})$ can not be separable.

Exercise 4.
Let $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$d(x,y) := |\arctan(x) - \arctan(y)|.$$

Prove that
a) d is a metric on $\mathbb{R}$ *(1 Point)*
b) the metric space $(\mathbb{R}, d)$ is not complete *(1 Point)*
c) the topology induced by d is the same as the euclidean topology (I.e. a set is open w.r.t. d if and only if it is open w.r.t. $|\cdot|$). *(2 Points)*