SS 2021
M. Röckner

Exercises for Functional Analysis

Exercise 7
Submission date: Friday, 04.06.2021
Digital submission via the E-Learning site of the tutorial

Exercise 1.
Let L be a one-dimensional subspace of a Hilvert space H and $0 \neq a \in L$. Prove that for every $x \in H$

$$\operatorname{dist}(x, L^\perp) = \frac{|(x,a)|}{\|a\|}$$

holds. *(4 Points)*

Exercise 2.
Let $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi \geqslant 0$, $\varphi(-x) = \varphi(x)$, $\operatorname{supp}(\varphi) \subseteq B_1(0)$ and $\int \varphi = 1$. For $\varepsilon > 0$ let $\varphi_\varepsilon(x) := \varepsilon^{-1}\varphi(\frac{x}{\varepsilon})$ the corresponding Dirac sequence. Addtionally, let

$$f_\varepsilon(x) := \begin{cases} -\varepsilon, & x < \varepsilon \\ x, & x \in [-\varepsilon, 1+\varepsilon] \\ 1+\varepsilon, & x > 1+\varepsilon \end{cases}$$

and $\Phi_\varepsilon := \varphi_\varepsilon * f_\varepsilon$. Prove that

$$\Phi_\varepsilon(x) = x, \quad \forall x \in [0,1],$$
$$-\varepsilon \leqslant \Phi_\varepsilon(x) \leqslant 1+\varepsilon, \quad \forall x \in \mathbb{R}$$

and for $y \leqslant x$

$$0 \leqslant \Phi_\varepsilon(x) - \Phi_\varepsilon(y) \leqslant x - y.$$

(4 Points)

Exercise 3.
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with $a_n \geqslant 0$. Let $M_a := \{(x_n)_{n\in\mathbb{N}} \in \ell^p \mid |x_n| \leqslant a_n,\ \forall n \in \mathbb{N}\}$. Prove for $1 \leqslant p < \infty$ and $(a_n)_{n\in\mathbb{N}} \in \ell^p$ that the set M_a is compact. *(4 Points)*

Exercise 4.
Prove: A subset A of the metric space (X,d) is precompact (same as relatively compact) if and only if the closure of A is compact in the completion $(\bar{X}, \bar{d})$ of (X,d). *(4 Points)*