SS 2021
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Exercises for Functional Analysis

Exercise 8
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Exercise 1.
Let L be subvector space of a Hilbert space H and

$$L^{\perp} := \{x \in H \mid (x,y)_H = 0,\ \forall y \in L\}.$$

Prove that
a) For every $x \in H$ there is exactly one $x_L \in \bar{L}$ and one $x_{L^\perp} \in L^\perp$ such that

$$x = x_L + x_{L^\perp}$$

b) $H = \bar{L} \Leftrightarrow L^\perp = \{0\}$
c) $\overline{C_c^0(\mathbb{R}^n)}^{\perp} = \{0\}$ in $L^2(\mathbb{R}^n)$ *(4 Points)*

Exercise 2.
Let $a < b$ and E be the unit ball of $H^1((a,b))$. Prove that E precompact in $L^2((a,b))$. *(4 Points)*

Exercise 3.
Let $O(n)$ be the group of orthogonal matrices in $\mathbb{R}^{n\times n}$. Let $f \in L^p(\mathbb{R}^n,\mathbb{R})$ with $1 < p < \infty$ and $f_A(x) := f(A^{-1}x)$. Prove that

$$K_f := \{f_A \mid A \in O(n)\}$$

is a compact set in $L^p(\mathbb{R}^n,\mathbb{R})$. *(4 Points)*
Hint: $O(n)$ is compact. If $(f_{A_n})_{n\in\mathbb{N}}$ converges to a h in L^p then there is a $A \in O(n)$ with $h = f_A$

Exercise 4.
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and $0 < \alpha < \beta \leqslant 1$. Prove that every bounded set in $C^{0,\beta}(\bar{\Omega})$ is precompact in $C^{0,\alpha}(\bar{\Omega})$. *(4 Points)*