Exercises for Functional Analysis

Exercise 9
Submission date: Friday, 18.06.2021
Digital submission via the E-Learning site of the tutorial

Exercise 1.
Let $a < b$, $k \geqslant 1$ *und* $C^k([a,b]) := \{f\colon [a,b] \to \mathbb{R} | f$ *k-time continous differentiable*$\}$ *with the norm* $\|f\|_\infty := \sup\{|f(x)|\colon x \in [a,b]\}$. *Prove that the differential operator* $\frac{d}{dx}\colon C^{k+1}([a,b]) \to C^k([a,b])$, $f \mapsto \frac{df}{dx}$ *is a linear and non-continuous operator.* *(4 Points)*

Exercise 2.
Consider the space $(C([0,1]), \|\cdot\|_\infty)$. *We define the operators* $A_n\colon C([0,1]) \to C([0,1])$ *by*

$$(A_n x)(t) = t^n(1-t)x(t).$$

Does $(A_n)_{n\in\mathbb{N}}$ *converge in der operator norm?* *(4 Points)*

Exercise 3.
Consider the space $(C([0,1]), \|\cdot\|_\infty)$. *Let* $f \in C([0,1])$. *Define the operator* $A_f\colon C([0,1]) \to C([0,1])$ *by*

$$(T_f x)(s) := f(s) \cdot x(s) \qquad s \in [0,1].$$

Prove that T_f *is a bounded operator and calculate the operator norm of* T_f. *(4 Points)*

Exercise 4.
Let X, Y *be normed spaces. Let* $T\colon X \to Y$ *be a linear and bounded operator. Prove that the following definitions of the operator norm are equivalent.*

- $\|T\| = \inf\{c \geqslant 0\colon |Tx| \leqslant c|x| \ \forall x \in X\}$
- $\|T\| = \sup_{x\neq 0} \frac{|Tx|}{|x|}$
- $\|T\| = \sup_{|x|\leqslant 1} |Tx|$
- $\|T\| = \sup_{|x|=1} |Tx|$

(4 Points)

Hint: Some of the implications have already been proven in the lecture notes.