Summer term 2023
Prof. Dr. Michael Röckner and Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 1
Total points: 12
Submission before: Friday, 14.04.2023, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1. (4 Points)

Let X, Y be jointly Gaussian random variables on $\mathbb{R}^d$. Prove that X and Y are independent if and only if $\mathrm{cov}(X, Y) = 0$.

Let $(U, (\cdot,\cdot)_U)$ be a separable Hilbert space.

Problem 2 (Rotational invariance)**.** (4 Points)

Let μ be a Gaussian measure on $(U, \mathcal{B}(U)$ with mean zero. Let $\theta \in \mathbb{R}$ and define the rotation $R_\theta : U \times U \to U \times U$ by

$$R_\theta(x, y) = (x \sin(\theta) + y \cos(\theta), x \cos(\theta) - y \sin(\theta)), \quad x, y \in U.$$

Prove that $((\mu \otimes \mu) \circ R_\theta^{-1})(A) = (\mu \otimes \mu)(A)$ for all $A \in \mathcal{B}(U) \otimes \mathcal{B}(U)$.

Hint: Use the characterisation of Gaussian measures through their Fourier transform (Theorem 2.1.2).

Problem 3. (4 Points)

Let $Q \in L(U)$ be nonnegative, symmetric, with finite trace and $\mathrm{Ker}Q = 0$. Prove that for all $0 < r < \infty$

$$0 < N(m, Q))(B_r(x)) < 1.$$

Here $B_r(x)$ denotes the open ball in U with radius r.

Hint: Use Proposition 2.15.