Summer term 2023
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Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 10
Total points: 14
Submission before: Friday, 16.06.2023, 12:00 noon

Problem 1. (2+2+2 Points)

Consider the framework of Section 4.1. Let (V, H, V^*) be a Gelfand triple.

(i) Prove that

$$_{V^*}\langle z, v\rangle_V = \langle z, v\rangle_H \text{ for all } z \in H, v \in V.$$

(ii) Let $d \in \mathbb{N}$ and set $(H, ||\cdot||_H) = (\mathbb{R}^d, ||\cdot||_{\mathbb{R}^d})$ for some fixed norm $||\cdot||_{\mathbb{R}^d}$ on $\mathbb{R}^d$. Show that the embedding of V into H is a linear continuous isomorphism. In particular, $||\cdot||_V$ is equivalent to $||\cdot||_{\mathbb{R}^d}$.

(iii) Show that in general (H1) does not imply that $x \mapsto A(t, x, \omega)$ is continuous for fixed $t \in [0, T]$ and $\omega \in \Omega$.

Hint: Find a counterexample, e.g. in the finite dimensional setting $V = H = V^* = \mathbb{R}^2$.

Problem 2. (4 Points)

Prove Exercise 4.1.2 in the lecture notes.

Problem 3 (cf. between Remark 4.1.6 and Exercise 4.1.7)**.** (4 Points)

Let $\Lambda \subset \mathbb{R}^d$ be open. Let $H_0^1(\Lambda)$ be defined as in (4.1.6). Show that the canonical inclusion of $H_0^1(\Lambda)$ into $L^p(\Lambda)$ is one-to-one if and only if the linear operator

$$\nabla : C_c^\infty(\Lambda) \subset L^p(\Lambda) \to L^p(\Lambda)$$

is closable.