Summer term 2023
Prof. Dr. Michael Röckner and Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 5
Total points: 14
Submission before: Friday, 12.05.2023, 12:00 noon

Let $(E, ||\cdot||_E)$ be a separable Banach space and $(U, \langle\cdot,\cdot\rangle_U)$ be a separable Hilbert space.

Problem 1 (cf. proof of Proposition 2.2.6). (3+1 Points)

Show that there exist $l_n \in E^*, n \in \mathbb{N}$, such that

$$||x||_E = \sup_{n\in\mathbb{N}} l_n(x), \text{ for all } x \in E.$$

Use this equality to conclude that $\mathcal{B}(E) = \sigma(E^*)$.

Problem 2 (Prove the details). (2+2 Points)

(i) (cf. Remark 2.2.8) Let $f \in C([0,T]; E)$. Show that

$$\sup_{t\in[0,T]} ||f(t)||_E = \operatorname{ess\,sup}_{t\in[0,T]} ||f(t)||_E.$$

(ii) Let $n, m \in \mathbb{N}$ and $f \in L(\mathbb{R}^m, \mathbb{R}^n)$. Let $A = (a_{ij})_{i=1,\dots,n, j=1,\dots,m}$ be the transformation matrix of f with respect to orthonormal bases $\{b_1, ..., b_n\}$ of $\mathbb{R}^n$ and $\{b'_1, ..., b'_m\}$ of $\mathbb{R}^m$. Show that

$$||f||^2_{L_2(\mathbb{R}^m,\mathbb{R}^n)} = \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}|^2.$$

Problem 3. (4 Points)

Prove Proposition 2.3.4. in the lecture notes.

Problem 4. (4 Points)

Let $M \in \mathcal{M}^2_T(U)$. M is a Q-Wiener process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ if and only if $(\langle M(t), u\rangle_U \langle M(t), v\rangle_U - t\langle Qu, v\rangle_U)_{t\in[0,T]}$ is a martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ for all $u, v \in U$.