Summer term 2023
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Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 6
Total points: 16
Submission before: Friday, 19.05.2023, 12:00 noon

Problem 1. (4 Points)

Prove Proposition A.2.2 in the lecture notes.

Hint: A good exercise for 'measure theoretic induction'.

Problem 2 (Prove the details)**.** (2+1+1 Points)

Consider the situation of the proof of Proposition 2.3.8.

(i) Show that $\mathcal{K}$ is an algebra and that every element in $\mathcal{K}$ can be written as a finite disjoint union of elements in $\mathcal{A}$.

(ii) Let $A \in \mathcal{G}$. Prove that $A^{\complement} \in \mathcal{G}$.

Consider the situation of the proof of Lemma 2.3.9.

(iii) Show that the penultimate equality in Step 1 holds.

Problem 3. (4 Points)

Assume that W is a Q-Wiener process, $\Phi \in N_W^2$, and $a, b > 0$. Prove that

$$P\left(\sup_{t\in[0,T]} \left|\left|\int_0^t \Phi(s)dW(s)\right|\right|_H > a\right) \leqslant \frac{b}{a^2} + P\left(\int_0^T ||\Phi(s)||_{L_2^0}^2 ds > b\right).$$

Problem 4. (4 Points)

Assume that W is a Q-Wiener process and $\Phi_1, \Phi_2 \in N_W^2$. Prove that

$$E\int_0^t \Phi_i(s)dW(s) = 0, \quad i = 1, 2.$$

Moreover, calculate the following covariance operators[1]

$$Cov\left(\int_0^t \Phi_1(r)dW(r), \int_0^s \Phi_2(r)dW(r)\right), \quad s, t \in [0, T].$$

[1] $Cov(X, Y) : H \to H, Cov(X, Y)h := E[(X - E[X])\langle Y - E[Y], h\rangle_H]$ for $h \in H, X, Y \in L^2(\Omega, \mathcal{F}, P; H)$.