Winter term 2023/2024
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Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 11
Total points: 13
Submission before: Friday, 12.01.2023, 12:00 noon

Problem 1 (cf. Proof of Lemma 5.2.19 (and Proof of Theorem 4.2.4))**.** (3 Points)

Let $V \subset H \subset V^*$ be a Gelfand triple. Let $e_n \in V, n \in \mathbb{N}$, be an orthonormal basis in H such that $\{e_i : i \in \mathbb{N}\}$ is dense in V. Define $H_n := \mathrm{span}\{e_i : i \in \{1, ..., n\}\}, n \in \mathbb{N}$. Let $T > 0$, $p \in]1, \infty[$. Prove that the set

$$\left\{ \sum_{i=1}^{m} \rho_i(t) v_i : \rho_i \in L^\infty([0,T]), v_i \in \bigcup_{n\in\mathbb{N}} H_n, i \in \{1, ..., m\}, m \in \mathbb{N} \right\}$$

is dense in $L^p([0,T]; V)$.

Problem 2. (3 Points)

Consider the situation of Lemma 5.2.21. Prove (5.2.26), i.e. use (H3') and (H4') to show that there exists $C_0 \in (0, \infty)$ such that

$$\begin{aligned} 2{}_{V^*}\langle A(t_0, u_{n_i}(t_0)), u_{n_i}(t_0) - u(t_0)\rangle_V \leqslant & -\frac{\theta}{2}||u_{n_i}(t_0)||_V^\alpha + C_0(f(t_0) + h(t_0)g(||u_{n_i}(t_0)||_H^2)) \\ & + C_0(1 + ||u_{n_i}(t_0)||_H^{\alpha\beta})||u(t_0)||_V^\alpha. \end{aligned}$$

Hint: Use Young's inequality.

Problem 3. (4 Points)

Consider the situation of Section 5.2. Assume that A satisfies (H2") and (H4'). Let $k : [0,T] \times V \to \mathbb{R}$ be a bounded function such that for every bounded set O there exists $L \in (0, \infty)$ with

$$|k(t,x) - k(t,y)| \leqslant L||x - y||_V \quad \forall (t,x,y) \in [0,T] \times O \times O.$$

Then $\tilde{A} := kA$ satisfies (H2") replacing A.

Problem 4. (3 Points)

Let $d \in \mathbb{N}$. Let $f : \mathbb{R}^d(/\{0\}) \to \mathbb{R}, x \mapsto |x|^\alpha$. Determine all $\alpha \in \mathbb{R}, p \in [1, \infty)$ such that $f \in H^{1,p}(B_1(0))$.