Exercises to Stochastic Analysis

Sheet10
Total points: 16
Submission before: Friday, 23.12.2022, 12:00 noon

(*[Parts of] Exercises marked with "*" are additional exercises.*)

Problem 1 (Completion of probability space). (2+2 Points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. On the previous sheet we particularly discussed the augmentation of a sub-σ-algebra $\mathcal{A} \subseteq \mathcal{F}$, which is $\sigma(\mathcal{A}, \mathcal{N})$, where $\mathcal{N} = \{N \in \mathcal{F} : \mathbb{P}(N) = 0\}$. This augmentation is not to be confused with the completion *of the basic σ-algebra $\mathcal{F}$ and with the corresponding* completion *of the space $(\Omega, \mathcal{F}, \mathbb{P})$. These terms are investigated in this exercise.*

Consider the system of subsets of zero sets $\mathcal{N}_\subseteq := \{M \subseteq N : N \in \mathcal{N}\}$. In general, it is not true that $\mathcal{N}_\subseteq \subseteq \mathcal{F}$, i.e. there are subsets of zero sets which are not measurable. However, by the monotonicity of $\mathbb{P}$ intuitively it seems plausible that one should have "$\mathbb{P}(M) = 0$" for $M \in \mathcal{N}_\subseteq$. Also for technical reasons it might be useful to know that any subset of a zero set is a zero set (i.e. measurable) itself. For instance, if a property P holds for $\mathbb{P}$-a.a. paths of a stochastic process, then the set of paths which violate P is a (possibly proper) subset of a zero set. To achieve this, one considers the *completion* $\bar{\mathcal{F}}$ of $\mathcal{F}$, i.e.

$$\bar{\mathcal{F}} := \sigma(\mathcal{F}, \mathcal{N}_\subseteq).$$

(i) Prove $\bar{\mathcal{F}} = \{F \cup M | F \in \mathcal{F}, M \in \mathcal{N}_\subseteq\}$.

Recall: in general for two set systems $\mathcal{C}, \mathcal{D}$ $\sigma(\mathcal{C}, \mathcal{D}) = \{C \cup D | C \in \mathcal{C}, D \in \mathcal{D}\}$ is wrong, even if $\mathcal{C}$ and $\mathcal{D}$ are σ-algebras!

It seems reasonable to define $\bar{\mathbb{P}}$ on $\bar{\mathcal{F}}$ by $\bar{\mathbb{P}}(F \cup M) := \mathbb{P}(F)$.

(ii) Prove that $\bar{\mathbb{P}}$ is well-defined and a probability measure on $\bar{\mathcal{F}}$ which extends $\mathbb{P}$ on $\mathcal{F}$.

The probability space $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is called the completion *of $(\Omega, \mathcal{F}, \mathbb{P})$ and by construction: if $N \in \bar{\mathcal{F}}$ with $\bar{\mathbb{P}}(N) = 0$ and $M \subseteq N$, then $M \in \bar{\mathcal{F}}$ and $\bar{\mathbb{P}}(M) = 0$. Sometimes one assumes to work on a complete probability space. If one considers a filtration $(\mathcal{F}_t)_{t \geqslant 0}$ on such a complete space and considers its augmentation, then it is of course true that every subset of any zero set belongs to the augmented filtration. So far, we have never explicitly assumed to work on a complete probability space in this lecture, but we will, for instance, do so in next semester's lecture.*

Problem 2 (Deterministic motion without memory as example of Markov process). (4 Points)

*The main feature of a Markov process can be described as "*the future evolution of the process after time $t_0 > 0$ only depends on its state at time t_0, and not on its past $[0, t_0)$*". Typically, Markov processes are stochastic processes. However, for deterministic systems, the concept of '*independence of the future from the past, given the present*' is a causality principle that applies to large parts of classical physics, described by ODEs as in the exercise below: here the change of a curve y at t_0 depends only on its current position (via $b(y(t_0))$), but not on $y_{|[0,t_0)}$. Hence, it seems plausible that solutions to such an ODE form a Markov process or, strictly speaking, not the curves themselves, but their path laws (which in the deterministic case are just the Dirac measures over the paths) are* *expected to be Markov (because by def. a Markov process is a family of probability measures $\mathbb{P}_x$. In this exercise, you prove that this is indeed the case if the ODE is well-posed.*

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field such that the ODE on $[0, \infty) \times \mathbb{R}^d$

$$\frac{d}{dt} y(t) = b(y(t)), \quad y(0) = x$$

is well-posed, i.e. for every $x \in \mathbb{R}^d$ there is a unique C^1 solution $y^x : [0, \infty) \to \mathbb{R}^d$ with $y^x(0) = x$, and $x \mapsto y^x \in C(\mathbb{R}_+, \mathbb{R}^d)$ is continuous wrt. the locally uniform topology on $C(\mathbb{R}_+, \mathbb{R}^d)$.

A sufficient condition for this is that b is locally Lipschitz and satisfies the *one-sided growth condition* $\langle b(x), x\rangle \leqslant C(1 + |x|)$ for all $x \in \mathbb{R}^d$ and some $C > 0$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$.

Consider the probability measures $\mathbb{P}_x(\cdot) := \delta_{y^x}(\cdot)$ on $(\Omega, \mathcal{F})$, where

$$\Omega = C(\mathbb{R}_+, \mathbb{R}^d), \ \mathcal{F} = \sigma(\pi_t, t \geqslant 0), \ \pi_t : \Omega \to \mathbb{R}^d, \pi_t(f) = f(t).$$

Denoting $X_t := \pi_t$ (i.e. we consider the *canonical model* of Section 3.1.), prove that $(\Omega, \mathcal{F}, (\mathbb{P}_x)_{x \in \mathbb{R}^d}, (X_t)_{t \geqslant 0})$ is a Markov process wrt. $(\mathcal{F}_t)_{t \geqslant 0} = (\sigma(X_s, 0 \leqslant s \leqslant t))_{t \geqslant 0}$.

Problem 3 (Basics on Markov processes). (2+1+2+3 Points)

Markov processes are one of the most important classes of stochastic processes used in pure stochastic analysis and its applications in e.g. finance and physics. It is therefore important that you develop an intuition for the features of these processes. The following exercise shall help you to develop an intuitive understanding.

(i) Let $n = 3$. Write out equation (3.1.1.) in full length and simplify as much as possible for a) the general case and b) for $f(x_0, x_1, x_2, x_3) = \mathbb{1}_{A_0}(x_0) \cdot \mathbb{1}_{A_1}(x_1) \cdot \mathbb{1}_{A_2}(x_2) \cdot \mathbb{1}_{A_3}(x_3)$, $A_i \in \mathcal{S}$.

(ii) Show that in the situation of Thm.3.1.4 (i)(a) one has $\mathbb{P}_x \circ X_t^{-1} = p_t(x, \cdot)$ for all $t \geqslant 0$
(that this is true in 3.1.4. (i)(b) is clear by (3.1.2.). Hence, altogether one sees that for a Markov process $((\mathbb{P}_x)_{x \in S}, (X_t)_{t \geqslant 0})$, one always has that $p_t(x, \cdot)$ equals $\mathbb{P}_x \circ X_t^{-1}$.)

(iii) Use (ii) to note that the right-hand side of (M2) is equal to $p_t(X_s, B)$ (where $p_t(x, dy)$ denotes the Markovian kernel of the given Markov process). Then show that (M2) implies for all $t, s \geqslant 0, B \in \mathcal{S}$ and $x \in S$

$$\mathbb{E}_x[X_{t+s} \in B] = \int_S \mathbb{P}_y(X_t \in B) \, p_s(x, dy)$$

and interpret this equality in the case that the Markov process is *normal*, i.e. $\mathbb{P}_x(X_0 = x) = 1$ for all $x \in S$.

(iv) Prove (3.1.1) in the situation of Thm.3.1.4(i)(b) explicitly for the case $n = 0$, $n = 1$ and $n = 2$, i.e. first prove it for $n = 0$ (you can use (ii)) and then iteratively for the other cases.
In other words, this proves the first three iterates of the inductive argument used in the proof of (3.1.1.) in Thm.3.1.4(i)(b).
You can first use functions of tensor type and then argue (in detail!) by a monotone class-argument, as indicated in the proof of Thm.3.1.4(i)(b).