Winter break exercises to Stochastic Analysis

Sheet11
Total points: 16+8*
Submission before: **Thursday, 12.01.2023, 12:00 noon**

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (Path measures). (2+2+3* Points)

Measures on spaces of paths are, at least implicitly, present everywhere in stochastic analysis, since every stochastic process is associated with such a path measure, namely its law. *Many features of a process can be studied via its law. An important feature of such path measures is that even though they are measures on a "huge" space of paths, they are already uniquely determined by their family of finite-dimensional distributions, as we learn in this exercise.*

Let $(S, \mathcal{S})$ be a measurable space and $\Omega \subseteq S^{[0,\infty)}$ a subset of the space of all maps $y : [0, \infty) \to S$.

Usual cases are $\Omega = C([0, \infty), S)$ or $\Omega =$ space of càdlàg paths, if S is a topological space, or $\Omega = S^{[0,\infty)}$.

Consider probability measures $P, \tilde{P}$ on the measurable space $(\Omega, \mathcal{F})$, $\mathcal{F} := \sigma(\pi_t, t \geqslant 0)$, where $\pi_t : \Omega \to S$, $\pi_t(y) := y(t)$ for $y \in \Omega$ denotes the canonical projection from Ω at t.

(i) Prove: If P and $\tilde{P}$ have the same *finite-dimensional marginals*, i.e. if

$$P \circ (\pi_{t_0}, \dots, \pi_{t_n})^{-1} = \tilde{P} \circ (\pi_{t_0}, \dots, \pi_{t_n})^{-1}, \quad \forall n \in \mathbb{N}_0, 0 \leqslant t_0 < \dots < t_n$$

(also called *finite-dimensional distributions*), then $P = \tilde{P}$ on $\mathcal{F}$.

Measures P on $(\Omega, \mathcal{F})$ are also called path measures *or* path laws, *since Ω can be thought of (as a subset of) the space of paths $y : [0, \infty) \to S$. An important special case of such path measures are* laws *of stochastic processes:*

(ii) Let X be an $(\mathcal{A}_t)$-adapted S-valued [continuous] stochastic process on a filtered probability space $(\tilde{\Omega}, \mathcal{A}, (\mathcal{A}_t)_{t\geqslant 0}, \mathbb{P})$, then it can be considered as a map $X : \tilde{\Omega} \to \Omega$, $X(\omega) = (t \mapsto X_t(\omega))$ with $\Omega = S^{[0,\infty)}$ [with $\Omega = C([0, \infty), S)$].

Prove that X is $\mathcal{A}/\mathcal{F}$-measurable. Then consider the *law* of X, i.e. the image measure $\mathbb{P}\circ X^{-1}$ on $(\Omega, \mathcal{F})$, and deduce that it is uniquely determined by the distributions of the random vectors $(X_{t_0}, \dots, X_{t_n})$, $n \in \mathbb{N}_0, 0 \leqslant t_0 < \dots < t_n$.

(iii)* Prove that if (S, d) is a separable metric space such that $\mathcal{S} = \mathcal{B}(S)$ and $\Omega = C([0, \infty), S)$ is equipped with the topology of locally uniform convergence, then $\mathcal{B}(\Omega) = \mathcal{F}$.

You may use: since (S, d) is separable, Ω with the top. of loc. unif. conv. can be metrized by a metric $\tilde{d}$ so that $(\Omega, \tilde{d})$ is separable.

Hence $\mathcal{F}$ is not some strange artificial σ-algebra, but also the natural one if one studies the path space Ω from a topological point of view. Furthermore, note that $\mathcal{F}$ is obviously also equal to $\mathcal{B}(\Omega)$ if Ω is considered with the topology of pointwise convergence.

Problem 2 (The canonical model). (1,5+1,5 Points)

Markov processes are families of probability measures $\mathbb{P}_x$. The usual case is that one has a stochastic process X^x on a common probability space for each $x \in S$, and $\mathbb{P}_x$ is the law of X^x (see ex.1). The Markov property is not concerned with pathwise *properties of X^x (such as path regularity), but only with its law and finite-dimensional distributions. These information can be read off from $\mathbb{P}_x$, and the paths $t \mapsto X^x(\omega)$ can be neglected. Since the laws $\mathbb{P}_x$ are measures on $(\Omega, \mathcal{F})$ ($\Omega \subseteq S^{[0,\infty)}$ and see the previous ex. for the def. of $\mathcal{F}$), it is reasonable to choose this pair as the measurable space in the definition of the Markov process. Choosing $X_t = \pi_t$ on Ω, clearly $\mathbb{P}_x \circ X^{-1} = \mathbb{P}_x$ for all x, and we arrive at the canonical model. Note that we have replaced the process X^x for all x by a common process X, and "all information is stored in the laws $\mathbb{P}_x$"!*

(i) Consider the situation of the "canonical model' in Section 3.1. Prove that the shift operator ϑ_t (cf. Def.3.1.5) is $\hat{\mathcal{F}}_t^0/\mathcal{F}$-measurable, and $\vartheta_t^{-1}(\mathcal{F}) = \hat{\mathcal{F}}_t^0$ for all $t \geqslant 0$.

(ii) Prove Lemma 3.1.6 (i).

Problem 3 (Construction of a Poisson process). (1+2*+3*+2 Points)

Poisson processes are key examples of stochastic processes with càdlàg paths. In this and the following exercises, we want to study these processes and see that - to a certain degree - among the càdlàg processes they play a role as central as the Brownian motion in the class of continuous processes. Recall that Brownian motion B can be defined by independence of $B_t - B_s$ from $\sigma(B_r, 0 \leqslant r \leqslant s)$ and the distribution of $B_t - B_s$ being $N(0, t-s)$. Similarly, a Poisson process N (with intensity $\lambda > 0$) *is characterized as follows: $N_t - N_s$ is independent of $\sigma(N_r, 0 \leqslant r \leqslant s)$ and $N_t - N_s$ is Poisson distributed with parameter $\lambda(t-s)$ for all $0 \leqslant s < t$.*

Let $T_i, i \in \mathbb{N}$, be a sequence of independent exponentially distributed random variables with parameter $\lambda > 0$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $\mathbb{P}(T_i \leqslant t) = 1 - \exp(-\lambda t)$, $t \geqslant 0$. Let $S_n := \sum_{i=1}^n T_i$,

$$N_t := \sum_{i=1}^{\infty} \mathbb{1}_{S_i \leqslant t},$$

and $\mathcal{F}_t := \sigma(N_s, 0 \leqslant s \leqslant t)$.

Intuition: Each T_i represents a ticking clock that rings some time after it is started (and is "dead" afterwards), and T_i denotes the duration between its start and the moment it rings. T_{i+1} is started when T_i rings.

(i) Based on this intuition, what is the meaning of S_n and N_t? Draw a "typical' path $t \mapsto N_t(\omega)$.

Moreover, prove the following:

(ii)* For $A \in \sigma(T_1, \ldots, T_n)$ and $s < t$

$$\mathbb{P}(S_{n+1} > t, N_s = n, A) = \exp(-\lambda(t-s))\mathbb{P}(N_s = n, A).$$

Now you may assume the following identify for $A \in \sigma(T_1, \ldots, T_n)$ and $s < t$:

$$\mathbb{P}(N_t - N_s \leqslant k, N_s = n, A) = \exp(-\lambda(t-s)) \sum_{j=0}^{k} \frac{(\lambda(t-s))^j}{j!} \mathbb{P}(N_s = n, A).$$

(iii)* $\mathbb{P}(N_t - N_s = k, A) = \exp(-\lambda(t-s)) \frac{(\lambda(t-s))^k}{k!} \mathbb{P}(A)$ for $s < t$, $k \geqslant 0$ and **all** $\mathbf{A} \in \mathcal{F}_\mathbf{s}$.

Argue why this implies that $N_t - N_s$ is Poisson distributed with parameter $\lambda(t-s)$ and independent of $\mathcal{F}_s$, i.e. N is a Poisson process with intensity λ.

Hint: Show first that for each $A \in \mathcal{F}_s$ there is $A' \in \sigma(T_1, \ldots, T_n)$ such that $A \cap \{N_s = n\} = A' \cap \{N_s = n\}$. Dynkin-system!

(iv) $(N_t - \lambda t)_{t \geqslant 0}$ and $((N_t - \lambda t)^2 - \lambda t)_{t \geqslant 0}$ are $(\mathcal{F}_t)$-martingales, where $\mathcal{F}_t := \sigma(N_s, 0 \leqslant s \leqslant t)$.

Consequently, $N_t - \lambda t$ is an important example of a discontinuous martingale with respect to which stochastic integrals are defined!

Problem 4 (Characterization of Poisson processes). (2 Points)

Note the analogy to Lévy's characterization of Brownian motion!

A process N is called a jump process with jumps of height $+1$, *if $\mathbb{P}$-a.e. path is piecewise constant, and* $\lim_{\varepsilon \to 0}(N_t - N_{t-\varepsilon}) \in \{0, 1\}$ *for all $t \geqslant 0$ $\mathbb{P}-a.s.$*

Let $N = (N_t)_{t \geqslant 0}$ be a monotonically non-decreasing stochastic process with càdlàg paths and $N_0 = 0$ and let $\mathcal{F}_t := \sigma(N_s, 0 \leqslant s \leqslant t)$. Consider the following statements:

(i) N is a right-continuous jump process with jumps of height $+1$ such that $(N_t - \lambda t)_{t \geqslant 0}$ is a $(\mathcal{F}_t)$-martingale.

(ii) N is a Poisson process with intensity $\lambda > 0$ (see the text before the previous exercise for the definition).

Prove 'ii) $\implies$ i)'.

As a matter of fact, (i) and (ii) are even equivalent, which can be proven similarly to Lévy's characterization of Brownian motion.

Problem 5. (2+2 Points)

(i) Let $\lambda > 0$. Does there exist a stochastic process $(X_t)_{t \geqslant 0}$ with continuous paths such that for $0 \leqslant s \leqslant t$ the increments $X_t - X_s$ are independent of $\sigma(X_r, 0 \leqslant r \leqslant s)$ and Poisson distributed with parameter $\lambda(t-s)$?

(ii) Does there exist a stochastic process $(X_t)_{t \geqslant 0}$ with continuous paths such that $X_t - \lambda t$ is a martingale wrt. $(\sigma(X_r, 0 \leqslant r \leqslant t))_{t \geqslant 0}$?

We wish you a merry christmas and a healthy and happy transition to 2023!