Winter term 2022/2023
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Exercises to Stochastic Analysis

Sheet12
Total points: 16
Submission before: Friday, 20.01.2023, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Throughout this exercise sheet, let B be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $(\mathcal{F}_t)_{t\geqslant 0}$ the right-continuous version of its natural filtration. On the same space, W denotes a d-dimensional standard Brownian motion, i.e. $W = (W_1, \dots, W_d)$, where $\{W_i\}_{i\leqslant d}$ are independent one-dimensional standard Brownian motions.

Problem 1 (Markov property of Brownian motion and tail events). (1+2+2 Points)

Let $\mathbb{P}_x$, $x \in \mathbb{R}^d$, be the distribution of $W + x$ on $\big(C(\mathbb{R}_+, \mathbb{R}^d), \sigma(\pi_t, t \geqslant 0)\big)$, where $\pi_t : C(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^d$ are the canonical projections given by $\pi_t(w) := w(t)$, and let $\mathcal{F}^* := \cap_{t\geqslant 0}\sigma(W_s|s \geqslant t)$.

(i) Interpret the σ-algebra $\mathcal{F}^*$.

(ii) Prove that for any $A \in \mathcal{F}^*$ and $x \in \mathbb{R}^d$ one has $\mathbb{P}_x(A) \in \{0, 1\}$.

Hint: Use that $(tW_{\frac{1}{t}} + x)_{t>0}$ is again a d-dim. Brownian motion (why?) and show that one can apply Blumenthal's $0-1-$law.

(iii) Using (ii), prove the following stronger statement: For any $A \in \mathcal{F}^*$, one either has

$$\mathbb{P}_x(A) = 1, \quad \forall x \in \mathbb{R}^d$$

or

$$\mathbb{P}_x(A) = 0, \quad \forall x \in \mathbb{R}^d.$$

Hint: First show $\mathbb{P}_x(A) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2}} \mathbb{P}_y(A)\, dy$ for any $A \in \mathcal{F}^*$ and $x \in \mathbb{R}^d$. For this, you may use without proof the identity $\mathbb{P}_x(A) = \mathbb{E}_x[\mathbb{P}_{W_1}(A)]$ (which can be proven by the Markov property of $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$).

Problem 2 (Girsanov theorem for Brownian motion I). (2 Points)

This is a basic application of Girsanov's theorem for Brownian motion, which should help you to get a feeling for the theorem and its assumptions.

Let $T > 0$. Find a probability measure Q on $\mathcal{F}_T$ which is equivalent to $\mathbb{P}$ such that $(Y_t)_{t\leqslant T}$, $Y_t := -3t + B_t$, is a Brownian motion wrt. Q and $(\mathcal{F}_t)_{t\leqslant T}$.

Problem 3 (Girsanov theorem for Brownian motion II). (2+3 Points)

Set $Y_t := t + B_t$, $t \geqslant 0$.

(i) For each $T > 0$, find a probability measure Q_T on $\mathcal{F}_T$ such that Q_T is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ and such that $(Y_t)_{0 \leqslant t \leqslant T}$ is a Brownian motion wrt. Q_T.

(ii) Prove that there exists a probability measure Q on $\mathcal{F}_\infty := \sigma(\mathcal{F}_t, t \geqslant 0)$ such that $Q_T = Q$ on $\mathcal{F}_T$ for all $T > 0$. Also show

$$\mathbb{P}(\lim_{t\to\infty} Y_t = \infty) = 1,$$

but

$$Q(\lim_{t\to\infty} Y_t = 0) = 0.$$

Why does this not contradict the equivalence of Q_T and $\mathbb{P}$ on each $\mathcal{F}_T$?

Problem 4 (Wald-type inequality). (4 Points)

Later on, in Thm.4.4.16, you will learn that the inequality below is always an equality, provided $\mathbb{E}[\exp(\frac{1}{2}T)] < \infty$, *but this we do not assume in this exercise.*

Let $T : \Omega \to \mathbb{R}_+$ be an $(\mathcal{F}_t)$-stopping time. Prove

$$\mathbb{E}\big[\exp(B_T - \frac{1}{2}T)\big] \leqslant 1,$$

and provide an example for T such that the inequality is strict.