Winter term 2022/2023
Prof. Dr. Michael Röckner and Dr. Marco Rehmeier

Exercises to Stochastic Analysis

Sheet 4
Total points: 14
Submission before: Friday, 11.11.2022, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1. (3 Points)

The quadratic variation process of a process X is related to the sample path structure of X, as the next two exercises show. The statement of the following exercise appears reasonable, if one recalls the intuitive meaning of the quadratic variation $\langle f\rangle_t$ of a function $f : \mathbb{R}_+ \to \mathbb{R}$ as a refined measurement of the total vertical displacement of the graph of f on $[0, t]$, compare the introductory text of Ex. 1 and 2 from Sheet 1.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space with a right-continuous filtration such that $\mathcal{F}_0$ contains all $\mathbb{P}$-zero sets. Let X be a continuous local martingale up to $T > 0$ with pathwise continuous quadratic variation $t \mapsto \langle X(\omega)\rangle_t$ on $[0, T(\omega))$ for a sequence of partitions $(\tau_n)_{n\in\mathbb{N}}$ with the usual conditions. Let $0 \leqslant S_1 \leqslant S_2 < T$ be stopping times. Show

$$\langle X\rangle_{S_1} = \langle X\rangle_{S_2} \implies X \text{ pathwise constant on } [S_1, S_2].$$

The left-hand side of the above implication is understood ω-wise, i.e. $\langle X(\omega)\rangle_{S_1(\omega)} = \langle X(\omega)\rangle_{S_2(\omega)}$ for P-a.e. $\omega \in \Omega$.

Problem 2. (3 Points)

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, T and X be as in the previous problem. Show: for $\mathbb{P}$-a.e. ω, the path $t \mapsto X_t(\omega)$ is either constant on $[0, T(\omega))$ or of unbounded variation.

Problem 3. (3 Points)

Let $d \in \mathbb{N}$ and let $(X^j_t)_{t\geqslant 0}, 1 \leqslant j \leqslant d$, be independent continuous standard Brownian motions on a common probability space. Prove that

$$t \mapsto \frac{1}{\sqrt{d}} \sum_{j=1}^{d} X^j_t$$

is a continuous standard Brownian motion as well.

Problem 4. (5 Points)

Prove Proposition 1.4.29.