Winter term 2022/2023
Prof. Dr. Michael Röckner and Dr. Marco Rehmeier

Exercises to Stochastic Analysis

Sheet 8
Total points: 16
Submission before: Friday, 09.12.2022, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (Stochastic integrals wrt. Brownian motion I). (4 Points)

Stochastic integrals wrt. Brownian motion are a central class of stochastic processes, e.g. within the theory of stochastic differential equations, and it is important to note that one can calculate their basic characteristics, namely its expectation and variance.

Let $T > 0$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filterered probability space as specified at the beginning of Ch.2.3. and let $B = (B_t)_{t\geqslant 0}$ be a standard $(\mathcal{F}_t)$-Brownian motion (i.e. B_t is $\mathcal{F}_t$-measurable and $B_t - B_s$ is independent of $\mathcal{F}_s$). Also, let $H \in L^2(]0,T] \times \Omega, \mathcal{B}(]0,T]) \otimes \mathcal{F}, dt \otimes \mathbb{P})$ be $(\mathcal{F}_t)_{t\leqslant T}$-adapted (here dt denotes Lebesgue measure on $\mathbb{R}_+$).

(i) First, briefly review Ch.2.3. and explain why the stochastic integral $G_t := \int_0^t H_s dB_s$, $t \in [0,T)$, is well-defined. Then, calculate the expectation and variance of G_t.

(ii) Now assume additionally that H is deterministic, i.e. independent of $\omega \in \Omega$. Show that in this case G_t is normally distributed.

Problem 2 (Lemma 2.4.35). (4 Points)

The assertion of this exercise is used for the proof of Cor.2.4.37, which is an important property of stochastic integrals. Compare also to ex.1 on sheet 3, where you proved a "deterministic" version of the latter.

Let $M, N \in \mathcal{M}^2$ and $T : \Omega \to [0,\infty]$ be a stopping time (wrt. the same filtration for which $\mathcal{M}^2$ is considered). Prove that

$$\langle M, N^T \rangle_t = \langle M, N \rangle_{T\wedge t} \quad \forall t \geqslant 0$$

holds $\mathbb{P}-$a.s.

Hint: You may use the Doob-Meyer decomposition for cadlag martingales (cf. beginning of Sect.2.2), even though we proved it only for continuous local martingales in Prop.2.2.8.

Problem 3 (Stochastic integrals wrt. Brownian motion II). (4 Points)

By solving the next two exercises, you will become more experienced with stochastic integrals with Brownian motion as integrators.

Let $B = (B_t)_{t\geqslant 0}$ be a Brownian motion,

$$sgn : \mathbb{R} \to \{-1, 1\}, \quad sgn(x) := \begin{cases} 1 & , x \geqslant 0 \\ -1 & , x < 0, \end{cases}$$

and set $M_t := \int_0^t sgn(B_s)dB_s$, $t \geqslant 0$.

Prove that M is well-defined (as a stochastic integral) and that M is a Brownian motion. Then, compute the covariance of M_t and B_t (not the *covariation*!).

(As the underlying filtration, the most natural choice is the right-continuous and zero-set-augmented version of the filtration generated by B.)

Problem 4 (Stochastic integrals wrt. Brownian motion III). (4 Points)

Let B be as in the previous exercise and consider $Y_t := \int_0^t \mathbb{1}_{\{B_s>0\}} dB_s$, $t \geqslant 0$. Explain why Y is well-defined as a stochastic integral. Then, show that Y is a global (!) martingale (wrt. the filtration mentioned in the previous exercise) and find the unique increasing, adapted cadlag process A such that $Y_t^2 - A_t$ is a martingale.