Winter term 2021/22
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Exercises to Probability Theory I

Sheet 8
Submission before: Friday, 10.12.2021, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 29. (Example 2.4.3) (1+1+2 points)
Let X, Y be independent, $N(0, \sigma^2)$-distributed random varibales and define

$$R := \sqrt{X^2 + Y^2}, \quad \Psi := \arctan \frac{Y}{X}.$$

Show that

(a) R and Ψ are independent.

(b) Ψ is uniformly distributed on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

(c) The distribution of R is absolutely continuous with density

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), & \text{if } r \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

To this end, calculate

$$P\left(\{R \in A\} \cap \{\Psi \in B\}\right), \quad A, B \in \mathcal{B}(\mathbb{R}).$$

Problem 30. (Example 2.4.7 (ii)) (4 points)
Let X_i, $i = 1, 2$ be independent, $N(m_i, \sigma_i^2)$-distributed random variables. Determine the distribution of the sum $X_1 + X_2$.

Problem 31. (Proposition 2.4.7, Example 2.4.8 (iii)) (4 points)
According to Example 2.4.8 (iii), the density of the **Gamma-distribution** $\Gamma_{\alpha,p}$ for $\alpha > 0, p > 0$ is given by

$$g_{\alpha,p}(x) = \begin{cases} \frac{1}{\Gamma(p)} \alpha^p x^{p-1} \exp(-\alpha x) & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

Let X and Y be independent random varibales with distributions Γ_{α,p_X} and Γ_{α,p_Y}, respectively. Show that the sum $X + Y$ is Γ_{α,p_X+p_Y}-distributed.

Problem 32. (Proposition 2.5.2) (0.5 + 0.5 + 0.5 + 1.5 + 1 points)
Prove Proposition 2.5.2 from the lecture, i.e. show that the Fourier transform $\mu \in \mathcal{M}_+^1(\mathbb{R}^n)$ satisfies the following propoerties:

(i) $\hat{\mu}(0) = 1$;

(ii) $|\hat{\mu}| \leqslant 1$;

(iii) $\hat{\mu}(-u) = \overline{\hat{\mu}(u)}$;

(iv) $\hat{\mu}$ is uniformly continuous;

(v) $\hat{\mu}$ is positive definite.