Summer term 2022
Prof. Dr. Michael Röckner and Dr. Andre Schenke

Exercises to Probability Theory II

Sheet 2
Submission before: Friday, 22.04.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

For the whole sheet, let $(\Omega, \mathcal{A}, P, T)$ be dynamical system.

Problem 5. (Almost surely invariant sets)
Let $A \in \mathcal{A}$ be a *P-a.s. invariant set*, i.e. $P(A\Delta(T^{-1}A)) = 0$, where Δ denotes the symmetric difference (disjoint union) of sets. Show that there exists a set $A' \in \mathcal{J}$ such that $P(A\Delta A') = 0$, i.e. the set A' differs from A only by a P-nullset. (6 points)

Hints: You can proceed along the following steps: (i) Convince yourself that the mapping $d_P(A, B) = P(A\Delta B)$, $A, B \in \mathcal{A}$ is a *pseudo-metric*, i.e. it satisfies the axioms of a metric except for the definiteness. (ii) Convince yourself that $|P(A) - P(B)| \leqslant P(A\Delta B)$ and of the following elementary properties of Δ: $A\Delta B = A^c\Delta B^c$ and $\left(\bigcup_\alpha A_\alpha\right) \Delta \left(\bigcup_\alpha B_\alpha\right) \subset \bigcup_\alpha (A_\alpha \Delta B_\alpha)$, and $T^{-1}(A\Delta B) = (T^{-1}A)\Delta(T^{-1}B)$. (iii) Show that for any set A, the set $B := \bigcup_{n=0}^\infty T^{-n}A$ satisfies $T^{-1}B \subset B$. (iv) Show that if B satisfies $T^{-1}B \subset B$, then $C := \bigcap_{k=0}^\infty T^{-k}B$ satisfies $T^{-1}C = C$. Use this to construct A' and to show that $P(A\Delta A') = 0$.

Problem 6. (Rotations on the circle, Example 6.1.5 (ii))
Let $\Omega = S^1$ and $T(\omega) = e^{2\pi i\alpha}\omega$, i.e. we consider the situation of Example 6.1.5 (ii). Let α be irrational. Show that the equidistribution P on Ω is the only stationary distribution for T. (6 points)

Hint: Recalling that $[0, 1[\simeq S^1$ via $[0, 1[\ni x \mapsto e^{2\pi ix} =: \omega \in S^1$, consider the functions $\tilde{f}_n \colon [0, 1[\to S^1$, $\tilde{f}_n(x) = e^{2\pi inx}$, which can be understood on S^1 as $f_n(z) = z^n$. Show that both the equidistribution and any other stationary distribution must agree on these functions and use the Stone–Weierstraß theorem and monotone classes.