Summer term 2022
Prof. Dr. Michael Röckner and Dr. Andre Schenke

Exercises to Probability Theory II

Sheet 3
Submission before: Friday, 29.04.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

For the whole sheet, let $(\Omega, \mathcal{A}, P, T)$ be dynamical system.

Problem 7. (Characterisation of ergodicity)
Show that the following statements are equivalent:

(i) P is ergodic.

(ii) For all $A, B \in \mathcal{A}$ it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P(A \cap T^{-k}B) = P(A)P(B)$$

(i.e. the dynamical system is *weakly mixing*.)

(iii) Any decomposition $\Omega = A \cup A^c$ into invariant sets A and A^c has the property that $P(A) = 0$ or $P(A^c) = 0$. (6 points)

Problem 8. (On the Kac recurrence theorem, Remark 6.3.5 (iii))
Let $A \in \mathcal{A}$.

(i) Show that for P-a.e. $\omega \in \{E_A < \infty\}$ we have $T^k\omega \in A$ for infinitely many $k \in \mathbb{N}$.

(ii) Let $\omega \in \Omega$ with $T^k\omega \in A$ for infinitely many $k \in \mathbb{N}$. Show that

$$\lim_{M\to\infty} \frac{1}{n_M} \sum_{k=0}^{n_M-1} \left((R_A 1_A) \circ T^k\right)(\omega) = 1$$

for an increasing sequence $(n_M)_{M\in\mathbb{N}}$. Conclude that P-a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (R_A 1_A) \circ T^k = 1_{\{E_A<\infty\}}.$$

(6 points)

Hint for (i): This has basically been shown in the lecture already. Use the proof of Proposition 6.2.3 (i).
Hint for (ii): It might help to show first the following: if additionally $\omega \in A$, then one even has for infinitely many $n \in \mathbb{N}$ that $\frac{1}{n}\sum_{k=0}^{n-1}(R_A 1_A) \circ T^k(\omega) = 1$.