Summer term 2022
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Probability Theory II

Sheet 4

Exercise 11. (Example of a Markov Chain)
Let $Y_1, Y_2, \dots$ be independent and identically distributed random variables on a probability space $(\Omega, \mathcal{A}, P)$ with values in $[0, \infty[$. Let $X_0 = x_0 \in [0, \infty[$, and $X_n := X_0 Y_1 \cdots Y_n$, $n \in \mathbb{N}$. Show that $X_0, X_1, \dots$ is a Markov chain. To this end, give the right transition kernel p on $[0, \infty[$ and prove that the distributions $P_{(X_0, \dots, X_n)}$ coincide with $\delta_{x_0} \otimes p \otimes \cdots \otimes p$. (4 points)

Exercise 12. (Equilibrium distribution of a Markov chain)
Let ν be a probability measure on $(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0))$, and $\nu_n := \nu(n), n \in \mathbb{N}_0$. Let infinitely many of the ν_n be nonzero. We define a stochastic kernel on $\mathbb{N}_0$ by

$$p(i,j) := \begin{cases} \nu_n, & i = 0, j = n, \\ 1, & i = n, j = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Show that an equilibrium distribution for the associated Markov chain exists if and only if

$$\sum_{n=0}^{\infty} n \nu_n < \infty.$$

Furthermore, determine this distribution, if it exists and show that it is unique. (4 points)

Exercise 13. (Strong mixing for the Ornstein–Uhlenbeck Markov chain)
We consider again the Ornstein–Uhlenbeck process from Chapter 4: let $S = \mathbb{R}$ with associated Borel-σ-algebra $\mathcal{S} = \mathcal{B}(\mathbb{R})$. Define the transition kernel by

$$p(x, \cdot) := N(\alpha x, \sigma^2), \quad x \in \mathbb{R},$$

where $|\alpha| < 1$ and $\sigma^2 > 0$. Let Ω and $\mathcal{A}$ be defined as in Chapter 7.

(i) Show that for the iterated kernels p^k and all $x \in \mathbb{R}$ it holds that

$$p^k(x, \cdot) \to N\left(0, \frac{\sigma^2}{1-\alpha^2}\right) =: \mu$$

weakly as $k \to \infty$.

(ii) Conclude using (i): for every bounded measurable function $f \colon \mathbb{R} \to \mathbb{R}$ and every $x \in \mathbb{R}$:

$$\mathbb{E}_x[f(X_k)] \to \int f d\mu \quad \text{for } k \to \infty.$$

(it might help to note that pf is continuous and bounded.)

(iii) Show that the associated Markov chain is **strongly mixing**, i.e. for every $A, B \in \mathcal{A}$ it holds that

$$P_\mu(A \cap (\theta^{-k} B)) \to P_\mu(A) P_\mu(B)$$

for $k \to \infty$.

Hint for (iii): Show that statement first for $A \in \mathcal{A}_n$, $n \in \mathbb{N}$ using (ii) and the Markov property. Then extend the statement using the üsualärguments to general $A \in \mathcal{A}$. (4 points)