Summer term 2022
Prof. Dr. Michael Röckner and Dr. Andre Schenke

Exercises to Probability Theory II

Sheet 5
Submission before: Friday, 13.05.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

Problem 12. (Another Markov chain; uniqueness of invariant measures)
Let $S = S^1$ (i.e. the ring of Example 6.1.5 (ii)) with Borel σ-algebra $\mathcal{S}$. Let $T_\alpha(x) := e^{2\pi i\alpha}x$ be the rotation, where $\alpha \in \mathbb{R}$ is arbitrary. We define on S the transition kernel[1]

$$p(x, A) := \frac{1}{\sqrt{2\pi\varepsilon^2}} \int 1_A(T_{\alpha+s}(x)) e^{-\frac{s^2}{2\varepsilon^2}} \, ds.$$

(i) Show that p satisfies the **strong Feller property**, i.e. for $g \colon S^1 \to \mathbb{R}$ bounded and measurable, we have that pg is continuous. (1.5 points)
Hint: The main difficulty lies in understanding the expression $p(x, dy)$ in the definition of $(pg)(x)$. To this end, apply the change of variables formula and the identification of S^1 with $[0, 1[$ from the Lecture Notes.

(ii) Show that p satisfies Liouville's theorem (i.e. the Liouville property). Conclude that there is at most one equilibrium distribution μ and that P_μ is ergodic for every equilibrium distribution μ. (1.5 points)
Hint: Argue by contradiction. Try to follow the proof of Example 7.3.5 (ii) using the strong Feller property and compactness of S^1!

(iii) Furthermore, show that the uniform distribution on S^1 is the invariant measure. (1 point)

Problem 13. (A special process)
Let $(\Omega, \mathcal{A})$, X_n, $n \in \mathbb{N}_0$ and p be defined as in the beginning of Chapter 7. Let $f \colon S \to \mathbb{R}$ be measurable and (for simplicity) bounded. Define

$$Y_n := f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (p-1) f(X_k), \quad n \in \mathbb{N},$$

where $(p-1)f := pf - f$.

(i) Determine $\mathbb{E}_x[Y_n]$ for $x \in S$. (2 points)

(ii) Show that for $m < n$, $\mathbb{E}_x[Y_n \mid \mathcal{A}_m] = Y_m$. (2 points)
Hint: Show the statement first for $m = n - 1$.

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[1] This kernel, for small $\varepsilon > 0$, describes almost exactly the rotation about the angle α we already know, but α is "slightly perturbed", so our jump does not exactly "hit" its target.

Problem 14. (A Non-Markovian process)
Let $S = \{-1, 0, 1\}$, $\mathcal{S} = \mathcal{P}(S)$ and $(\Omega, \mathcal{A})$, X_n, $\mathcal{A}_n$, $n \in \mathbb{N}_0$ be defined as in the beginning of Chapter 7. We define the paths $\omega_1, \ldots, \omega_4 \in \Omega$:

$$\omega_1 := (-1, 0, -1, 0, -1, 0, \ldots), \quad \omega_2 := (0, -1, 0, -1, 0, -1, \ldots),$$
$$\omega_3 := (0, 1, 0, 1, 0, 1, \ldots), \quad \omega_4 := (1, 0, 1, 0, 1, 0, \ldots).$$

Let $P = \frac{1}{4}(\delta_{\omega_1} + \ldots + \delta_{\omega_4})$. We want to show that the process given in this way is *not* Markovian.

(i) Show first that for a Markov chain, Lemma 7.1.1 implies that for any initial distribution μ, any $B \in \mathcal{A}$ and $n \in \mathbb{N}$ the following holds: (1 point)

$$\mathbb{E}_\mu \left[1_B \circ \theta^n \mid \mathcal{A}_n\right] = \mathbb{E}_\mu \left[1_B \circ \theta^n \mid \sigma(X_n)\right], \quad P_\mu - a.s.$$

(ii) Explain in words why the above process cannot be Markovian. (1 point)

(iii) Find a $B \in \mathcal{A}$, $n \in \mathbb{N}$ such that the equality of (i) is not satisfied. (1 point)
Hint: Once you have found B and $n \in \mathbb{N}$, use e.g. Remark 5.1.3 (iv).

(iv) Show that P is stationary for the shift θ. (1 point)