REMARKS ON JORDAN ALGEBRAS (DIM 9, DEG 3), CUBIC SURFACES, AND DEL PEZZO SURFACES (DEG 6)

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`The purpose of these notes is to record some formulas and remarks. Everything deserves a check.`

1. A construction of Jordan algebras (deg 3, dim 9)

The base field F has char $\neq 3$.

Let us ask for a functorial construction

$$\mathbb{B}\colon (L,K)\mapsto B(L,K)$$

which associates to an ordered pair of separable degree 3 extensions a 9-dimensional Jordan algebra of degree 3. Consider the split cases $L=K=F\oplus F\oplus F$ and let $\tilde{B}=B(L,K)$. The functoriality of $\mathbb{B}$ then yields a homomorphism

$$\Psi_{\mathbb{B}}\colon S_3\times S_3=\operatorname{Aut}(L)\times\operatorname{Aut}(K)\to\operatorname{Aut}(\tilde{B}).$$

Clearly $\mathbb{B}$ is determined by $\tilde{B}$ and $\Psi_{\mathbb{B}}$.

Here is an example: Let

$$Z=F[x]/(x^2+x+1),\quad \sigma\colon Z\to Z,\ \sigma(x)=x^2$$

and let

$$A=M_3(Z),\quad \tau\colon A\to A,\ \tau(a)=\sigma(a)^t.$$

Then (A,τ) is an algebra with involution of second kind. Put

$$\tilde{B}=A^\tau.$$

There are the (split) subalgebras

$$L=\begin{pmatrix}F&&\\&F&\\&&F\end{pmatrix}$$

and

$$K=\frac{1}{3}(1+\beta+\beta^2)F\oplus\frac{1}{3}(1+x\beta+x^2\beta^2)F\oplus\frac{1}{3}(1+x^2\beta+x\beta^2)F$$

where

$$\beta=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

Let G be the subgroup of $\operatorname{Aut}(\tilde{B})$ (as Jordan algebra) leaving the subalgebras L and K invariant. The natural homomorphism $G\to\operatorname{Aut}(L)\times\operatorname{Aut}(K)$ turns out to be an isomorphism.

Date: December 17, 1996, minor changes on July 29, 2003.

Lemma 1. *Let $\mathbb{B}$ be as above and suppose that $\Psi_{\mathbb{B}}$ is injective. Then $\mathbb{B}$ is isomorphic to the functor given by the example.* □

Lemma 2. *The last lemma can be extended to* ***unordered*** *pairs of cubic extensions, that is to say to a cubic extension over a quadratic extension. The underlying group is then $(S_3 \times S_3) \rtimes \mathbb{Z}/2$.*

One would like to see a rational description of $B(L,K)$ for arbitrary K and L. Here it is for $\operatorname{char} F \neq 2, 3$:

Put

$$B = B(L,K) = L \otimes K.$$

Let $L_0 \subset L$, $K_0 \subset K$ be the subspaces of trace 0 elements. Define a Jordan product $\bullet$ on B by the following formulas with $\alpha \in L_0$ and $\beta \in K_0$.

$$\begin{aligned}
(1 \otimes 1)^2 &= 1 \otimes 1 \\
(\alpha \otimes 1)^2 &= \alpha^2 \otimes 1 \\
(1 \otimes \beta)^2 &= 1 \otimes \beta^2 \\
(\alpha \otimes \beta) \bullet (\alpha \otimes 1) &= \frac{1}{4}\big(\operatorname{trace}(\alpha^2) - 2\alpha^2\big) \otimes \beta \\
(\alpha \otimes \beta) \bullet (1 \otimes \beta)^2 &= \frac{1}{4}\alpha \otimes \big(\operatorname{trace}(\beta^2) - 2\beta^2\big) \\
(\alpha \otimes \beta)^2 &= -\frac{1}{2}\alpha^2 \otimes \beta^2 + \frac{1}{8}\big(\operatorname{trace}(\alpha^2) \otimes \beta^2 + \alpha^2 \otimes \operatorname{trace}(\beta^2)\big)
\end{aligned}$$

(One could clean these formulas a bit, by using the adjoint $\alpha^{\#} = \alpha - \frac{1}{2}\operatorname{trace}(\alpha^2)$.)

If L is cyclic and K is a Kummer extension, then $B(L,K) = A^+$ where A is the usual crossed product.

From this it is not difficult to see that the H^2-mod3 invariant of B is the cup product of the H^1-mod3 invariants of L and K (all of these invariants are defined only up to sign).

Also concerning the "mod2-part" of $B(L,K)$ there is a sort of product.

Lemma 3.

$$\operatorname{trace}_{B(L,K)/F} \simeq 3 \operatorname{trace}_{L/F} \otimes \operatorname{trace}_{K/F}$$

Proof. This follows from the above formulas. There might be a better proof. □

Note that the trace form of a cubic extension with discriminant δ is $\langle 1, 2, 2\delta\rangle$. The associated 2-fold Pfister form is $\langle\langle -2, -\delta\rangle\rangle$.

Lemma 4. *Let δ_L, δ_K be the discriminants of L and K, respectively. Then the $H^3(\mathbb{Z}/2)$-invariant of B is*

$$(-2, -\delta_L, -\delta_K) \in H^3(F, \mathbb{Z}/2).$$

This follows from Lemma 3. Before I was aware of Lemma 3 I used the following arguments.

Proof. To check this one looks at our split example $\tilde{B}$ (which has $H^3(\mathbb{Z}/2)$-invariant $(-3,-1,-1)$) and restricts to a $\mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup of G.

To be specific, introduce the following coordinates

$$\tilde{B} = \left\{ \begin{pmatrix} a & \bar{u} & \bar{w} \\ u & b & \bar{v} \\ w & v & c \end{pmatrix} \;\middle|\; a, b, c \in F,\ u, v, w \in Z \right\}$$

in $\tilde{B}B$ (with $^- = \sigma$). Moreover let

$$\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\epsilon, \theta \colon B \to B, \quad \epsilon = \mathrm{ad}_\alpha,\, \theta = {}^t.$$

Then $\epsilon\theta$ is an element of order 2 in $\mathrm{Aut}(L) \subset G$ and θ is an element of order 2 in $\mathrm{Aut}(K) \subset G$.

The trace form of B has the diagonal form

$$\mathrm{trace}_B = \langle 1, 1\rangle \perp \langle 1\rangle \perp 2\langle 1, 3\rangle \perp 2\langle 1, 3\rangle \otimes \langle 1, 1\rangle$$

with respect to the coordinates

$$\big((a, b), c, u, (v, w)\big) \in F^2 \oplus F \oplus Z \oplus Z^2 \text{ and } Z = F \oplus \sqrt{-3}F.$$

After twisting one has

$$\mathrm{trace}_B = 2\langle 1, \delta_L\rangle \perp \langle 1\rangle \perp 2\langle 1, 3\delta_K\rangle \perp 2\langle 1, 3\delta_L\delta_K\rangle \otimes 2\langle 1, \delta_L\rangle.$$

This gives

$$\begin{aligned} \mathrm{trace}_B &= \langle 1, 1, 1\rangle \perp 2\langle\langle -3\delta_K\delta_L\rangle\rangle \otimes \langle 2, 2\delta_L, \delta_L\rangle \\ &= \langle 1, 1, 1\rangle \perp 2\langle\langle -3\delta_K\delta_L\rangle\rangle \otimes \langle\langle -2, -\delta_L\rangle\rangle' \end{aligned}$$

Finally note $\langle\langle -3\delta_K\delta_L, -2, -\delta_L\rangle\rangle = \langle\langle -2, -\delta_L, -\delta_K\rangle\rangle$. □

2. Twisting sums of four cubes

Consider the cubic form

$$\Phi \colon L_0 \otimes K_0 \to F, \quad \Phi = (\mathrm{N}_{L/F} | L_0) \otimes (\mathrm{N}_{K/F} | K_0).$$

It turns out that Φ is also given by the norm form of $B(L, K)$:

$$\Phi = \mathrm{N}_{B(L,K)/F} |(L_0 \otimes K_0).$$

Let

$$C = C(L, K) = \{\Phi = 0\} \subset \mathbb{P}(L_0 \otimes K_0)$$

be the associated cubic surface.

Suppose that $K = F[x]/(x^3 - b)$. Then

$$\Phi(\alpha \otimes x + \alpha' \otimes x^2) = \mathrm{N}_{L/F}(\alpha)b + \mathrm{N}_{L/F}(\alpha')b^2$$

In particular, for $L = F[x]/(x^3 - a)$ this gives the diagonal cubic form

$$\Phi = ab\langle 1, a, b, ab\rangle$$

If $b = 1$ and $L = F \oplus F \oplus F$, then Φ has the form

$$uv(u + v) + st(s + t)$$

Lemma 5. *The surface $C(L, K)$ has a rational point if and only if the H^2-mod3-invariant of $B(L, K)$ is trivial (i.e., $B(L, K)$ has zero divisors).*

Proof. A cubic form is isotropic if and only it is isotropic over a quadratic extension. We may therefore assume that $L = F[x]/(x^3 - a)$ and $K = F[x]/(x^3 - b)$. In this case the algebras is (L, b) and the cubic form is

$$\Phi = ab\langle 1, a, b, ab\rangle$$

which is isotropic if and only if the equation

$$b = N_{L/F}\big(\frac{u + vx}{w + tx}\big)$$

has a solution. But any element in $L = F[x]/(x^3 - a)$ is of the form

$$\frac{u + vx}{w + tx}.$$

So the cubic form is isotropic if and only if $b \in N_{L/F}(L^\times)$, i.e., the algebra has zero divisors. □

Lemma 6. *The surface $C(L, K)$ has a rational point if and only if it is rational.*

Proof. If L is split, $L = e_1F \oplus e_2F \oplus e_3F$, then $(e_i - e_j) \otimes K_0$ give 3 disjoint lines in the cubic surface. Hence in general there is a *set* of three lines in C defined over F. As Colliot-Thelene informed me, in this case C is rational if and only if C has a rational point. The reference is:

[1] Swinnerton-Dyer, H. P. F. , The birationality of cubic surfaces over a given field. Michigan Math. J. 17 1970 289–295.

This paper is not available to me till now. □

Corollary 7. *The stable birational equivalence class of $C(L, K)$ depends only on the H^2-mod3-invariant of $B(L, K)$ (defined up to sign).*

Proof. Indeed, if C is rational over $F(C')$, and vice versa, then $C \times C'$ is stable birational to C and C'. □

Question 1. *What about birational equivalence?*

3. A construction of (all) del Pezzo surface of degree 6

A del Pezzo surface of degree 6 is a form of $\mathbb{P}^2$ blown up in 3 points in general position. They may be constructed as follows. Let B be a Jordan algebra (of the type as above) and let $L \subset B$ be a separable associative subalgebra of degree 3. Define

$$Y(B, L) = \{\, [b] \in \mathbb{P}(B) \mid \{b, L_0, b\} = 0 \,\}$$

Here $\{b, \lambda, b\}$ denotes the Jordan triple product.

In the split situation $B = M_3^+$ and $L = \Delta$ (diagonal matrices) this gives

$$Y(M_3, \Delta) = \{\, [X] \in \mathbb{P}(M_3) \mid X\Delta_0X = 0 \,\}$$

Any matrix X with $[X] \in Y(M_3, \Delta)$ has rank 1, so that $X = v \cdot w^t$ for some 3-vectors v, w. This gives an identification

$$Y(M_3, \Delta) = \big\{\, \big([v], [w]\big) \in \mathbb{P}^2 \times \mathbb{P}^2 \;\big|\; v_1w_1 = v_2w_2 = v_3w_3 \,\big\}$$

In other words, $Y(M_3, \Delta)$ is the "quadratic correspondence of $\mathbb{P}^2$", as described in Hartshorne's book.

Let's discuss the corresponding automorphism groups.

On M_3 the group $\mathrm{PGL}_3 \rtimes \mathbb{Z}/2$ acts by conjugation and transposition. The subgroup leaving Δ_0 invariant is of the form

$$H = T^2 \rtimes (S_3 \rtimes Z/2)$$

with T^2 a 2-dimensional torus (=projective diagonal matrices).

So H acts on $Y = Y(M_3, \Delta)$, and one finds that $H = \mathrm{Aut}(Y)$, since on such a del Pezzo surface the hexagon consisting of the 6 exceptional lines is left invariant under all automorphisms of Y—and so $\mathrm{Aut}(Y)$ consists of the automorphisms of the toric structure on Y.

Corollary 8. *There is a bijection between (isomorphism classes of) pairs (B, L) and del Pezzo surfaces of degree 6.*

Question 2. *What about the (stable) birational classification of the Y's?*

The stable question this is not difficult to answer by using the toric structure. There are classifying H^2-mod3 and H^2-mod2 invariants.

Let (A, τ) be an algebra with involution of second kind with center Z such that $B = A^\tau$. Then the imbedding of Y to $\mathbb{P}^2 \times \mathbb{P}^2$ twists to an embedding

$$Y(B, L) \subset R_{Z/F}\big(\mathrm{SB}(A)\big)$$

If Z is split, i.e., $B = A'^{+}$ for a central simple algebra A', then

$$Y(B, L) \subset \mathrm{SB}(A') \times \mathrm{SB}(A'^{\mathrm{op}})$$

The projection to any of the factors is the blow down of 3 lines $R_{L/F}(\mathbb{P}^1) \subset Y(B, L)$.

Let still $B = A^\tau$ and let

$$S = \{\, [b] \in \mathbb{P}(B) \mid \operatorname{rank} b = 1,\ b^2 = 0 \,\}$$

If $B = {M_3}^{+}$, then

$$S = \Big\{\, \big([v], [w]\big) \in \mathbb{P}^2 \times \mathbb{P}^2 \;\Big|\; v_1 w_1 + v_2 w_2 + v_3 w_3 = 0 \,\Big\}$$

The intersection $S \cap Y$ is exactly the hexagon on Y.

4. Blowing down the cubic surface

We return to the case $L = F \oplus F \oplus F$ and $K = F[x]/(x^3 - 1)$. Note that then $B(L, K) = M_3^{+}$. Then the cubic surface is given by

$$C = \{uv(u + v) + st(s + t) = 0\}$$

Consider the map

$$C \to \mathbb{P}^2 \times \mathbb{P}^2, \quad (u, v, s, t) \mapsto \big([-vs, st, uv], [-ut, uv, st]\big).$$

If I am not mistaken, this map is everywhere defined, maps to Y and the map $C \to Y$ is a blow down of 3 lines. The map $C \to \mathbb{P}^2$ (given by any of the two projections) should be the blow up in the 6 points

$$[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1], [1, \zeta, \zeta^2], [1, \zeta^2, \zeta]$$

with ζ a primitive 3^{rd} of unity.

Question 3. *How to describe these blow downs in the non split situation?*