THE VARIETY OF ANGLES

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`preliminary version`

Introduction

This text grew out of an attempt to understand the derived triangle construction, see Section 5. We consider the variety of (algebraic) angles of an Euclidean quadrangle and an action of S_6 on it. This action is induced from an action on a 5-dimensional torus.

1. An action of S_6

Let T be the torus with the 6 characters

$$u_{ij} = u_{ji} \colon T \to \mathbf{G}_\mathrm{m}, \quad 0 \le i < j \le 3$$

as a base of $\mathrm{Hom}(T, \mathbf{G}_\mathrm{m})$. Let $U = T/d(\mathbf{G}_\mathrm{m})$ where

$$d \colon \mathbf{G}_\mathrm{m} \to T, \quad d^*(u_{ij}) = \mathrm{id}$$

is the diagonal.

In the following $ijk\ell$ stands for a permutation of 0123. The lattice $\mathrm{Hom}(U, \mathbf{G}_\mathrm{m})$ is generated by the elements

$$\alpha_{ijk} = u_{ij}^{-1} u_{jk}$$

One has

$$\begin{aligned} \alpha_{ij0} &= \alpha_{0ji}^{-1} \\ \alpha_{i0j} &= \alpha_{0ij}\alpha_{ij0} \\ \alpha_{ijk} &= \alpha_{0jk}\alpha_{ij0} \end{aligned}$$

It follows that $\mathrm{Hom}(U, \mathbf{G}_\mathrm{m})$ is generated by the 6 elements

$$\alpha_{0ij}, \quad 1 \le i, j \le 3$$

They are subject to the relation

$$\alpha_{012}\alpha_{023}\alpha_{031} = \alpha_{013}\alpha_{032}\alpha_{021} \tag{1}$$

Since $\dim U = 5$, this gives a complete presentation of U.

We consider the following automorphisms of U. Here the first three types are defined on T and factor to U.

- The elements $g \in S_4$ which act by permutations of the indices 0123.
- The inverse ι with $\iota(u_{ij}) = u_{ij}^{-1}$. One has

$$\iota(\alpha_{ijk}) = \alpha_{kji}$$

- The involution τ with $\tau(u_{ij}) = u_{k\ell}$. One has

$$\tau(\alpha_{ijk}) = \alpha_{k\ell i}$$

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- The automorphism Φ of U defined on the generators α_{0ij} by

$$\Phi(\alpha_{0ij}) = \alpha_{0jk}$$

  It is easy to see that this assignment preserves the relation (1). It is clear that $\Phi^3 = 1$. In Section 5 we will describe a geometric interpretation of Φ.

Let G be the subgroup of $\mathrm{Aut}(U)$ generated by these elements.

One considers the sets of characters

$$\begin{aligned} M_a &= \{\alpha_{ijk}\}, \qquad \alpha_{ijk} = u_{ij}^{-1}u_{jk} \\ M_c &= \{\gamma_{ijk\ell}\}, \qquad \gamma_{ijk\ell} = u_{ij}^{-1}u_{k\ell}^{-1}u_{i\ell}u_{jk} \\ M &= M_a \cup M_c \end{aligned}$$

One has $|M_a| = 24$ and $|M_c| = 6$ and therefore $|M| = 30$. (Later the sets M_a, M_c will be interpreted in terms of angles and cross ratios in a quadrangle, respectively.)

It is clear that the automorphisms ι, τ and $g \in S_4$ preserve M_a and M_c. The element Φ preserves M. Indeed, one has

$$\begin{aligned} \Phi(\alpha_{0ij}) &= \alpha_{0jk} \\ \Phi(\alpha_{ij0}) &= \alpha_{ki0} \\ \Phi(\alpha_{ijk}) &= \Phi(\alpha_{0jk}\alpha_{ij0}) = \alpha_{0ki}\alpha_{ki0} = \alpha_{k0i} \\ \Phi(\alpha_{i0j}) &= \Phi(\alpha_{0ij}\alpha_{ij0}) = \alpha_{0jk}\alpha_{ki0} = \gamma_{0jki} \\ \Phi(\gamma_{0ijk}) &= \Phi(\alpha_{0ij}\alpha_{jk0}) = \alpha_{0jk}\alpha_{ij0} = \alpha_{ijk} \end{aligned}$$

In the following we denote by $S_3 \subset S_4$ the isotropy group of the index 0.

Lemma 1. *One has the following relations in* $\mathrm{Aut}(U)$*:*

- *The element ι lies in the center of* $\mathrm{Aut}(U)$*. One has $\iota^2 = 1$.*
- *The element τ commutes with S_4. One has $\tau^2 = 1$.*
- *The element Φ commutes with S_3. One has $\Phi^3 = 1$.*
- *One has $(\tau\Phi)^2 = 1$.*
- *Let $h = (01) \in S_4$ (the transposition). One has $(h\Phi^{-1}\tau\Phi)^3 = \iota$.*

Proof. To see $(\tau\Phi)^2 = 1$ one observes

$$\begin{aligned} \tau\Phi(\alpha_{0ij}) &= \tau(\alpha_{0jk}) = \alpha_{ki0} \\ \tau\Phi(\alpha_{ki0}) &= \tau(\alpha_{jk0}) = \alpha_{0ij} \end{aligned}$$

with ijk a permutation of 123

To see $(h\Phi^{-1}\tau\Phi)^3 = \iota$ one observes first (with ij a permutation of 23)

$$\begin{aligned} h\Phi^{-1}\tau\Phi(\alpha_{01i}) &= h\Phi^{-1}(\alpha_{j10}) = h(\alpha_{1i0}) = \alpha_{0i1} \\ h\Phi^{-1}\tau\Phi(\alpha_{0i1}) &= h\Phi^{-1}(\alpha_{ji0}) = h(\alpha_{i10}) = \alpha_{i01} \end{aligned}$$

Next note that τ leaves every $\gamma_{ijk\ell}$ fixed and therefore $\Phi^{-1}\tau\Phi$ leaves all elements α_{i0j} fixed. Hence

$$h\Phi^{-1}\tau\Phi(\alpha_{i01}) = h(\alpha_{i01}) = \alpha_{i10} = \alpha_{01i}^{-1}$$

Moreover

$$\begin{aligned}
h\Phi^{-1}\tau\Phi(\alpha_{0ij}) &= h\Phi^{-1}(\alpha_{1i0}) = h(\alpha_{ij0}) = \alpha_{ij1}\\
h\Phi^{-1}\tau\Phi(\alpha_{ij1}) &= h\Phi^{-1}\tau(\alpha_{10i}) = h\Phi^{-1}(\alpha_{ij1}) = h\Phi^{-1}(\alpha_{ij0}\alpha_{0j1})\\
&= h(\alpha_{j10}\alpha_{0ij}) = \alpha_{j01}\alpha_{1ij}\\
h\Phi^{-1}\tau\Phi(\alpha_{j01}\alpha_{1ij}) &= h\Phi^{-1}\tau(\alpha_{01i}\alpha_{ij0}\alpha_{j01}) = h\Phi^{-1}\tau(\alpha_{ji1})\\
&= h\Phi^{-1}(\alpha_{10j}) = h(\alpha_{ji1}) = \alpha_{ji0} = \alpha_{0ij}^{-1}
\end{aligned}$$

(There might be a simpler way to present this using the relation with the root system A_5, see below). □

Corollary 1. *The group G is isomorphic to $S_6 \times \mathbf{Z}/2$, with S_6 generated by S_4, $\tau\iota$ and Φ and with $\mathbf{Z}/2$ generated by ι.*

Proof. This follows from Lemma 1 and, for instance, the known presentation of S_6 as the Weyl group of A_5. □

It seems that the action of S_6 on M is the standard action of S_6 on the 2-element subsets of a 6-element set followed by a non-inner automorphism of S_6.

2. Relation with the root system A_5

Consider the set of characters on U:

$$A = \left\{\frac{u_{ij}}{u_{kl}}\right\} \cup \left\{\left(\frac{u_{ij}u_{kl}}{u_{ik}u_{il}}\right)^{\pm 1}\right\}$$

Here $ijk\ell$ runs through the permutations of 0123.

The set A is also invariant under the group G, it has $6 + 24 = 30$ elements and is in fact the root system A_5. A simple base of positive roots in the order of the Dynkin diagram is given by

$$\frac{u_{01}}{u_{23}},\quad \frac{u_{03}u_{12}}{u_{01}u_{02}},\quad \frac{u_{02}}{u_{13}},\quad \frac{u_{01}u_{23}}{u_{02}u_{03}},\quad \frac{u_{03}}{u_{12}}$$

The lattice generated by A has index 3 in the lattice generated by M. Thus U is the torus of $\mathrm{SL}(6)/\mu_2$.

Let us describe $U \subset \mathrm{SL}(6)/\mu_2$ directly. Let

$$X = \ker(\mathbf{G}_\mathrm{m}^6 \to \mathbf{G}_\mathrm{m}) \subset \mathrm{SL}(6)$$

be the standard diagonal torus in SL(6) with coordinates $(z_1, z_2, z_3, z_4, z_5, z_6)$ subject to $z_1z_2z_3z_4z_5z_6 = 1$, $z_i \neq 0$. One finds that

$$\begin{gathered}
\rho\colon X \to U\\
(z_1, z_2, z_3, z_4, z_5, z_6) \mapsto\\
[u_{01}, u_{02}, u_{03}, u_{23}, u_{31}, u_{12}] = [z_2, z_1z_2z_4, z_5^{-1}, z_1, z_1z_2z_3, z_6^{-1}]
\end{gathered}$$

is the corresponding map of tori. It is equivariant with respect to $S_6 \times \mathbf{Z}/2 = G$. (If I had known this earlier, I could have saved a lot of typing in the proof of Lemma 1.)

3. A cubic form in 6 variables

Let F be a ring, let K be a flat F-algebra of rank 3 and let L be a flat K-algebra of rank 2. Then $\Lambda^3(K/F)$ is an invertible F-module. We have the cubic map

$$f = f_{L/K/F}\colon L \to \Lambda^3(K/F)$$
$$f(x) = 1 \wedge T_{L/K}(x) \wedge N_{L/K}(x)$$

where $T_{L/K}$, $N_{L/K}\colon L \to K$ is the trace resp. norm.

Lemma 2.
$$f(x^{-1}) = \frac{-f(x)}{N_{L/F}(x)}$$

Proof. This can be seen from the explicit formulas below. □

For our later purposes it suffices to consider case $K = F \times F \times F$ and $L = K \times K$. In this case we get a cubic form f in 6 variables, well defined up to a sign (depending on the choice of a generator of $\Lambda^3(K/F)$). It is obviously invariant under

$$\operatorname{Aut}(L/K/F) = \mathbf{Z}/2 \times S_4 = \mathbf{Z}/2 \wr S_3$$

One finds

$$f(x_1, x_2, x_3, y_1, y_2, y_3) = \sum_{i=1,2,3} x_i x_{i+1}(y_i - y_{i+1}) + y_i y_{i+1}(x_i - x_{i+1})$$

or

$$f(u_{01}, u_{02}, u_{03}, u_{23}, u_{31}, u_{12}) = -\sum_{g \in S_4/\{1,(01)(23)\}} \operatorname{sgn}(g) g(u_{01} u_{02} u_{13})$$

If we compose this with the map ρ, then a direct calculation gives the following amazing identity:

$$f(z_2, z_1 z_2 z_4, z_5^{-1}, z_1, z_1 z_2 z_3, z_6^{-1}) =$$
$$-\frac{z_1 z_2}{z_5 z_6}\left(z_1 + z_2 + z_3 + z_4 + z_5 + z_6 - z_1^{-1} - z_2^{-1} - z_3^{-1} - z_4^{-1} - z_5^{-1} - z_6^{-1}\right)$$

Here $(z_1, z_2, z_3, z_4, z_5, z_6)$ are subject to $z_1 z_2 z_3 z_4 z_5 z_6 = 1$, $z_i \neq 0$.

This shows that f has a $(S_6 \times \mathbf{Z}/2)$-symmetry. This will find an explanation later on.

Here is another way of writing the map ρ: The collection of the 12 elements $z_r^{\pm 1}$ is the collection of the 12 elements

$$\frac{u_{ij} u_{ik} u_{j\ell}}{t}$$

where t is subject to

$$t^2 = \prod_{ij} u_{ij} \quad \text{(6 factors)}$$

Since $z_r \in S^1$, one has $z_r^{-1} = \overline{z}_r$. Hence, if we write

$$z_r = \cos\varphi_r + i \sin\varphi_r$$

then

$$z_r - z_r^{-1} = 2i \sin\varphi_r$$

and we get the following fundamental relation:

$$\sum_{r=1}^{6} \sin\varphi_r = 0$$

Here the φ_r are certain angles (well defined up to $2\pi\mathbf{Z}$ and a common constant in $\pi\mathbf{Z}$) associated to an Euclidean quadrangle. It would be desirable to have a geometric interpretation of this relation.

4. The fundamental relation for angles in a quadrangle

Let $x_0, x_1, x_2, x_3 \in \mathbf{C}$ be four distinct points in the Euclidean plane. Define the quantities ($i \neq j$)

$$u_{ij}(x) = u_{ji}(x) = \frac{x_i - x_j}{\bar{x}_i - \bar{x}_j}$$

where $\bar{x}$ is the complex conjugate of x.

We call $u_{ij}(x)$ the "algebraic side" given by x_i, x_j. We wish to understand which values $u_{ij}(x)$ are possible. Note that the $u_{ij}(x)$ are 6 complex numbers of norm 1. They are also invariant under translations of the quadrangle and under multiplication of the quadrangle with a real number. Therefore the (real) dimension of the space of possible u_{ij} is at most $8 - 2 - 1 = 5$. It is in fact 5 and here is the corresponding basic equation in $(\mathbf{S}^1)^6$:

Lemma 3. *Let $L = \mathbf{C}^6$ with coordinates u_{ij}, let τ be the involution of L with $\tau(u_{ij}) = u_{k\ell}$ and let $K = L^\tau$ be the invariant subalgebra. Then for every $x \in \mathbf{C}^4$ outside the general diagonal one has*

$$f_{L/K/F}\big(u_{ij}(x)\big) = 0$$

Proof. Omitted. □

Lemma 4. *Given 6 (generic) complex numbers $u_{ij} \in \mathbf{S}^1$ of norm 1 with*

$$f_{L/K/F}\big(u_{ij}\big) = 0$$

Then there exists a quadrangle $x \in \mathbf{C}^4$ with $u_{ij} = u_{ij}(x)$.

Proof. Omitted. □

These observations translate to similar considerations for the "algebraic angles"

$$\alpha_{ijk}(x) = u_{ij}(x)^{-1} u_{jk}(x)$$

Corollary 2. *Let $U' = (\mathbf{S}^1)^6/\mathbf{S}^1$ be the quotient by the diagonal. Let further*

$$V_4 \subset U'$$

be the image of

$$\{f = 0\} \cap (\mathbf{S}^1)^6$$

Then for a generic quadrangle $x \in \mathbf{C}^4$ the elements $\alpha_{ijk}(x)$ define a point $\alpha(x) \in V_4$ and x is uniquely determined by $\alpha(x)$ up to Euclidean similarities.

In other words, V_4 is the variety of algebraic angles of an Euclidean quadrangle. It is also the moduli space of (general) Euclidean quadrangles with ordered vertices up to Euclidean similarities.

Or: An Euclidean quadrangle has 24 (algebraic) angles. The relations among them are given by the straightforward multiplicative relations and the form f.

For instance, suppose that for a quadrangle $ABCD$ the vertices A, B, and the angles CAB, ABC, DAB, ABD are known. Then C, D are known as well. Using the form f one can express the angle ACD in terms of the given 4 angles. All the other angles of $ABCD$ are then easy to compute.

To be more algebraic, let $W_4 \to \operatorname{Spec} \mathbf{Z}$ be the subvariety of the torus U from Section 1 given by $f = 0$. The variety W_4 is invariant under the involution ι. Now V_4 is the set of real points of the twisted form of W_4 with $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ acting via ι.

5. The derived triangle

Cf. [1, p. 16, Section 1.5, Exercise 12]

For a generic quadrangle $x = (x_0, x_1, x_2, x_3) \in \mathbf{C}^4$ define

$$\phi(x) = y = (y_0, y_1, y_2, y_3) \in \mathbf{C}^4$$

as follows: $y_0 = x_0$ and y_i is the reflection of x_0 at the side $x_j x_k$ where ijk stands for a permutation of 123.

Then the angles $x_0 x_i x_j$ and $y_0 y_j y_k$ are equal (this is Coxeter's exercise and left to the reader). It follows that the quadrangles x and $\phi^3(x)$ are similar.

Therefore ϕ defines an automorphism (possibly rational) of V_4. It turns out that ϕ is the restriction of Φ to V_4. The considerations of Section 1 yield therefore an action of $G = \mathbf{Z}/2 \times S_6$ on V_4.

References

[1] H. S. M. Coxeter, *Introduction to geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989, Reprint of the 1969 edition, 90a:51001.

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