Norm Residue Homomorphism

Markus Rost
Universität Bielefeld
Postfach 100131
33501 Bielefeld
Germany

Mathematische Arbeitstagung June 22-28, 2007
Max-Planck-Institut für Mathematik, Bonn, Germany

In the introductory part of the talk we formulated the norm residue homomorphism

$$\begin{aligned} h_{(n,p)}\colon K_n^M k/p &\to H^n_{\mathrm{et}}(k,\mu_p^{\otimes n}) \\ \{a_1,\dots,a_n\} &\mapsto (a_1)\cup\dots\cup(a_n) \end{aligned}$$

from Milnor's K-groups to Galois cohomology. The *generalized Milnor conjecture* (aka Milnor-Bloch-Kato conjecture, aka ...) states the bijectivity of this map for any prime p, any n, and any field k with $\operatorname{char} k \neq p$.

Further we discussed norm varieties and their relation to characteristic numbers and cobordism. See [2].

Finally we considered what we call the "basic correspondence of a splitting variety". It is obtained by the following diagram, which is essentially due to Voevodsky:

$$u \in \ker\big[H^n_{\mathrm{et}}(k,\mu_p^{\otimes(n-1)}) \longrightarrow H^n_{\mathrm{et}}(k(X),\mu_p^{\otimes(n-1)})\big]$$

$$\simeq \Big\uparrow j$$

$$H^{n,n-1}_{\mathcal{M}}(\mathcal{X},\mathbf{Z}/p)$$

$$\Big\downarrow \beta\circ Q_1\circ\cdots\circ Q_{n-2}$$

$$\mu \in H^{2b+1,b}_{\mathcal{M}}(\mathcal{X},\mathbf{Z})$$

$$\Big\downarrow \mathrm{proj}$$

$$\text{homology of }\big[\,\mathrm{CH}^b(X)\to\mathrm{CH}^b(X^2)\to\mathrm{CH}^b(X^3)\big]$$

Here

$$u = (a_1) \cup \cdots \cup (a_n) \in H^n_{\text{et}}(k, \mu_p^{\otimes n})$$

is a symbol (we assume $\mu_p \subset k$) and X is a smooth variety over k over which the symbol is split, i.e., $u_{k(X)} = 0$.

Furthermore, $\mathcal{X}$ is the simplicial scheme

$$\mathcal{X} : X \leftleftarrows X^2 \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, X^3 \cdots$$

The map j relating motivic cohomology of $\mathcal{X}$ to Galois cohomology is an isomorphism if one assumes the generalized Milnor conjecture in weight $n-1$. For this one uses results from [5].

Then one applies the Milnor operations Q_i in motivic cohomology (these can be expressed in terms of the motivic Steenrod operations similarly as in topology) and the Bockstein homomorphism β.

One obtains the class

$$\mu \in H^{2b+1,b}_{\mathcal{M}}(\mathcal{X}, \mathbf{Z}), \qquad b = \frac{p^{n-1}-1}{p-1}$$

which plays an essential role in Voevodsky's work on the generalized Milnor conjecture, cf. [6]. If X is a norm variety for the symbol u, Voevodsky uses the class μ to show that X is a generic (up to extensions of degree prime to p) splitting variety for u and to split off from X a certain motive, the so-called generalized Rost motive. (For $p = 2$ genericity and the construction of the motive can be obtained in a much more elementary way using quadratic forms.) All this is essential for the final proof of the conjecture (involving, as for $p = 2$, Margolis homology and the so-called "injectivity", settled in [1], see also [4]).

An important step in handling μ is to verify a certain nontriviality condition. Some ingredients for this part of Voevodsky's work have not been written up in details yet, but it seems that they will appear soon, cf. [7].

Last year I was able to derive genericity and the construction of the motive in a more ad hoc fashion, cf. [3]. One considers the standard spectral sequence for the simplicial scheme $\mathcal{X}$ which leads to the map proj as indicated in the diagram. Then one picks a representative

$$\rho \in \mathrm{CH}^b(X^2)$$

of $\mathrm{proj}(\mu)$. I call any such element a *basic correspondence of the norm variety X of u*. Working with ρ, I could verify the necessary nontriviality condition "by hand", so to speak, namely by investigating the specific examples of norm varieties I had constructed earlier in [1].

For an illustration, let us look at the case $n = 2$. In this case $b = 1$. For X we take a Severi-Brauer variety (of dimension $p-1$). Thus ρ is an element in the Picard group of X^2:

$$\rho \in \mathrm{CH}^1(X^2) = \mathrm{Pic}(X^2)$$

If we pass to the algebraic closure $\bar{k}$ of k, then

$$X_{\bar{k}} = \mathbf{P}^{p-1}_{\bar{k}}$$

and one finds

$$\rho_{\bar{k}} = \pi_0^*[\mathcal{O}(1)] - \pi_1^*[\mathcal{O}(1)] \mod p \operatorname{Pic}(X^2_{\bar{k}})$$

where

$$\pi_0, \pi_1 \colon X \times X \to X$$

are the projections.

References

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